Young Children’s Algebraic Reasoning Abilities

by

Mary Clare Cavanagh

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved November 2016 by the
Graduate Supervisory Committee:

Carole Greenes, Chair
Ray Buss
Elaine Surbeck

ARIZONA STATE UNIVERSITY

December 2016
ABSTRACT

The purpose of this study was to identify the algebraic reasoning abilities of young students prior to instruction. The goals of the study were to determine the influence of problem, problem type, question, grade level, and gender on: (a) young children’s abilities to predict the number of shapes in near and far positions in a “growing” pattern without assistance; (b) the nature and amount of assistance needed to solve the problems; and (c) reasoning methods employed by children.

The 8-problem Growing Patterns and Functions Assessment (GPFA), with an accompanying interview protocol, were developed to respond to these goals. Each problem presents sequences of figures of geometric shapes that differ in complexity and can be represented by the function, \( y = mf + b \): in Type 1 problems (1 - 4), \( m = 1 \), and in Type 2 problems (5 - 8), \( m = 2 \). The two questions in each problem require participants to first, name the number of shapes in the pattern in a near position, and then to identify the number of shapes in a far position. To clarify reasoning methods, participants were asked how they solved the problems.

The GPFA was administered, one-on-one, to 60 students in Grades 1, 2, and 3 with an equal number of males and females from the same elementary school. Problem solution scores without and with assistance, along with reasoning method(s) employed, were tabulated.

Results of data analyses showed that when no assistance was required, scores varied significantly by problem, problem type, and question, but not grade level or gender. With assistance, problem scores varied significantly by problem, problem type, question, and grade level, but not gender.
Participants employed recursive reasoning more frequently than covariational reasoning for all problems, all problem types, and at all grade levels. Use of recursive reasoning decreased by grade level from Grade 1. Use of covariational reasoning increased by grade level from Grade 1.

Surprising results, in terms of the lower performance scores in Grade 3 than Grade 2, and the lack of difference in performance by gender at all grade levels, suggest the need for additional study.
ACKNOWLEDGMENTS

I am deeply indebted to Carole Greenes for her phenomenal guidance throughout my long doctoral program. As the ultimate role model and mentor, she helped me grow as a researcher. Her exacting standards and copious notes on her advisees’ drafts are legendary. I appreciate the generous use of her time and focused attention devoted to detailed editing of each aspect of the instrument, proposal, and finally, my dissertation. Her confidence in me, constant encouragement, and expert direction and focus couldn’t be rivaled. I cannot imagine anyone else with her specific in-depth knowledge, academic expertise, and skills who could guide me to this end.

I am grateful to my other two committee members. Ray Buss was instrumental in overseeing my research design, analyses, and interpretation of the data. His critical feedback and insights were invaluable. Elaine Surbeck urged me to dig deeper into the research of cognitive psychologists and the learning potential and needs of young learners. Her careful reading and critiques directed me to keep the teacher in mind while attending to appropriate content and activities for children. Thank you to Eli Chambers, my note taker, for his keen observations, as well as to Shuchi Sharma and Kacey Clark who efficiently transcribed and organized data about participants’ reasoning methods.

Thank you to my daughter, Nicole, who cross-checked citations with references and was my constant cheerleader, as well as to my family for their ongoing encouragement: Nicole; Pat & Dorothy; Jack & Dee Dee; Paul & Jill; Therese & Vince; Danielle & Dan; Natalie & Kirk; Christine & Scott; Michelle & Jim; Paul; Patrick & Jack; Daniel & John; Jacob & Ethan; Luke, Sophia & Bridget; Addison, Claire & Julia; and Kaylie. Their photos on my desktop kept me smiling.
# TABLE OF CONTENTS

| LIST OF TABLES | vii |
| LIST OF FIGURES | viii |

## CHAPTER 

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 OVERVIEW OF THE STUDY</td>
</tr>
<tr>
<td>Introduction</td>
</tr>
<tr>
<td>Statement of the Problem</td>
</tr>
<tr>
<td>Research Questions</td>
</tr>
<tr>
<td>Rationale</td>
</tr>
<tr>
<td>Chapter Summary</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 REVIEW OF THE LITERATURE</td>
</tr>
<tr>
<td>Introduction</td>
</tr>
<tr>
<td>An Historical Overview of “Function” in Mathematics</td>
</tr>
<tr>
<td>Function in School Mathematics</td>
</tr>
<tr>
<td>Policies and Recommendations by Professional Organizations</td>
</tr>
<tr>
<td>Research Focusing on Young Students’ Functional Reasoning Talents</td>
</tr>
<tr>
<td>Assessment with Assistance</td>
</tr>
<tr>
<td>Mathematics Content of the GPFA</td>
</tr>
<tr>
<td>Chapter Summary</td>
</tr>
</tbody>
</table>

iv
<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>METHODS AND PROCEDURES</td>
<td>62</td>
</tr>
<tr>
<td>Introduction</td>
<td>62</td>
</tr>
<tr>
<td>Instrument Development</td>
<td>62</td>
</tr>
<tr>
<td>Pilot Studies</td>
<td>62</td>
</tr>
<tr>
<td>Population and Sample</td>
<td>73</td>
</tr>
<tr>
<td>Study Instruments</td>
<td>74</td>
</tr>
<tr>
<td>Study Questions and Methods of Analysis</td>
<td>83</td>
</tr>
<tr>
<td>Administering the Growing Patterns and Functions Assessment</td>
<td>85</td>
</tr>
<tr>
<td>Timeline for Instrument Development, Recruitment of Participants, Data Collection and Analyses</td>
<td>87</td>
</tr>
<tr>
<td>Chapter Summary</td>
<td>87</td>
</tr>
<tr>
<td>4</td>
<td>89</td>
</tr>
<tr>
<td>DATA ANALYSIS</td>
<td>89</td>
</tr>
<tr>
<td>Introduction</td>
<td>89</td>
</tr>
<tr>
<td>Types of Data</td>
<td>89</td>
</tr>
<tr>
<td>Research Results</td>
<td>90</td>
</tr>
<tr>
<td>Chapter Summary</td>
<td>107</td>
</tr>
<tr>
<td>5</td>
<td>109</td>
</tr>
<tr>
<td>DISCUSSIONS AND RECOMMENDATIONS</td>
<td>109</td>
</tr>
<tr>
<td>Introduction</td>
<td>109</td>
</tr>
</tbody>
</table>
5  Discussions and Conclusions ................................................................. 110
   Summary of Results and Conclusions .................................................. 116
   Limitations of the Study ..................................................................... 117
   Recommendations for Future Research ............................................. 118
   Recommendations for Teaching Practice ........................................... 118

REFERENCES ........................................................................................... 122

APPENDIX

A  IRB APPROVAL .................................................................................. 143
B  STATEMENT OF INFORMED CONSENT ....................................... 145
C  GROWING PATTERNS AND FUNCTIONS ASSESSMENT ............. 147
D  INTERVIEW PROTOCOLS ................................................................. 158
E  PARTICIPANT ASSESSMENT SHEET FOR NOTE TAKER AND
   INTERVIEWER .................................................................................. 163
F  DOCUMENTATIONS OF PARTICIPANTS’ COMMENTS .................. 166
G  PILOT STUDY A ............................................................................... 170
H  PILOT STUDY B ............................................................................... 174
I  PILOT STUDY C ............................................................................... 185
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.01 Correct Responses to Tasks 3, 4, 5 and 6</td>
<td>34</td>
</tr>
<tr>
<td>3.01 Timeline for Instrument Development, Recruitment of Participants, Data Collection and Analyses</td>
<td>86</td>
</tr>
<tr>
<td>4.01 Means of Scores (RQ 1) without Assistance by Problem, Problem Type, Question, Grade, and Gender</td>
<td>91</td>
</tr>
<tr>
<td>4.02 Means of Scores (RQ 2) with Assistance by Problem, Problem Type, Question, Grade, and Gender</td>
<td>96</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.01</td>
<td>What’s My Rule? Table and Function</td>
<td>21</td>
</tr>
<tr>
<td>2.02</td>
<td>Representations of Functions</td>
<td>22</td>
</tr>
<tr>
<td>2.03</td>
<td>Peanuts Cartoon Ridicules New Math</td>
<td>24</td>
</tr>
<tr>
<td>2.04</td>
<td>Towers of Cubes and Table</td>
<td>27</td>
</tr>
<tr>
<td>2.05</td>
<td>Kim and Sloane’s Pattern Assessment</td>
<td>31</td>
</tr>
<tr>
<td>2.06</td>
<td>Item 1: Relate Number of Eyes to The Number of Dogs</td>
<td>37</td>
</tr>
<tr>
<td>2.07</td>
<td>Item 2: Follow a Function Rule</td>
<td>37</td>
</tr>
<tr>
<td>2.08</td>
<td>Item 6: Functions as Algebraic Equations</td>
<td>37</td>
</tr>
<tr>
<td>2.09</td>
<td>Item 11: Identify the Next Value in a Number Pattern</td>
<td>37</td>
</tr>
<tr>
<td>2.10</td>
<td>Pretest Item</td>
<td>44</td>
</tr>
<tr>
<td>2.11</td>
<td>WPT Item 1 for the Function $b = a + 1$</td>
<td>47</td>
</tr>
<tr>
<td>2.12</td>
<td>WPT Item 2 for the Function Mike = Leslie + 4</td>
<td>47</td>
</tr>
<tr>
<td>2.13</td>
<td>CTT Function Machine</td>
<td>48</td>
</tr>
<tr>
<td>2.14</td>
<td>CTT Recording Board for $y = 3n + 1$</td>
<td>49</td>
</tr>
<tr>
<td>2.15</td>
<td>Year 1, Task 1 for Number of Squares = Figure Number + 1</td>
<td>52</td>
</tr>
<tr>
<td>2.16</td>
<td>Three Students’ Drawings for Task 1 Figure 5</td>
<td>52</td>
</tr>
<tr>
<td>2.17</td>
<td>Pretest</td>
<td>54</td>
</tr>
<tr>
<td>3.01</td>
<td>Problem 1, Pilot Study A</td>
<td>63</td>
</tr>
<tr>
<td>3.02</td>
<td>Problem 2, Pilot Study A</td>
<td>64</td>
</tr>
<tr>
<td>3.03</td>
<td>Problem 3, Pilot Study A</td>
<td>65</td>
</tr>
<tr>
<td>3.04</td>
<td>Problem 4, Function Table, Pilot Study A</td>
<td>66</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>3.05</td>
<td>Problem 1, Pilot Study B</td>
<td>68</td>
</tr>
<tr>
<td>3.06</td>
<td>Problem 3: (2 shapes/colors) ( T = d + 2 ), Pilot Study B</td>
<td>69</td>
</tr>
<tr>
<td>3.07</td>
<td>Problem 4: (3 shapes/colors) ( T = d + 2 ), Pilot Study B</td>
<td>70</td>
</tr>
<tr>
<td>3.08</td>
<td>Problem 2, Pilot Study C</td>
<td>72</td>
</tr>
<tr>
<td>3.09</td>
<td>Problem 6, Pilot Study C</td>
<td>72</td>
</tr>
<tr>
<td>3.10</td>
<td>Problem 1 ( GPFA )</td>
<td>76</td>
</tr>
<tr>
<td>3.11</td>
<td>Problem 2 ( GPFA )</td>
<td>77</td>
</tr>
<tr>
<td>3.12</td>
<td>Problem 3 ( GPFA )</td>
<td>77</td>
</tr>
<tr>
<td>3.13</td>
<td>Problem 4 ( GPFA )</td>
<td>78</td>
</tr>
<tr>
<td>3.14</td>
<td>Problem 5 ( GPFA )</td>
<td>78</td>
</tr>
<tr>
<td>3.15</td>
<td>Problem 6 ( GPFA )</td>
<td>79</td>
</tr>
<tr>
<td>3.16</td>
<td>Problem 7 ( GPFA )</td>
<td>79</td>
</tr>
<tr>
<td>3.17</td>
<td>Problem 8 ( GPFA )</td>
<td>80</td>
</tr>
<tr>
<td>4.01</td>
<td>Means of Problem Scores by Problem Number and Type</td>
<td>93</td>
</tr>
<tr>
<td>4.02</td>
<td>Trends in Problem Scores by Problem Pairs and Problem Types</td>
<td>94</td>
</tr>
<tr>
<td>4.03</td>
<td>Means of Assistance Scores by Problem Number and Problem Type</td>
<td>98</td>
</tr>
<tr>
<td>4.04</td>
<td>Trends in Assistance Scores by Problem Pairs and Problem Types</td>
<td>99</td>
</tr>
<tr>
<td>4.05</td>
<td>Grade 2 Participant’s Recording for Problem 8</td>
<td>102</td>
</tr>
<tr>
<td>4.06</td>
<td>Use of Recursive and Covariational Reasoning by Need for Assistance</td>
<td>103</td>
</tr>
<tr>
<td>4.07</td>
<td>Use of Covariational Reasoning by Problem Number and Problem Type</td>
<td>104</td>
</tr>
<tr>
<td>4.08</td>
<td>Use of Recursive Reasoning by Problem and Problem Type</td>
<td>105</td>
</tr>
<tr>
<td>4.09</td>
<td>Percent of Use of Recursive Reasoning Methods by Problem</td>
<td>106</td>
</tr>
</tbody>
</table>
CHAPTER ONE

OVERVIEW OF THE STUDY

Algebra is a major domain of mathematics and widely regarded as the “gateway” to the study of advanced mathematics, higher education, and a wide selection of careers (Chazan, 2000; Kilpatrick & Izsák, 2008; Moses, 1995). Knowledge of algebra’s concepts and skills is also foundational to the study of other content areas (e.g., the sciences, economics, business, engineering, and the visual arts), because it provides the language and structure for analyzing and generalizing relationships and solving problems (Ball, 2003; Roth, 2006).

The National Council of Teachers of Mathematics’ (NCTM) Curriculum and Evaluation Standards for School Mathematics (1989) and its Principles and Standards for School Mathematics (2000) include algebra as a strand for exploration by students in all grade levels, Kindergarten through Grade 12. Although the Common Core State Standards for Mathematics, released in 2010 by the National Governors Association Center for Best Practices (NGA) and the Council of Chief State School Officers (CCSSO), do not single out algebra or algebraic reasoning as separate content foci until Grade 6, the algebraic ideas of expressions and equation relationships and the use of different representations (e.g., tables, graphs, words, and symbols) are embedded in the recommendations for study in the number and operations and geometry standards.

Historically, two different approaches have been employed in the teaching of algebra: An equations approach focuses on operations, properties, manipulating symbols, and solving equations and systems of equations. A functions approach focuses on relationships between quantities and the generalization of patterns (Stroup, 2005).
Researchers have studied young children’s abilities to reason algebraically focusing on one or a combination of both approaches.

Researchers who focused on the functions approaches found that students, as young as 6 years, are capable of learning to extend growing patterns, make generalizations, and describe functional relationships. Few studies have examined young students’ functional reasoning talents prior to instruction and even fewer have documented the nature of assistance students need to become successful. Knowing what students know in advance of teaching is critical to building curricula (Falk, 2000; National Council of Teachers of Mathematics, 2000; Wiggins, 1988). “What children can do with the assistance of others is even more indicative of their mental development than what they can do alone” (Vygotsky, 1978, p. 85).

Statement of the Problem

The purposes of this study were to: 1) Investigate Grades 1 – 3 students’ abilities to solve and generalize increasingly more complex growing patterns, without and with assistance; 2) Document the nature of that assistance; and 3) Identify the reasoning methods students employ in the solution of problems.

Research Questions

Research Questions (RQ). This study addressed the following questions:

RQ 1: Does the Growing Patterns and Functions Assessment (GPFA) score vary by
a. Problem?
b. Problem Type?
c. Question?
d. Grade Level?
RQ 2: Does the Assistance Score vary by
a. Problem?
b. Problem Type?
c. Question?
d. Grade Level?
e. Gender?

RQ 3: Do Reasoning Methods (recursive reasoning and covariational reasoning), vary by:
  a. Problem?
  b. Problem Type?
  d. Grade Level?
  e. Gender?

Definition of Terms

Assistance. Assistance is provided through sets of pre-determined questions that are presented to participants who are experiencing difficulty during the solution process (Tsankova, 2003). In the present study, two types of assistance are provided: general and specific. General assistance focuses on the overall pattern and how it changes (e.g., “Tell me how the pattern changes;” “What part of the pattern stays the same?”). Specific assistance focuses on one figure or on relationships between consecutive figures in a pattern, (e.g., “Count the number of shapes in this figure;” “How is Figure 2 different from Figure 1?”).

Covariational reasoning. This type of reasoning entails the conception of two quantities varying in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), as for
example, when the position of a figure in a pattern is used to determine the number of shapes in that figure.

**Function.** A function pairs one domain of values \((x)\) with one range of values \((y)\). For every \(x\) value, there is a unique \(y\) value. Functions can be represented symbolically as \(y = mx+b\). When the exponent of \(x\) is 1, the graph is a straight line. When \(b = 0\), the graph contains the origin. When \(b \neq 0\), the line does not contain the origin. In the present study, both types of linear functions were used, those in which \(b = 0\) and \(b \neq 0\).

**Growing pattern.** This is a pattern in which the figures in the pattern change in a quantitatively predictable way (e.g., “plus 2” pattern: 0, 2, 4, 6…)

**Prompts.** A prompt is an encouragement to continue thinking, talking about, and solving. Examples of prompts are: “Keep going.” “Good.” “Yes.” “Tell me more.” “Okay.” “And?” “Tell me what you see.” Prompts also include nods or gestures of support.

**Recursive reasoning.** As Kenney and Bezuska (2015) describe in their book, *Number Treasury 3: Investigations, Facts and Conjectures About More Than 100 Number Families*, “The \(n\)th term of a sequence is in recursive form if each new member of the sequence is expressed in terms of the preceding member(s)” (p. 107). In the lower elementary grades, recursive reasoning occurs when students “describe patterns like 2, 4, 6, 8,… by focusing on how a term is obtained from the previous number – in this example, by adding 2” (NCTM, 1998, p. 38).
Rationale

The need for the present study is supported by the Vygotskian Theory of Learning, the importance of function in school mathematics, and research on young children’s learning.

Theoretical Foundation

The theoretical foundation for this research is based on the work of the Russian psychologist, Lev Vygotsky, and his Zone of Proximal Development (ZPD) (1978). The ZPD is the distance between the actual and the potential developmental levels of a student – that is, the distance between what students can do without assistance and what they can accomplish with varying degrees of assistance. According to Vygotsky, appropriate support or scaffolding will enable students to learn beyond their capacity but within their ranges of competence. However, to build on what students know, their potential must be assessed throughout the learning process, not at the end of learning. Vygotsky’s research also supports providing assistance through the use of probing questions and prompting gestures in both learning and assessment environments.

Vygotsky’s work and the idea of The Zone of Proximal Development (1978) has spurred researchers and educators in all areas of education to develop assessment instruments that measure students’ prior knowledge (Balac, 2001; Balazic, 1997; Behrend, 1994; Burns, 2010; Campione & Brown, 1984; Campione, Brown, & Ferrara, 1982; Dixon-Krauss, 1996; Ferrara, 1987; Ferrara, Brown, & Campione, 1986; Feuerstein, 2000; Ginsburg, Jacobs & Lopez, L.S., 1993; Mathews, 2002; Robb, 2014; Tsankova, 2003). Knowing what students know is important to curriculum developers and teachers who structure curricula that build on that knowledge in order to produce

“If we had to reduce all of educational psychology to just one principle, we would say this: the most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly” (Ausubel, Novak, and Hanesian, 1978, p. 163). Students’ proficiency and interest in mathematics thrive when new concepts make sense because they are connected with their prior knowledge (Ginsburg, 2002).

**Importance of Algebra and Algebraic Reasoning in School Mathematics**

Algebra, as a course of study, has been considered a cornerstone of school mathematics since the 1800s (Greenes, Cavanagh, Dacey, Findell, & Small, 2001; Kilpatrick & Izsák, 2008). The National Council of Teachers of Mathematics (NCTM) identified algebra as one of the domains in its *Curriculum and Evaluation Standards for School Mathematics* (1989) and in its *Principles and Standards for School Mathematics* (2000). In the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010), algebra is once again highlighted, because its concepts and reasoning methods are critical to not only the study of more mathematics (Beatty, 2014; Usiskin, 1995), but also to be a means of increasing success in the study of other fields, such as the sciences, economics, business, sports, and the workplace (Ball, 2003; Roth, 2006).

Recognizing the importance of algebra, policy makers and educators supported the *Algebra for All* movement in the 1990s, recommending that *all* students have the
opportunity to enroll in courses in algebra in Grade 9, and in some schools/communities, in Grade 8 (ACT 2008; Balfanz & Legters, 2004; Everson & Dunham, 1996; Steen, 1999; Usiskin, 1988). In the U.S. Department of Labor’s Secretary’s Commission on Achieving Necessary Skills (SCANS) Report (1991), algebraic thinking was identified as a skill young people need to succeed in a high performance workplace. In Algebra, The New Civil Right (1995), Robert Moses stated that “Algebra means access. It unlocks doors to productive careers and democratizes access to big ideas” (p. 53).

By 1995, the failure rates in Algebra I were increasing (Ball, 2003; Schmidt, 2012; Silver, 1995), and attitudes about the value of mathematics were plummeting (Silver, 1995). Despite increased efforts to enhance student performance, in Grades 8 and 9, high rates of failure persisted, ranging from 20% to 91% (Anderson, 2014; Carnoy & Rothstein, 2013; International Association for the Evaluation of Educational Achievement, 2012; National Science Foundation, 2011; National Center for Science and Engineering Statistics, 2014; Richmond (2009); Stoelinga & Lynn, 2013; U. S. Department of Education, 2015).

These higher failure rates were attributed to students’ lack of preparation for the study of algebra (Bottoms, 2003; Carpenter, Levi, Franke, Zeringue, 2005). The transition from arithmetic to the “more abstract” algebra was identified as a major contributing factor (Cai, Lew, Morris, Moyer, Ng, & Schmittau, 2005; Cai & Moyer, 2008; Carpenter & Levi, 2000; Tsankova, 2003). The NCTM pointed to the need for stronger connections between curriculum and instruction, and called for well-articulated connections across grade levels (NCTM, 2000, 2006). To lay a foundation for smoothing the transition to algebra, while concurrently strengthening understanding of arithmetic,
Carpenter, Franke, and Levi (2003) recommended the introduction of algebraic reasoning in the early grades.

Despite recommendations for early exposure to algebraic reasoning, there is a paucity of information about what young students are capable of learning, the content and progression of topics, and the teaching approach.

**Function in School Mathematics**

Of the two main foci in algebra instruction – equations and functions – the equation approach has received greater attention. Its focus is on learning computational techniques to solve equations and systems of equations in which variables are treated as unknowns. The job of the solver is to manipulate the equation(s) to solve for the value(s) that “satisfies” the equation(s), that is, makes the statement(s) true.

By contrast, with functions the focus of instruction is on identifying how pairs of variables “vary” in a systematic way. The role of the solver is to identify the rule that generates one value based on its relationship to the other. The present study focused on functions.

Researchers have emphasized the importance of functional reasoning, not only in algebra, but also in the other strands of mathematics, including number and operations, measurement, geometry, and statistics (Cooper & Warren, 2011; Dougherty, 2008; Kilpatrick & Izsák, 2008). In economics and the physical and biological sciences, all formulas are functions (NCTM, 2000; Saul, 2008).

In the curriculum for Grades 1 through 5, the concept of a function is seen in the various domains of mathematics, with particular emphasis on linear functions of the form, \( y = mx + b \). When \( b = 0 \), those linear functions are referred to as proportions.
the study of the base ten number system, when “regrouping” tens to ones, students observe that 1 ten = 10 ones, 2 tens = 2 x 10 or 20 ones, 3 tens = 3 x 10 or 30 ones, and so on. This is an example of a linear function, a multiplicative relationship (\( y = 10x \), where \( y \) represents the number of ones and \( x \) represents the number of tens). Likewise, 1 hundred = 10 tens, so 2 hundreds = 20 tens, and 3 hundreds = 30 tens. We can also say that the tens and ones (and hundreds and tens) are related “proportionally” (e.g., 1 is to 10 as 2 is to 20, and 3 is to 30).

Students see this same type of proportional relationship when they convert measurement units. For example, in the customary system, 1 foot = 12 inches, so 2 feet = 2 x 12, or 24 inches, etc., and 1 quart = 2 pints, so 2 quarts = 4 pints, etc. Likewise, in the metric system, 1 meter = 100 centimeters, so 2 meters = 2 x 100 or 1 quarter = 5 nickels, so 2 quarters = 2 x 5, or 10 nickels.

**Research on Young Children’s Knowledge of Functional Relationships**

Studies of students’ knowledge of functional relationships have employed various formats, the use of patterns of numbers or shapes has been used most frequently. Other researchers have used tables of related data or machines that “operate” on inputs to produce outputs (or vice versa).

**Studies employing growing patterns.** Kim & Sloane (2010) conducted a study of 64 Grade 2 students using both two repeating patterns and two growing patterns. For the growing patterns, the researchers used white and black congruent circles for one problem and two upper case letters as the figures for the other problem. There were no figure numbers and the groups of figures were not separated. Only 31% of students were successful with the letters pattern and only 39% of students successfully solved the shape
pattern. Participants had great difficulty recognizing what was changing in each growing pattern, and did not recognize the relationships between the position of a figure in the pattern and the number of letters/shapes in that position.

The lack of figure numbers, the nature of the figures and the spacing between figures may have contributed to poor performance. Furthermore, the limited number of problems made generalization of results of the Kim & Sloane (2010) study suspect.

As part of a teaching study, Radford (2014, 2012) studied 25 Grade 2 students’ understanding of growing patterns of congruent squares in only one item. That item showed the first four figures of a pattern with an increasing number of squares. Position numbers were identified under each figure. Students had to extend the pattern by drawing and recording the number of squares in Figures 5 and 6. Drawings indicated that Grade 2 students were not able to identify the correct number of squares in those two figures. Radford concluded that students lacked understanding of the meaning of the phrase, “growing pattern,” and the act of “extending” the pattern. Furthermore, the use of one item made generalization not possible.

Moss & London McNab (2011) conducted a teaching study with 120 Grade 2 students. Their pretest consisted of one problem. Position numbers were located under each figure of a growing pattern with increasing numbers of squares. The assessment was conducted as a one-on-one interview. Results revealed that, without assistance, none of the Grade 2 students correctly predicted the number of squares in the 4th position, or in later positions of the pattern. None of the students could describe the pattern rule.

Studies employing combinations of pictorial representations and tables of data. McEldoon & Rittle-Johnson (2010) assessed 231 Grades 2-6 students using two
formats for the presentation of growing patterns: 1) horizontal lists of numbers (e.g., 0, 2, 6, 12, . . .), and 2) vertical function tables with sequential entries in the input columns. The job for participants was to identify the next three figures in the list of numbers, complete the table, and select the rule from a set of options. The percentage of Grade 2 and Grade 3 students who successfully completed number patterns when a rule was provided was 71% for Grade 2 and 82% for Grade 3. By contrast, when identifying the function rules from sets of four multiple choice items, only 35% of Grade 2 and 45% of Grade 3 participants were successful. The greater percentages of student success, as compared to other studies, may be attributed to the function table format in which input and output numbers are displayed and the number of given inputs/outputs was four or more.

McEldoon & Rittle-Johnson (2010) developed an assessment guide to provide specific wording to enable those conducting the study to assist participants who requested help/clarification. The Assessment Guide is valuable as a model for other studies. Unfortunately assistance provided to participants during the study was not tracked.

Blanton, Stephens, Knuth, Gardiner, Isler, and Kim (2015) assessed 106 Grade 3 students as part of an algebraic reasoning teaching study. Of the 11 pretest items, only Item 10 focused on functional relationships. That item contained 5 questions. The context was people seated at tables. The problem was to identify how the number seated is related to the number of “connected” tables. Nineteen percent of Grade 3 students were successful. They used strategies such as drawing pictures of tables or extending the pattern in a function table. Only 3% of participants could describe the relationship between the number of tables and the number of people who could be seated at the tables. As the researchers acknowledged, solution methods employed by participants were not
clear. Furthermore, the connected tables format, with joined sides caused visualizing and counting difficulties.

**Studies employing function machines.** Sorkin (2011) conducted one-on-one interviews with 97 Grade 2 students to assess their abilities to identify function rules as part of a teaching study. Sorkin used a tri-fold presentation board with a picture of a machine with two doors, one labeled “Input” and the other door labeled “Output,” to the right of the input door. Participants chose a tower, made from 2 to 6 cubes, placed the tower in the input door, and out “popped” a new tower. Participants had to figure out what the machine did to the cubes inside. Sixty-four percent of all participants succeeded in identifying function rules and describing them. However, articulating the rules clearly was challenging, particularly for the more difficult items (e.g., Output = $n(n + 1)$). The researcher scaffolded problems when participants did not articulate or show the correct answer with cubes. That “scaffolding” assistance was not documented.

Warren, Miller, & Cooper (2011) conducted interviews with 10 students in Grades 2 and 3 to assess their functional reasoning abilities. The researchers used six items: two language, one shape, and three number tasks, all in conjunction with the function machine, “Rosie.” For the shape items, an attribute block such as a red, large, thin triangle went into Rosie’s ear and a red, large thin, triangle came out of the opposite ear. After three trials, participants were asked to predict the output value for each new input value, and to then identify the rule. Data revealed that 90% of all participants correctly predicted the output values from given input values, and that 83% were able to describe the function rules. However, while the findings are impressive, they are not generalizable because of the small sample and limited number of items.
Chapter Summary

The need for students’ increased proficiency in algebra has been called for by professional education associations, higher education, and industry. The importance of the concept of function stems from its centrality in all branches of mathematics, related fields (e.g., the sciences) and in everyday life. To better prepare students for the study of algebra, early experience with algebraic concepts is recommended. Recent research has demonstrated that young children are capable learners, and know more than what is expected. However, there is a paucity of research on young students’ knowledge of functions and functional reasoning. Research on the nature and amount of assistance students need to solve algebraic reasoning problems will provide valuable information in the design of curricula that will facilitate efficient learning.

To date, none of the studies of students’ functional reasoning talents focus on the entire grade span, Grades 1-3. Five studies examined performance at only one grade level. Two studies investigated two levels, relevant to the present study. Studies that offered assistance to participants did not document nor analyze that assistance. The influence of gender has not been studied.

Chapter Two presents an historical overview of the importance of function in mathematics and school mathematics, as well as policies and recommendations for incorporating function in the preK-12 curricula. The chapter continues with detailed accounts and analyses of research studies of young students’ knowledge of function prior to instruction or without assistance. The chapter concludes with a description of the mathematics content of the Growing Patterns and Functions Assessment (GPFA).
CHAPTER TWO

REVIEW OF THE LITERATURE

The present study was designed to gain insight into Grades 1, 2, and 3 students’ knowledge of aspects of the concept of function prior to instruction, and to determine the amount and type of assistance they need to solve function problems. In this chapter are summaries of the role of function in mathematics education, from both historical and contemporary perspectives. Current literature on the inclusion of the topic of functions in elementary school curricula, and studies of young students’ understanding of function are described next. This is followed by descriptions and critiques of research studies on the learning of patterns and functions that are relevant to the present study.

An Historical Overview of “Function” in Mathematics

Function is a key concept in the study of calculus as it is central to the study of change (e.g., quantities, growth, time, speed, and acceleration). Although Sir Isaac Newton (1642-1727) was one of the first mathematicians to use calculus to solve physics problems concerning motion and velocity, it was Gottfried Leibniz (1646-1716), a German mathematician and philosopher, who developed the system of notation for calculus that is widely used in mathematics today (Cajori, 1928; Mastin, 2010). Leibniz introduced the mathematical term function to describe the relationship between two quantities that he called x and y. He also introduced the terms constant, variable, and parameter (Ponte, 1984).

Peter Dirichlet (1805 - 1859), a German mathematician noted for his major contributions to number theory, is recognized as one of the first mathematicians to provide a definition of function that is still used in mathematics today: “to any x there
corresponds a single finite \( y \)” (Elstrodt, 2007, p. 19). In 1917, Carathéodory provided this definition of function: “a rule of correspondence from a set \( A \) to the set of real numbers” (Malik, 1980, p. 491). In 1939, a group of primarily French mathematicians who collectively wrote using the pseudonym Nicolas Bourbaki, provided this definition of function: “a rule of correspondence between two sets.”

Despite variations in definitions and descriptions, functions were used primarily to model the generality of physical phenomena, like the concept of motion. Today, functions are used to describe, explain, and predict phenomena in the life sciences, human and social sciences, and engineering. As noted by a great many mathematicians and mathematics educators, the concept of function undergirds the various domains of mathematics, such as measurement relationships and formulas, proportions, the base ten system, and ideas related to variation and constant rates of change (Chazan & Yerushalmy, 2003; Ponte, 1992; Tall, 1992; Weber, 2012).

**Function in School Mathematics**

The role of function in school mathematics became the focus of attention of mathematics education reformists in the latter part of the 19th century, and has continued to the present.

**The early years: 1890-1950.** The focus on the concept of function in school mathematics began with secondary curricula innovation and later reached the elementary school level. In 1892, a subcommittee of the Committee of Ten, formed by the National Education Association (NEA), advanced the idea of equation as a unifying theme in mathematics in their College Entrance Requirements Committee report in 1899. The American Mathematical Society in its 1903 *Report of the Committee of the American*
Mathematical Society on Definitions of College Entrance Requirements in Mathematics, concurred with the NEA. In 1893, in a speech at the University of Chicago, Felix Klein, a German professor of mathematics noted for his work in function theory recommended that function be the “central theme of school mathematics” (Hamley, 1934, p. 49).

As leaders in mathematics were advocating for an emphasis on algebra and function in school mathematics, societal issues posed threats to mathematics education. Secondary school enrollments were exploding and pressure was building to tie students’ education more closely to future employment, putting academic subjects on the defensive. Some schools dropped mathematics as a graduation requirement (Jones, 1972; Schiller & Muller, 2003). Responses to these attacks resulted in the founding of professional associations. The Mathematical Association of America (MAA) was founded in 1915 and the National Council of Teachers of Mathematics (NCTM) was founded in 1920.

In 1916, Earle Raymond Hendrick, a mathematician at the University of Missouri, became the first president of the Mathematical Association of America. In that capacity, he formed the National Committee on Mathematical Requirements (NCMR) to make recommendations for the reform of the teaching of mathematics. In 1923 the NCMR published the 600-page *The Reorganization of Mathematics in Secondary Education*. Chapter VII of the report is devoted entirely to the concept of function in secondary school mathematics. This was considered to be “the first authoritative statement of the case for functional thinking to be found in American mathematical literature” (Hamley, 1934, p. 78). Hendrick, along with David Eugene Smith, editor of the National Council of Teachers of Mathematics’ (NCTM) first yearbook, argued for greater attention to the
function concept in school curricula in the United States (Hamley, 1934). They both recommended the development of materials for teachers “combining courses unified by one or more central ideas such as functionality and graphic representations” (NCMR, 1923, p. 38).

Crisis in mathematics education: 1950-1960. In the 1950s, it became evident that a greater number of engineers and other technically-prepared workers were needed for the United States to remain a global leader. In response to development of atomic weapons in the 1950s and to the Soviets’ launch of Sputnik in 1957, the United States government increased funding to overhaul education so that more scholars, teacher educators, and highly prepared mathematics teachers were produced (Divine, 1993). Those experts would be needed to help the country compete internationally in the technological revolution.

The National Science Foundation (NSF), an independent federal agency, was created by Congress in 1950 “to promote the progress of science; to advance the national health, prosperity, and welfare; to secure the national defense…” (NSF, 2011, p. 1). Institutes for teachers of mathematics, technology, and science were funded with large sums of money from the NSF. Likewise, the National Defense Act of 1958 (NDEA) provided funds for the teaching of mathematics, science, and foreign language in public schools at all levels (Schwegler, 1982).

The “new mathematics” movement. In 1951, before Sputnik raised mathematics education to the level of a national priority, Max Beberman, a professor of mathematics at the University of Illinois, established the University of Illinois Committee on School Mathematics (UICSM) to address the poor mathematics preparation of University of
Illinois incoming freshmen majoring in engineering. As Beberman noted, most freshmen were “unable to expand a power of a binomial, or graph an elementary function” (Raimi, 2004, Chapter 1, paragraph 44). Beberman, often referred to as the “Father of the New Math,” and his UICSM committee evaluated existing high school mathematics programs and identified serious problems with both the curricula and the teaching practices.

UICSM addressed both issues by developing new instructional materials and training teachers to test the materials with their students. Financial support was provided by the Carnegie Corporation and the (then) U.S. Office of Education. The UICSM program emphasized precise language. For example, in their treatment of function they stated:

The semantics notion that a noun ought to have a referent has led us to give precise descriptions of relations and functions. The customary vagueness that surrounds the word 'function' in conventional courses vanishes when a student realizes that a function is an entity, a set of ordered pairs in which no two figures have the same first component. (Howson, Keitel, & Kilpatrick, 1981, p. 22).

These Institutes were designed to first upgrade the mathematical competence of liberal arts college mathematics teachers, and then secondary school teachers. By 1957 the Institutes tended to be associated with “New Math.” Only teachers who had received training were permitted to use project materials, High School Mathematics, Course 1, 1964 and Course II, 1965), published by D. C. Heath and Company. UICSM’s heavy emphasis on set theory, number bases other than base ten, and its abstract and formal treatment of other topics tended to confuse teachers, students, and parents (Raimi, 2004).
Succeeding Beberman as director of UICSM was Robert B. Davis, a noted mathematician and educator. He led UICSM from 1970 until its demise in 1975. Davis is best known for his leadership of the Madison Project (Klein, 2005), described later in this chapter.

**School Mathematics Study Group.** A prominent K-12 “New Math” curriculum project was the School Mathematics Study Group (SMSG), which operated from 1958 through 1977, with funding from the National Science Foundation (NSF). The project was directed by Edward G. Begle, Professor of Mathematics at both Yale and Stanford Universities. Initially, SMSG emphasized the structure of mathematics rather than routine computation (Sloan, 1974). The publication of the SMSG elementary program (SMSG, 1967) trailed behind the secondary program by several years (Raimi, 1995). Functions were informally introduced with exercises, such as, “The members of Set A are found by adding 9 to each of the numbers in the set \{0, 1, 2, 3, …, 9\}” (SMSG, 1965, Grade 4, p. 81, Exercise 6a). SMSG books were placed in the public domain as “exemplary” textbooks for publishers to study and copy freely when producing their own textbook programs.

Approximately 50% of public high school mathematics teachers participated in at least one of the SMSG Institutes. The Institutes were designed to familiarize them with the new curriculum and have them rehearse teaching strategies under the guidance of SMSG authors and master teachers (Klein, 2005). Unfortunately, resources were not available to provide adequate professional development for the great number of elementary teachers interested in the training, who outnumbered the high school mathematics teachers ten to one (Raimi, 1995). Furthermore, elementary teachers, who
typically had weak backgrounds in mathematics, “ignored anything demanding in their textbooks and took refuge in teaching the algorithms they had themselves learned as children” (Raimi, 1995, p. 1).

By the mid-1970s, SMSG was considered a commercial failure because of its excessive focus on language, the “theory of sets,” number systems other than base 10, and the large number of teachers and parents unable to adjust to the text materials (Duren, Askey & Merzbach, 1989; Grouws, 1988; Klein, 2005; Roberts, 2004; Wooton, 1965). Despite the program’s ups and downs, its inclusion of function in secondary school mathematics has been identified as one of the most important curriculum recommendations of the 20th century (Blanton & Kaput, 2011; Cai & Knuth, 2011; Freudenthal, 1982; Hamley, 1934; Schwartz, 1990; Sheehy, 1996).

**The Madison Project.** In 1957, Robert B. Davis founded the Madison Project at Syracuse University with funding from the National Science Foundation that continued until 1975. The Madison Project was identified as a “curriculum revision” project that broadened the scope of elementary school mathematics from attention to not only algorithms in arithmetic but also to: 1) fundamental concepts of algebra, such as variable, function, and signed numbers; 2) fundamental concepts of coordinate geometry, such as the graph of a function; 3) ideas of logic, such as implication; and, 4) relations of mathematics to the physical sciences (Davis, 1964). Davis identified the need for creating interest in mathematics by employing a greater variety of rich open-ended learning activities and using physical models and mathematical symbols that enable understanding of mathematics. As noted by Davis, “Interesting patterns must lurk under the surface of every task” (1964, p. 8).
One activity Davis used to introduce linear functions was “Guess My Rule.” In that activity, a leader provides students with values for the input of a “Function Machine” and the corresponding output. The job for students is to try to figure out the rule that relates all pairs of values, the output given the input, and then the input when given the output (Davis, 1967, 1985). Figure 2.01 shows a function table that uses a square for the input variable and a triangle for the output variable. The input values are listed in the column under the square. The output values are listed under the triangle. In this example, the value for the triangle is always three times that of the square.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
</tbody>
</table>

*Figure 2.01. What’s My Rule? Table and function (Page, 1964, p.194)*

The use of geometric shapes to represent variables, such as those shown in Figures 2.01 and 2.02, was encouraged by David Page, a contributor to the Madison Project. Page referred to the shapes as “frames” (Page, 1964, p.194), stating that they “provide a place for the numbers to go” (Page, 1964, p. 194), as opposed to using letters, such as $x$ and $y$. Using letters, students are “much more likely to think $x$ is really a (specific) number” (Page, 1964, p. 195). Figure 2.02 shows the two representations of $3 + 6 = 9$ with variables, one using frames and the other using letters.
Figure 2.02 Representations of functions (Page, 1964, p. 195)

Reactions to the “New Math.” During the 1960s, teachers, administrators, and parents were struggling with the “New Math.” Teachers were uncomfortable with a curriculum they had not been prepared to implement. Administrators were concerned about plummeting test scores. Parents felt unprepared to help their children and did not believe that the curriculum was practical. When would their children see “clock arithmetic,” bases other than ten, or set theory in the workplace? (Schoenfeld, 2004). At a symposium sponsored by the Thomas Alva Edison Foundation in Pittsburgh in November 1960, Edward G. Begle, Director of the School Mathematics Study Group (SMSG) stated, “In our work on curriculum, we did not consider the pedagogy” (Kline, 1974, Chapter 9). The New York Times ran a front page article covering the 1964 National Council of Teachers of Mathematics meeting in Montreal where Max Beberman, “Father of the New Math,” stated that the introduction of the new mathematics into elementary school programs was “hasty and unwise.” He also acknowledged that the “excessive… emphasis on esoteric branches of mathematics was at the expense of fundamentals.” He warned that New Math programs could produce “a generation of kids who can't do computational arithmetic” (1964, Schwartz, p. 1; see also Klein, 1974, Chapter 6).

Professors and mathematicians from other universities and corporations, including
California Institute of Technology, Harvard University, IBM, New York University, Northwestern University, Rand Corporation, Stanford University, University of California Los Angeles, University of Colorado, University of Maryland, University of North Carolina, University of Toronto, and the University of Wisconsin, drafted a memorandum expressing their concern for the need for change in the mathematics curricula, and their opposition to the “New Math” in its current form. In their memorandum entitled, “On the Mathematics Curriculum of the High School” (American Mathematical Monthly, 1962 and The Mathematics Teacher, 1962), they proposed a redirection of efforts to construct mathematics curricula by following principles and practical guidelines that: 1) provide for all students, including future mathematicians; 2) introduce mathematical concepts through concrete situations before premature abstract formalizations; and, 3) link mathematics with the other sciences and other subjects (Kline, 1974).

In 1965, Nobel Prize winning physicist Richard Feynman served as a Commissioner for the 10-member California State Curriculum Commission to choose the new textbooks for the state. He carefully read all mathematics textbooks submitted by publishers for adoption in California. In his article, “New Textbooks for ‘New’ Mathematics” (1965), he wrote about the over-emphasis on precise language and pure mathematics, offering little in the way of using mathematics to discover something interesting. He said, “I don't think it is worthwhile teaching such material” (p. 15). He asserted that, “We must pay more attention to the connection between mathematics and the things to which they apply” (p. 10).
New Math was lampooned publicly in the media. In 1965, for example, cartoonist Charles Schulz detailed a kindergartener’s frustrations with New Math in his *Peanuts* series (Figure 2.03). After puzzling over “one-to-one matching, equivalent and non-equivalent sets, subsets, joining sets…,” she bursts into tears and exclaims, "All I want to know is, how much is two and two?" (p. 1).

*Figure 2.03 Peanuts cartoon ridicules New Math (Schulz, 1965)*

Throughout his career, American mathematics professor and author Morris Kline was a critic of how mathematics was taught at all levels. In his book, *Why Johnny Can't Add: The Failure of the New Math* (1974), Kline describes Max Beberman’s concession that the New Math was a failure, that the media coverage of mathematical difficulties was shared by adults and children, and that there was a lack of relevance of the New Math to the students’ world. Kline argued for the use of real world applications that were of interest to students. For example, students could be taught statistics by studying baseball batting averages.

Following the New Math era, the pendulum swung to another extreme, the *Back to the Basics* Movement, a basic skills oriented era in education attempting to return to the *Three Rs: Readin’, Ritin’, and ‘Rithmetic.*
Policies and Recommendations by Professional Organizations

Several professional organizations responded to the Back to the Basics movement of the 1970s with support of many key concepts from the New Math. In January 1977, The National Council of Supervisors of Mathematics issued their NCSM Position Paper on Basic Mathematical Skills. In that document, basic mathematics skills included: problem solving; the application of mathematics to model and solve “everyday” mathematics problems; alertness to the reasonableness of results; estimation and approximation; reading, interpreting, and constructing tables, charts, and graphs; using mathematics to predict; and computer literacy.

The National Council of Mathematics (NCTM) elaborated on the basic skill areas identified by NCSM (1977). In its Agenda for Action (1980), NCTM addressed decreasing student performance on national and international assessments, and described the nature of mathematics that should be considered essential for primary and secondary school students with a goal of developing effective problem solvers.

In their ongoing effort to place more emphasis on conceptual understanding and promote higher order thinking in K-12 mathematics instruction, the NCTM published Curriculum and Evaluation Standards for School Mathematics in 1989. That document recognized that the current K-4 mathematics curriculum is “narrow in scope; fails to foster mathematical insight, reasoning and problem solving; and emphasizes the rote activities” (p. 15). To alleviate those problems, NCTM recommended that a variety of content areas, such as measurement, geometry, statistics, probability, and algebra, along with arithmetic, be taught in all grade bands (e.g., K-2, 3-5). They also advocated for greater focus on patterns and relationships, and the teaching of problem solving strategies
The four standards of Problem Solving, Communicating, Reasoning and Making Mathematical Connections, were emphasized as a means of teaching mathematics cohesively with connections among procedures and ideas (NCTM, 1989).

In 2000, the National Council of Teachers of Mathematics revised its 1989 publication and produced *Principles and Standards for School Mathematics* (PSSM). Unlike the previous publication, prekindergarten goals were included. The *Standards* are organized topically by: Number and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability. PSSM was quickly accepted and used to inform the development of state curriculum frameworks, mathematics textbook programs, and instructional practices.

*Principals and Standards for School Mathematics* (NCTM, 2000) clearly states that the study of Algebra (patterns, relations and functions) must be made available to all students because “systematic experience with patterns can build up to an understanding of the idea of function …(it) lays a foundation for later work with symbols and algebraic expressions” (p. 37). In order for students to develop functional reasoning, they must “first understand how quantities change in relation to each other” (NCTM, 2000, p. 37).

For Prekindergarten through Grade 2, *Principals and Standards for School Mathematics* (NCTM, 2000) recommends that students get experience describing patterns, as for example, 2, 4, 6, 8…, by observing how a new term is obtained from the previous term, and by using a “plus 2” function. In Grades 3 to 5, students should investigate patterns and express them in mathematical sentences, make predictions about what comes next, and “develop generalizations about the mathematical relationships in the pattern” (NCTM, 2000, p. 159).
One example provided by NCTM is the “tower of cubes” problem (See Figure 2.04). Grade 3 students should be able to figure out the surface area of each tower and predict the surface area of the next figure in the sequence. Grade 4 students should be able to construct the table. Grade 5 students should also be able to generalize the relationship between surface area \((S)\) and number of cubes \((n)\) in a tower, and predict the number of cubes in remote positions such as the 50\(^{th}\) position (242 cubes).

![Towers of cubes and table](NCTM, 2000)

To provide additional examples and activities for each content strand of the PSSM (NCTM, 2000), the *Navigations* series of books was created and published for each grade band: K-2, 3-5, 6-8, and 9-12. *Navigating through Algebra in Prekindergarten-Grade 2* (Greenes, Cavanagh, Dacey, Findell, Small, 2001), describes how fundamental ideas of algebra, (e.g., repeating and growing patterns, variable and equation, and relations and functions) can be introduced, developed, and extended. *Navigating through Problem Solving and Reasoning in Prekindergarten-Grade 2* books (Greenes, Cavanagh, Dacey, Findell, Sheffield, Small, 2001) activities for strengthening students’ problem solving
strategies and methods for guiding them to identify mathematical relationships; identify, continue and describe patterns and their rules; and, to reflect on probing questions posed by the teacher, are presented. These probing questions influenced the formation of assistance in the present study.

Adding it up: Helping Children Learn Mathematics (National Research Council, 2001) was written at the request of the National Science Foundation because of concern about the lack of reliable information on how young children learn mathematics. Such information was needed to guide curriculum development and pedagogical decisions about best practices. The book synthesizes research on prekindergarten through Grade 8 mathematics learning and makes research-based recommendations for teaching, teacher education, and curricular foci. The book describes studies of students’ growth in understanding early algebraic ideas including nature of function machines, and the treatment of variables in equations as input-output pairs (National Research Council, 2001, p. 264).

In 2006, the National Council of Teachers of Mathematics published Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence. That publication elaborated on the Principals and Standards for School Mathematics (PSSM, 2000) by identifying key areas of emphasis within the mathematics curriculum at each grade level. For Grades 1, 2 and 3, Focal Points addresses patterns, functional reasoning, and what students should be doing and learning by grade level:

- Grade 1: Through identifying, describing, and applying number patterns and properties in developing strategies for basic facts, children learn about other
properties of numbers and operations, such as odd and even numbers (PSSM, 2000, p. 32).

- **Grade 2**: Children use number patterns to extend their knowledge of properties of numbers and operations. For example, when skip counting, they build foundations for understanding multiples and factors (PSSM, 2000, p. 29).

- **Grade 3**: Students build a foundation for later understanding of functional relationships by describing relationships in context with such statements as, “The number of legs is 4 times the number of chairs” (PSSM, 2000, p. 15).

**Common Core State Standards in Mathematics.** The Common Core State Standards (NGA & CCSSO, 2010) is the result of collaborations of state leaders, including governors and state commissioners of education in 48 states. Through their membership in the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO), leaders wished to develop real-world learning goals for all students with the goal of improving achievement in mathematics and other subjects. In the development of these standards, the authors drew on high-quality mathematics standards from states across the country, and on input from researchers, state departments of education, assessment developers, professional organizations, teacher educators, parents and students, and other members of the public.

The CCSS in Mathematics reserves the study of growing patterns and functions until Grade 4: “Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself” (NGA & CCSSO, 2010, p. 29). Furthermore, in preparation for Grade 4, students need preliminary experiences to set the stage for this later exploration (Blanton, Stephens, Knuth, Gardiner,
However, researchers have shown that young students are capable of understanding these ideas (Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon, 2015; Cooper & Warren, 2011; Moss and London McNab, 2011; Radford (2014, 2012); Russell, 2016; Sorkin 2011).

**Research Focusing on Young Students’ Functional Reasoning Talents**

In this section, studies of young students’ functional reasoning abilities are presented. Studies designed to gain insights into students’ understanding that are not tied to teaching studies are reviewed first. Reviews of assessments that are part of teaching experiments are presented next.

**Kim & Sloane (2010).** In their study, *Second Grade Children’s Understandings and Difficulties with Patterns*, the authors investigated Grade 2 students’ understanding of repeating and growing patterns. The population consisted of 64 students from three Grade 2 classrooms in a school with more than 50% of the population identified as English Language Learners. The assessment consisted of four items – two with repeating patterns and two with growing patterns. Participants were asked to describe and extend each pattern. The patterns employed both geometric shapes and upper case letters (see Figure 2.05).
Researchers first administered the four-item paper-pencil test to all 64 students in May, near the end of Grade 2. Subsequently, 18 of those were selected to be interviewed individually, including both students who extended patterns correctly and those who did not. The latter group, with incorrect solutions, was selected because they demonstrated particular types of errors. In the interview, participants were first asked, “Why did you make it longer like this?” (Kim & Sloane, 2010, p. 254). Follow-up questions were based on participants’ responses.

Results on Kim and Sloane’s (2010) written assessment showed that participants had significantly greater success completing the repeating patterns (RP) than the growing patterns (GP). Of the 64 participants, 61 correctly continued RP 1 and 51 correctly completed RP 2. By contrast, only 19 correctly completed GP 1 and 8 correctly completed GP 2. Only five participants correctly identified and continued all four patterns. Most errors occurred because students extended the patterns as repeating patterns (e.g., by repeating the given pattern, repeating the simplest term, partly growing and then repeating the whole). Kim & Sloane concluded that participant difficulties

\begin{figure}
\centering
\begin{tabular}{|l|}
\hline
Repeating patterns (RP):
\hline
RP 1. ○Δ○Δ○Δ○Δ
\hline
RP 2. GGRRRGGRRRGRRR
\hline
Growing patterns (GP), showing changes in number of letters or shapes:
\hline
GP 1. ABABBABBBABBBB
\hline
GP 2. ●●○○●●○○○○●●○○○○○○●●
\hline
\end{tabular}
\caption{Kim and Sloane’s (2010, p. 2) pattern assessment}
\end{figure}
stemmed from not recognizing what was changing in each growing pattern, and not attending to the position of a figure or group of figures in that pattern.

Because Kim & Sloane (2010)’s assessment instrument had so few items (four), generalizing from their study is suspect. Furthermore, participants’ difficulties with identifying and continuing the patterns may have been due to the nature of each pattern with the close proximity of the figures and the lack of position/location numbers under each figure. The number of figures also may have influenced poor performance. As can be seen in Figure 2.05, GP1 displays four figures; 19 out of 64 students solved this correctly, as compared to GP2 that displays only 3 figures; 8 out of 64 students solved this correctly. Finally, no information was provided about the nature of the investigators’ questions and participants’ comments made during the interviews, which may have provided additional insight into participants’ depths of understanding, including areas of difficulty.

Results of Kim and Sloane’s study were taken into consideration in the design of GPFA by: 1) increasing the number of items in an assessment, 2) separating figures in a pattern, 3) providing position numbers under each figure in a pattern, and 4) using a note taker to record investigators’ questions and participants’ responses and comments.

Warren, Miller, & Cooper (2011, 2013). The Early Years Generalizing Project (EYGP) was designed to investigate how young children, ages 5 to 9, grasp and express the idea of function. The part of the study reported here focuses on the researchers’ 2011 study of students ages 7 to 9 years old, enrolled in a school in Australia. Ten participants were selected, five 7-8 year-olds and five 8-9 year-olds. The researchers sought to understand whether these students could generalize how input and output values co-vary,
by correctly 1) predicting output values from given input values, 2) describing the function rule, 3) predicting input values from given output values, and 4) describing the inverse function rule. In the EYGP, all tasks used letters or numbers printed on cards, or physical objects. A cardboard box, “Rosie,” represented the function machine. The tasks consisted of two language items (Tasks 1 and 2), one shape item (Task 3), and three number items (Tasks 4, 5, and 6). For each task, participants were shown a card or shape (attribute block) and directed to place it in Rosie’s ear (input). The output card or shape emerged from Rosie’s other ear.

Only tasks 3, 4, 5, and 6 are relevant to the present study and are reviewed here. 

Task 3. The function of how the characteristics of an attribute block change from input to output was examined. For example, given the input of a large, thick red triangle, and the output of a small, thin red triangle, the function is “Make it thinner and smaller.” After three examples, participants were asked to predict the output for another shape.

For the number tasks (Tasks 4, 5, and 6), three examples were given for each task, followed by a sequence of questions to examine participants’ depths of understanding by having them predict the input or output and explaining their answers. The function and the questions for each task are presented below.

Task 4. This task employs a Plus 2 function. Question 1: For an input of 6, what is the output? (8); Question 2: Identify the rule (Input plus 2); Question 3: 7 is the output, what is the input? (5). Question 4: Identify the rule (Output minus 2).

Task 5. This task employs a Subtract 3 function. Question 1: For an input of 9, what is the output? (6); Question 2: Identify the rule (Input minus 3); Question 3: 5 is the output, what is the input? (8). Question 4: Identify the rule (Output plus 3).
**Task 6.** This task employs a doubling function. Question 1: For an input of 4, what is the output? (8); Question 2: Identify the rule (Double the input); Question 3: 6 is the output, what is the input? (3). Question 4: Identify the rule (Half of the output).

All interviews conducted by the researchers were video-taped. The tapes were transcribed with special attention to both verbal responses and the use of concrete materials to identify how input and output co-vary. Data were collected about students’ successes as with identifying outputs when inputs were provided, the output function rule, inputs when outputs were provided, and the input (inverse function) rule. Table 2.01 displays the tasks and the numbers of students (n = 10) who were successful with the shape and number tasks. The tasks are shown in columns (T3), (T4), (T5), and (T6).

**Table 2.01.**

*Correct Responses to Tasks 3, 4, 5 and 6*

<table>
<thead>
<tr>
<th>Task</th>
<th>Shape Task (T3)</th>
<th>Number Tasks (T4)</th>
<th>Number Tasks (T5)</th>
<th>Number Tasks (T6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule</td>
<td>Make it smaller and thinner</td>
<td>Add 2</td>
<td>Subtract 3</td>
<td>Double</td>
</tr>
<tr>
<td>Identify Output</td>
<td>6</td>
<td>10</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>Identify Output Rule</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Identify Input</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Identify Input (Inverse) Rule</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

*Note.* See (Warren, Miller, & Cooper, 2011, p.755)

n = 10
As can be seen in Table 2.01, participants had greatest difficulty identifying the Inverse Rule for Task 6 (T6), with only four students able to identify the rule. Students had least difficulty Identifying the Output for Task 4 (T4) with 10 students successful. With regard to the shape item, researchers thought that students’ lack of familiarity with shapes and shape-attribute vocabulary may have affected their abilities to solve the problems and provide explanations.

Researchers categorized students’ generalizations of functional relationships by stages of complexity. A description of each stage and the number of participants at each stage follows. The asterisk next to number of students for Stages 2 and 3 indicates that one student was listed in two stages because of different generalizations based on the tasks.

- **Stage 1:** Use whole numbers to justify the rule (3 students).
- **Stage 2:** Express generalization without using the specific arithmetic operation. For example, when asked to generalize the Add 2 rule when inputting the word “finky,” responses included “2finky,” “It will turn into a 2,” and “finky add 2 letters” (3* students).
- **Stage 3:** Use of letters as variables. Researchers reiterated that “finky” meant any number (5* students).

In a later *Early Years Generalizing Project* study (2013), Warren, Miller, & Cooper interviewed six 5-year-old students using two attribute block shape tasks and one number task. The block shape tasks employed function rules: 1) Make it thinner and 2) Make it thinner and smaller. All six students predicted the outcomes for shape rule 1 and five of the six students were able to predict the outcome for shape rule 2. For the
number task, employing the Add 2 function, four of the six students successfully answered the questions about output and input when given the output, and stated the function rule. Only one of the six students succeeded in identifying the rules for all three tasks.

Results from both of Warren, Miller & Cooper’s (2011, 2013) studies indicate that young children have the capacity to identify inputs, outputs, and function rules when using a function machine. However, the paucity of items does not allow generalization. The researchers conducted the study, which may have created a bias in the interpretation of scoring.

McEldoon & Rittle-Johnson (2010). These researchers studied the recursive and covariational reasoning abilities of 231 students in Grades 2 through 6. Students were required to identify rules of correspondence among data presented in function tables, and then to use the rules to predict new values. The same 11 assessment items were used with students in all five grades.

Nine of the 11 items in the assessment require completing two-column function tables in which the solvers have to identify how the input and output co-vary. Samples of these types of items are presented in Figures 2.06, 2.07, and 2.08. The item in Figure 2.06, requires students to relate the number of eyes to the number of dogs to complete the function table. In the item in Figure 2.07, students have to apply the rule presented at the top of a table to complete the table.
In the item in Figure 2.08, students have to select the function rule that relates the inputs and outputs in the completed table.

Figure 2.09 presents an example of types of items used to assess students’ recursive reasoning talents. This is a +9 pattern. The next value in the pattern must be identified from a selection of four alternatives.

Figure 2.07. Item 2: Follow a function rule (McEldoon & Rittle-Johnson, 2010, p. 2)

In the item in Figure 2.08, students have to select the function rule that relates the inputs and outputs in the completed table.

Figure 2.08. Item 6: Functions as algebraic equations (McEldoon & Rittle-Johnson, 2010, p. 4)
All students in the McEldoon & Rittle-Johnson study (2010) were enrolled in two suburban schools. Of those students, 52 were in Grade 2 (24 females, 28 males), 50 in Grade 3 (30 females, 20 males), 25 in Grade 4 (15 females, 10 males), 60 in Grade 5 (28 females, 32 males), and 44 in Grade 6 (16 females, 28 males). Approximately 27% of all students were eligible for free or reduced lunch.

In February and March of 2009, the researchers administered the 11-item paper-pencil test to each group during one 60-minute class period. Directions and test items were read aloud to all Grade 2 and 3 students to reduce reading demands. Researchers circulated through all classrooms while students completed the assessments, and responded to individual questions with scripted prompts from the Functional Thinking (FT) Guide in the “Help Script” section. An example of a scripted prompt for Item 1a is: “Re-read the problem with the ‘kid’ while dragging your finger underneath the words and say, ‘Think of a way to use the table to help you. Here’s the column for number of dogs. (Gesture down along the column). And here is the column for the number of eyes (Gesture down along the column).” (McEldoon & Rittle-Johnson, 2010, Functional Thinking “Help Script,” p. 1).

Coding rubrics, from the “Coding Rubrics” section of the FT Guide (McEldoon & Rittle-Johnson, 2010), were used to identify the number of points to assign for various responses, along with samples of those responses. For example, the script for Item 1b emphasizes covariational reasoning: “Do you see a relationship between the number of dogs and the total number of eyes? How would you describe this relationship?” (Finding Patterns- Help Script, Section 1: p. 1). Coders were told to “Give 1 point (for a yes response) if the student mentions doubling, multiplying by 2, or a 2 to 1 relationship
between eyes and dogs” (Section 1: p. 1). For Item 8, focusing on function rules, Part 8d requires students to write the rule as a number sentence. In the researchers’ Coding Scheme for Functional Thinking Data Set, coders were instructed to give 1 point if the student response is, “A + 4 = B,” or a variation of this sentence. Give 0 points if the respondent leaves it blank, records a question mark, gives a specific number sentence instead of a general sentence with variables, or gives an incorrect answer. Examples of 0-point responses are: “I added them up;” “A add, B subtract;” or “2 + 4 = 6” (p. 1).

The researchers developed a hierarchy of levels or function skills that students progress through to reach a target level of generating a symbolic rule. Beginning with the lowest level, results by Level are described below along with the Level description, the jobs for students, grade level, and percent of students who successfully solved the problems (McEldoon & Rittle-Johnson, 2010, pp. 4-5).

• Level 1. Apply Rule: Complete the table for a function (e.g., \( y = x + 4 \)).
  - Grade 2: 71%
  - Grade 3: 82%
  - Grade 4: 94%
  - Grade 5: 91%
  - Grade 6: 91%

• Level 2. Recognize Rule: Identify the function rule from a set of rules (presented along with a completed function table).
  - Grade 2: 35%
  - Grade 3: 45%
  - Grade 4: 74%
Grade 5: 71%
Grade 6: 87%

- Level 3. Generate and Apply Verbal Rule: Describe the values and name the missing numbers in the (function) table.
  - Grade 2: 11%
  - Grade 3: 23%
  - Grade 4: 50%
  - Grade 5: 53%
  - Grade 6: 69%

- Level 4. Generate Symbolic Rule: Use symbols to formulate the rule that relates inputs to corresponding outputs (presented with a function table) Grade 2: 1%.
  - Grade 3: 5%
  - Grade 4: 32%
  - Grade 5: 38%
  - Grade 6: 68%

Of interest is that not all success rates increased with increases in grade level.

Ninety-four percent of Grade 4 students’ were successful with Level 1 problems, in contrast to 91% of students in Grades 5 and 6. For Level 4 and 5 problems, percents did increase by grade level. The researchers concluded that computational facility is a much greater factor in performance with function problems than was expected. They also pointed out that an assessment that covers a span of five grade levels is very difficult to accommodate in one 11-item test.

Of interest to the present study are McEldoon & Rittle-Johnson’s (2010)
Assessment Guide providing wording and gestures for each problem; the Coding Scheme that provides examples of different scores for different types of responses: and the Help Script that gives in depth “Help Prompts” for the various types of difficulties that students might encounter along with suggestions about the amount of time that different types of problems may require. These figures were incorporated into the design of Growing Patterns and Functions Assessment (GPFA) and Interview Protocol.

**Teaching experiments with pre-assessments.** The majority of studies of young children’s understanding of functional relationships were designed to evaluate instructional strategies that focused on enhancing students’ abilities to identify the regularities of patterns, extend those patterns, and determine how the ordered pairs (number of items in a figure and position of the figure in the pattern) co-vary (Blanton et al., 2015; Brizuela & Ernest, 2011; Fosnot & Jacob, 2010; Hawes et al., 2011; Isler, Blanton, Gardiner, Knuth, Stephens and Kang, 2014; Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon, 2015; Brizuela & Ernest, 2011; Fosnot & Jacob, 2010; Hawes, Moss & London McNab, 2011; Kim and Sloane, 2010; Lannin, 2005; McEldoon & Rittle-Johnson, 2010; Rivera, 2010; Schifter, Monk, Miller & Cooper, 2013; Sorkin, 2011). Four of these teaching studies (Blanton et al., 2014; Sorkin, 2011; Radford, 2014; Moss & London McNab, 2011) are relevant to the present study because they provide insights into the construction of items and types of assistance to offer with respect to: 1) incorporation of pretests that serve as indicators of what young children know prior to instruction; and 2) types of assistance (probing questions and prompts) provided and the nature of those.
Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon (2015). These researchers conducted a one-year teaching experiment with Grade 3 students to explore the influence of algebra instruction on their understanding of equations, and their abilities to generalize arithmetic relationships and to reason about functions. Thirty-nine Grade 3 students, from two intact classrooms in the same school served as the intervention group. Teachers of those classes had volunteered their students for the study. The 67 students in the control group were from four intact classrooms in the same school district – two at the same school as the intervention group and two at a second school with comparable demographics and academic performance. Of the 106 Grade 3 students, 10% were classified as minorities, 20% as low socioeconomic, and 5% as English Language Learners (ELL). Pretest results reported in the following discussion include both intervention and control group students.

An 11-item pretest to assess students’ understanding of what the researchers identified as the five big ideas of algebra: 1) generalized arithmetic; 2) equivalence, expressions, equations, and inequalities; 3) functional thinking; 4) variables; and, 5) proportional reasoning. Some pretest items assessed more than one big idea. Items were of three forms: true-false, multiple-choice, and short-answer. Most questions required students to explain their reasoning. A description of the items follows.

1. Four items (4, 6, 7 and 8) assessed abilities to generalize arithmetic operations or identify instances of the commutative property.

2. Five items (1, 2, 3, 5, and 9) assessed knowledge of equivalence, expressions, equations, and inequalities, and the abilities to solve missing-value problems and identify equivalent relationships.
3. Item 10, a five-part item, assessed abilities to reason with functions, specifically to observe growing patterns presented in words and symbols, and to predict “what comes next” as output, to identify a function rule, and to describe how the number of tables and number of people co-vary in words and with symbols (using variables).

4. Four items (6, 7, 8, and 10) assessed students’ abilities to represent a quantity with an algebraic expression involving variables.

5. Item 11 assessed students’ abilities to reason proportionally about data presented in a picture.

The pretest items had been piloted and refined through previous research, and had good psychometric properties (Blanton, 2008; Carpenter et al., 2003; Carraher, Schliemann, & Schwartz, 2008; Kenney, Lindquist, & Heffernan, 2002; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008; Knuth, Stephens, McNeil, & Alibali, 2006).

Of particular interest to the present study is Item 10, (see Figure 2.10). To solve the problem, the maximum number of people who could be seated at a table comprised of congruent square tables joined (showing sides) to form a row of tables had to be determined. After completing the function table, students were asked to demonstrate their reasoning by showing and describing the relationship between the number of tables and the number of people who could be seated at that table arrangement.

The pretest was administered to both the intervention and control groups at the beginning of the academic year. The researchers developed a coding scheme to include: 1) Answer for Item 10a; 2) Reasoning Method (Recursive or Covariational) for Items 10b, c, and d; and 3) Functional Rule (in words or in symbols) for Items 10b, c, and d;
44

and 4) Solution method for Item 10e.

<table>
<thead>
<tr>
<th>10. Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square tables. He can seat 4 people at one square table in the following way: If he joins another square table to the first one, he can seat 6 people</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) If Brady keeps joining square tables in this way, how many people can sit at: 3 tables? 4 tables? 5 tables? Record your responses in the table below and fill in any missing information:</td>
</tr>
<tr>
<td>Number of tables</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>b) Do you see any patterns in the table? Describe them.</td>
</tr>
<tr>
<td>c) Find a rule that describes the relationship between the number of tables and the number of people who can sit at the tables. Describe your rule in words.</td>
</tr>
<tr>
<td>d) Describe your relationship using variables. What do your variables represent?</td>
</tr>
<tr>
<td>e) If Brady has 10 tables, how many people can he seat? Show how you got your answer</td>
</tr>
</tbody>
</table>

**Figure 2.10.** Pretest item (Blanton et al., 2015, p. 66)

Researchers reported percentages of students who successfully solved each of the five parts of Item 10:

- Item 10a: 36% identified the number of people, who could sit at 1, 2, 3, 4… tables.
- Item 10b: 20% identified patterns in the function table.
- Item 10c: 3% described (in words) the relationship between the number of tables and the number of people who could sit at the tables. An example from their
Coding Scheme indicating covariational reasoning, was: “Everytime you add one more table, you add two more people” (p. 69).

- Item 10d: 0% used variables to describe the relationship between the number of tables and the number of people that could sit at those tables.
- Item 10e: 33% identified the number of people that could be seated with 10 tables.

The researchers concluded that low pretest scores represent the typical arithmetic-based curriculum that “does little to prepare students for the successful study of algebra in the later grades” (p. 71). However, the inclusion of the two illustrations of tables with people seated at those tables beside the function table provided a helpful visual connection between the two representations. Furthermore, the sequence of five questions for Item 10 provide a scaffolding learning sequence. Researchers decided that the “far” 10th position was close enough that it could be solved with covariational or recursive reasoning. Although none of the students solved Item 10d, (requiring a function rule with variable), 33% solved 10e. (requiring only identification of the number of people who could be seated at 10 tables).

Blanton et al.’s study (2015) revealed that only a limited number of Grade 3 students were able to solve function problems in a rich context. Those insights influenced the present study to focus on one skill at a broader range of grade levels and degrees of complexity.

Sorkin (2011). The researcher conducted a seven-week teaching experiment focusing on Grade 2 students’ abilities to generalize patterns by applying covariational reasoning. The population consisted of 97 students from 13 Grade 2 classrooms in three
urban schools. School A: (34 students) is an inner city public school with 66% of its students African American and 31% Hispanic. All students were from low SES families. School B: (34 students) is an inner city charter school with 87% of its students African American and 9% Hispanic. All students came from low SES families. School C: (48 student students) is a public school with 77% of of its students Caucasian, 9% African American, and 5% Hispanic. All students came from middle-income families.

Four pre-assessments were administered using one-on-one clinical interviews. Two assessments, the *Written Problem Test (WPT)* and the *Cube Tower Test (CTT)*, were designed by Sorkin to assess covariational reasoning. The WPT requires students to infer, explain, and interpret function rules without manipulatives. The items include stories, diagrams, and tables of numbers, all ordered by level of difficulty in terms of arithmetic operation, addition, subtraction and multiplication (e.g., $n + 1$, $n - 2$, $2n$), and by complexity of functions (e.g., $(n \times 2) + 1$). Figure 2.11 displays a function table employing the function $B = A + 1$. The numbers in Column A are not in numerical order to deter students’ use of recursive reasoning (e.g., by counting or skip counting) to determine the numbers in Column B. The interviewer’s script states: “Something happens to the numbers when you go from one column to the next: the 3 becomes a 4, the 1 becomes a 2, the 7 becomes an 8. What if a 4 was here (pointing to the empty space on the left), what would be here? (pointing to the empty space on the right)” (Sorkin, 2011, p. 134).
Interviewer Guides were provided for each interviewer for the WPT and CTT. The Guides included introductions, practice problems, and trial items for versions A and B; interviewer scripts; sample cube-tower displays; and, coding guidelines. For example, for Item 2 of the WPT (See Figure 2.12), interviewers were directed to read the story aloud: “Leslie and Mike have the same birthday. Mike is 6 and Leslie is 2. Mike wants to know how old he will be when Leslie is a certain age” (p. 136).

Using a function machine and linking cubes, the Cube Tower Test (CTT) was designed to assess students’ abilities to identify, express, and reason about function rules.
A recording board was made from a standing tri-fold foam presentation board with two “doors,” one for input and one for output. Students were presented with towers made from one to six connected unit cubes. Cube towers were named by the number of unit cubes (e.g., a tower with four cubes is called “4”). See Figure 2.13. Students were directed to pick one of the towers \( n \), to “feed” the function machine. This was accomplished by putting the cube tower into the input door. The teacher “opens” the Output door and presents the results of the application of the function.

![The Amazing Function Machine](image)

*Figure 2.13. CTT function machine (Sorkin, 2011, p. 113)*

The large recording board, resembling a T-table, was used to keep track of inputs and outputs (see Figure 2.14). If one “5” is inserted into the Input door, three 5’s plus 1 cube emerge from the Output door.
The functional reasoning tests were videotaped. Codes were developed to distinguish between full generalizations that were either “Clear” (CG) or “Unclear” (UG). The CG code was awarded when a student did a “decent” job of explaining the target rule. The UG code was given when a student appeared to mean a different rule altogether, or when the wording was too awkward to understand. The researcher and a research assistant met to review students’ explanations and to assign codes after each assessment session.

On the WPT pretest, 37.2% of students made generalizations using covariational reasoning. On the CTT pretest, 64% of students used covariational reasoning, correctly identified patterns, and provided clear explanations.

Two additional pretests were administered to all students: 1) *The Test of Early Mathematics Ability* – Third Edition (TEMA-3) (Ginsburg & Baroody, 2003), to identify
the level of mathematical ability of children, ages 3 through 8 years, and 2) *The Test of Language Development-Primary—Fourth Edition* (TOLD-P4) (Newcomer & Hammill, 2008) (grammar portion only), to determine the relationship between students’ oral language facilities and their abilities to explain rules in natural language. All tests were conducted as one-on-one clinical/flexible interviews by the researcher or by graduate students trained by the researcher.

Sorkin’s (2011) two function pretests were correlated to results on TEMA-3 and TOLD-P4. Students with greater TEMA-3 scores, achieved higher than average scores on Sorkin’s WPT and CTT assessments. Students’ performance on TOLD-P4 correlated with overall cube tower performance, particularly with explanation quality. There was a significant (p = .003) correlation between language facility and effectiveness in describing functional relationships. The close correlation of language fluency to quality of verbal explanations is of interest since 84% of students in the present study were Latino first generation U.S. citizens.

Sorkin’s research (2011) provides extensive data on Grade 2 students’ early knowledge of functions, from interpreting functional patterns, following function rules, inferring function rules, using different function table formats without and with manipulatives, to using variables to describe functional relationships. Some of her research was useful in the present study. Sorkin observed that many of her low performing students appeared to be distracted by the blocks, picking them up, connecting them in a manner that did not appear to be part of a thinking strategy. This observation reinforced the decision to exclude manipulatives as potential distractors in the design of the *GPFA*. Most of her items in the CTT were of similar levels of complexity, thus she
was not able to gauge depth of understanding. For this reason, the decision was made to provide items at varying levels of complexity in the GPFA.

Among the shortcomings of the Sorkin study are: 1) The relationship between scaffolding assistance and student success with describing functions was not analyzed. This information would have been useful in determining the types of assistance students need to successfully solve function problems. 2) Performance by gender was not analyzed, although spatial reasoning differences by gender have been studied and reported by many researchers (Casey, 1996, 2002; Casey, Pezaris, Anderson, & Bassi, 2004; Gong, He, & Evans, 2011; Costa, Terracciano, & McCrae, 2001; Gurian, 2011; Gurian & Stevens 2005; Gurian, Stevens, & King, 2010; Halpern, Benbow, Geary, Gur, Hyde, & Gernsbacher, 2007; Keller & Menon, 2009; Pomerantz, Altermatt, & Saxon, 2002; Sax, 2007, 2011).

Radford (2014). Luis Radford conducted a five-year longitudinal study to follow changes in 25 students’ algebraic thinking abilities, Grade 2 through Grade 6 (Radford, 2010, 2011, 2012, 2014). Over a one-week period during the fall of each academic year, students were engaged in a five-day series of lessons on pattern generalization that focused on describing geometric figures in a sequence as related to their positions in that sequence, and extend the sequence. The research team and participating teachers designed the lessons prior to instruction.

For the entire study, Radford used five video cameras. Each camera filmed one small group of students working together during their regular mathematics period, as well as during whole group discussions. Researchers focused on the selected filmed episodes,
along with the transcripts, to conduct frame-by-frame analyses of each student’s knowledge, and the role of gestures and words.

Of interest to the present study are the activities conducted by Radford with students in Grades 2 and 3 prior to instruction. Year 1, Task 1 is as an assessment of students’ abilities to extend growing patterns. Presented with four figures in a pattern (Figure 2.15), students were asked to draw Figures 5 and 6, and then to identify the number of squares in Figures 25 and 50 of the pattern. Students worked together in groups of three to share ideas about how to identify the number of squares in the remote figures (Radford, 2014, p. 263-214).

![Figures 1-3](image)

*Figure 2.15. Year 1, Task 1 for Number of Squares = 2 x Figure Number + 1 (Radford, 2014, p. 262)*

Examples of three student drawings of Figure 5 in Task 1 are presented in Figure 2.16. Student A did not demonstrate understanding of the figures as composed of two rows, nor the location of the shaded squares and the total number of squares. Student B displayed two rows, but did not show changes in the number of small squares in each row and the need for a shaded square. Student C did not attend to the total number of squares and the representation of two rows. It may be that in all of these cases, students did not understand the term “extend” that was provided in the directions.
As observed in one discussion, the challenge of a remote figure actually promoted the activation of covariational reasoning by Grade 2 student, Erica. She referred to the figure number and said, “Okay, What is 25 plus 25?” Then, smiling, she continued, “After that, you add one!” Erica generalized that the figure number plus the figure number plus 1 produced the total number of small squares in that figure (Radford, 2014, p. 264 and Radford, 2010, p. 4-75).

Difficulties students had with continuing patterns in Radford’s study (2014) may have been due to the use of only four figures, which did not clearly show changes between consecutive figures. This influenced the decision to include five figures in the patterns in the GPFA, and to make the figure differences more obvious. The confusion Radford’s students had with the language “to extend the pattern” also prompted the decision to leave space for numbers for figures in both near and far positions in the GPFA.

Moss & London McNab (2011). These researchers conducted a teaching experiment to explore student potential to understand linear functions and use covariational reasoning. Students were drawn from 20 inner-city public schools in Toronto and New York City. Six intact Grade 2 classrooms with 20 or 22 students per
class participated in the study. All schools served at-risk populations (high ESL, low SES). Classrooms were selected because of teacher interest in participating in the study. The length of the teaching experiment varied from 10 to 14 days, and was conducted separately in each of the six classrooms. Researchers collected classroom field notes, transcripts of videotaped lessons, and interviews with students during the lessons.

Prior to instruction, a short pretest interview was conducted using the geometric growing pattern shown in Figure 2.17. The goal of the pretest activity was to assess students’ initial abilities to identify “what comes next” in a pattern, to predict the number of items in a figure later in the pattern, and finally, to describe the pattern rule. Students were first asked, “If this pattern keeps growing in the same way, what would the next position look like?” (p. 281). Researchers used this first question to encourage students to study the spatial configurations of the tiles before being questioned about the number of tiles in the next position.

![Figure 2.17. Pretest (Moss & London McNab, 2011, p. 281)](image)

The pretest revealed that none of the students were able to identify what comes next in the pattern, much less the relationship between the position number and the number of tiles in the figure in that position. Their responses repeated the 3, 6, 9 sequence as a repeating pattern. In conversation between this dissertation researcher and Joan Moss (January 5, 2016), Moss stated that asking students to describe the visual
representation for near and far positions achieved better results than asking, “How many tiles would there be in the next position?” (Moss & London McNab, 2011, p. 281).

Moss & London McNab (2011) cite the work of Carraher et al. (2008), Lannin (2005), Rivera & Becker (2008), and others, who advocate the use of visual representations, such as geometric shapes, to enhance student understanding and generation of functional rules. Their assessments, using pictures of geometric growing patterns with position numbers below the figures, influenced the design of GPFA.

Difficulties these researchers encountered also influenced the design of the present study. The limited number of figures in a pattern (3) and no “space” for continuing the pattern may have been inadequate in establishing the “growing” nature of the pattern in the Moss and McNab study (2011). Furthermore, the identification of the position of the figure with a number in a square, may have caused confusion in the counting of squares.

These difficulties were taken into consideration in the design of GPFA by:
1) displaying discrete shapes in each figure of a growing pattern; 2) identifying the position of a figure in the pattern using the word Figure with a number; 3) using five figures to establish the pattern; and 4) allowing space for extending the pattern to indicate the missing figures with those spaces labeled Figure 6, Figure 7, ..., Figure 10.

Summary of Research on Young Students’ Knowledge of Functional Relationships

Few studies have examined young students’ understanding of functional relationships prior to instruction. In all studies reviewed, functional relationships were represented in various ways (e.g., function machine, patterns of shapes, tables of values) which may have affected solution success. Other factors that may have contributed to students’ difficulties include:
a. Pattern designs: The figures of patterns are not easily distinguishable, as they are not separated by space or other graphic dividers, such as lines (Kim and Sloane, 2010).

b. Position numbers: Position of figures in patterns are not labeled (Kim and Sloane, 2010).

c. Types of figures (manipulatives): In Warren, Miller, & Cooper’s studies (2011, 2013), students had lower scores for items requiring use of attribute blocks. Some of Sorkin’s (2011) low performing students appeared excited with tasks using the cubes but some were distracted by the cubes. Some of her higher performing students became bored with the cube activities.

d. Insufficient number of figures in a pattern: Moss & London McNab (2011) presented students with one pattern consisting of three figures. None of the students extended this pattern successfully. Kim & Sloane (2010) displayed four figures in one pattern (success rate 19/64) and only three figures in the other growing pattern (success rate 8/64).


f. Sample size of studies of students in grades 1-3: Warren, Miller, & Cooper assessed only ten students, ages 7 – 9, in their 2011 study and six students, age 5, in their 2013 study.
g. Limited number of grade levels: Only two teaching experiments involved students in grades 2 and 3. Four studies were limited to Grade 2 and one to Grade 3. None of the studies included Grade 1 students.

h. Researcher bias: Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon (2015) and Kim & Sloane (2010), conducted their own studies, leading to potential bias for interpretation of results.

i. Gender effect was not studied. Although some studies identified the number of male and female students, they did not analyze their data by gender.

By contrast, in the present study, a greater number of students (60), 20 students in each of grades 1, 2, and 3, were interviewed, one-on-one. In the GPFA instrument, each growing pattern presents five figures. Disconnected congruent shapes of the same color/shape are arranged vertically within each figure of the pattern. Figures are separated by spaces and line segments, and each is identified by a figure number to enable identification of covariational relationships. Spaces for missing figures through Figure 10 are provided and are identified by their respective figure number. During interviews, questions and responses are documented and the number and nature of assistance are noted. All data are analyzed by problem, problem type, question, grade level, and gender.

Assessment with Assistance

Vygotsky (1978) points out the value of “supported” instruction to enhance learning. His “Zone of Proximal Development” (ZPD) is described as the distance between what children can do without assistance and what they can accomplish with varying levels of support. Vygotsky points out that children’s performance with assistance is even more indicative of their degrees of understanding than when they
perform alone, and that with that information, educators can challenge students with content beyond their current skill levels, but within the range of their abilities.

Assessing student-understanding using supported methods, referred to as “supported assessment,” is consistent with Vygotsky’s ZPD theory. This type of supported assessment is also in concert with the National Board for Professional Teaching Standards for Early Childhood Education, which recommends that educators “elicit what a child knows by reassuring, probing, and rephrasing” (2012, p. 71) in order to meet that child’s needs.

In 2003, Tsankova conducted a study to examine the abilities of 60 students in grades 1, 2, and 3, to solve systems of equations with two variables (unknowns). In one-on-one interviews during the solution process, when students got stuck she provided assistance with a set of pre-determined hints that enabled students to continue. Depending on the degree of assistance provided, each type of hint was assigned a point value.

- Zero points for *Look*: A question or request made to a student to draw attention to key parts of the problem.
- One point for *Record*: Direction to draw or record numbers.
- Two points for *Solve*: Tell or show the solution steps student used to get to the answer.

Types of hints used were tabulated in order to reveal information about the amount and nature of assistance that students needed to solve the problems.

For the GPFA, Tsankova’s assessment was adapted and refined. When students were unable to correctly solve a problem, the interviewer asked guiding questions or provided “hints” to examine the degree of assistance they needed to solve the problems.
Mathematics Content of the *GPFA*

The mathematics content in the *GPFA* is in concert with the content of the two textbook programs used in the district where the present study was conducted. Both are published by Pearson: *enVisionMATH* (Charles, Cavanagh, Copley, Crown, Fennell, Ramirez, Sammons, Schielack, Tate & Van de Walle, 2011) and *Investigations in Number, Data and Space* (Russell, Economopoulos, Cochran, Murray, Hollister, Bastable, Bloomfield, Horowitz & Schifter., 2012). The *GPFA* content related to the mathematics topics presented in those two programs is described below.

**Mathematics in kindergarten.** Although kindergarten students are not part of the present study, content of the kindergarten programs is described to provide insight into Grade 1 skills that participants explored prior to November of Grade 1, when the *GPFA* was administered. Topics include: counting through 20; recognizing ordinals through 5th; adding and subtracting 1 or 2 through sums of 10; recognizing odd and even numbers; skip counting by 2 and 5; and recognizing, describing, extending, and recording simple growing patterns.

**Mathematics in Grade 1 – 3.** By November of Grade 2, students have explored how to identify rules for growing patterns, such as “Add one to the figure number” (Charles, et al. 2011, Grade 1, p. 36) or “Double the figure number and add one” (Charles, et al. 2011, Grade 2, p.485). Explaining the rule for a given numerical sequence and verifying that the rule works is introduced early in Grade 3.

Knowledge, concepts, and skills required for problem success on the *GPFA* are listed by the grade level in which they are first introduced in the two textbook programs used by participants in the *GPFA* study.
• Count by ones through 22 (Kindergarten and Grade 1)
• Skip count by 2s beginning from 1, 2, or 3, e.g., 3, 5, 7, 9, … (Grade 1)
• Recognize and compare ordinal numbers, 1<sup>st</sup> through 10<sup>th</sup> position (Grade 1)
• Recognize, describe, and extend growing patterns (Grade 1)
• Describe how the figure number and number of shapes in a figure co-vary (Grade 2)
• Multiply by 2 (Grade 2)

As can be seen above, all mathematics concepts and skills needed to solve GPFA problems have been introduced by the end of Grade 2 in the participants’ school programs.

**Chapter Summary**

In this chapter, an historical perspective on the growth of algebra as a topic in school mathematics is presented, with a focus on the concept of function and its development, beginning in the early grades. Developmental psychologist recommend that students have early experience with algebraic reasoning in order to better prepare them for the formal study of the subject.

There is a paucity of research investigating young students’ understanding of function prior to instruction. The research studies that have been conducted reveal that most students (Grades K – 3) need assistance to extend growing patterns, and to identify how the number of shapes in a figure is related to its position (location) in the pattern. As evidenced in the literature, functional reasoning performance is affected by test format, language employed in the directions and problems, and the arithmetic skills required.
In Chapter 3, the design of the present study including development of the instruments to be used to gather data and results of the three pilot studies are presented. The research instruments, including the *Growing Patterns and Functions Assessment (GPFA)*, scoring procedures, and *GPFA* interview protocol are described. The nature of the population and sample, the study questions, and methods of analyses, administration of the instrument, and the timeline conclude the chapter.
CHAPTER THREE

METHODS AND PROCEDURES

To examine young children’s depths of understanding of functional relationships, a mixed methods quantitative-qualitative study was designed and conducted with students in Grades 1, 2, and 3. The assessment was in the form of one-on-one interviews of students as they solved linear function problems involving patterns of geometric shapes. For participants who experienced difficulty, assistance in the form of pre-determined assistance questions, containing general and figure-specific information, was provided by the researcher-interviewer. Participant performance and assistance were scored and analyzed to answer research questions about performance on the GPFA without assistance, and with assistance, and methods employed to solve the problems.

In the first section of the chapter, the development of the instrument, informed by results of three pilot studies, is presented. The chapter continues with a description of the study instruments: the eight-problem assessment tool and the interview protocol. The demographics of the sample and the selection process are presented next, followed by the research questions, procedures used to collect the data, and processes applied to analyze of the data. The timeline for the conduct of the study concludes the chapter.

Instrument Development

Pilot Studies

Three pilot studies were conducted prior to construction of the Growing Patterns and Functions Assessment (GPFA): the first in March 2013, the second in June 2013, and the third in November 2013. All three pilot studies contributed to the content and format
of the final assessment instrument and to the nature of the assistance. Replicas of the
three pilot studies can be found in Appendices H, I, and J.

**Pilot Study A.** Pilot Study A, conducted in March 2013, was designed to gain
insight into how different problem formats affect solution successes, and the nature of
participant difficulties with the representations. The instrument contained four problems,
three focusing on pattern continuation and one using a function-table format. These types
of formats were suggested by the NCTM *Illuminations* (2008), and the research
publications by Greenes, Cavanagh, Dacey, Findell, & Small (2001), Radford (2012), and
Richardson (1999).

Problem 1, shown in Figure 3.01, presents a growing pattern with increasing
numbers of 2s. Students are instructed to extend the pattern to Row 5 (near position) and
to Row 10 (mid-far position) and then to Row 100 (remote position). The use of 2s, rather
than pictures of objects, was done to give the researcher the option of extending the
questions to ask participants for the sum of numbers in a row. That option was abandoned
because of the difficulty students experienced with the three questions.

```
<table>
<thead>
<tr>
<th></th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td></td>
</tr>
<tr>
<td>Row 2</td>
<td>2 2</td>
</tr>
<tr>
<td>Row 3</td>
<td>2 2 2</td>
</tr>
<tr>
<td>Row 4</td>
<td>2 2 2 2</td>
</tr>
</tbody>
</table>
```

Continue the pattern.

How many 2s are in Row 5? ______

How many 2s are in Row 10? ______

How many 2s are in Row 100? ______

*Figure 3.01* Problem 1, Pilot Study A
The design of Problem 2, shown in Figure 3.02, was influenced by the work of Richardson (1999), and Greenes, Cavanagh, Dacey, Findell, & Small (2001). A chart is shown with the counting numbers 1-25 arranged in rows, with five numbers in each row. The function relating the last number in a row \( (L) \) to the row number \( (r) \) is \( L = 5r \). The function relating the first number in a row \( (N) \) to the row number \( (r) \) is \( N = 5r - 4 \).

\[
\begin{array}{cccccc}
    \text{Row 1} & 1 & 2 & 3 & 4 & 5 \\
    \text{Row 2} & 6 & 7 & 8 & 9 & 10 \\
    \text{Row 3} & 11 & 12 & 13 & 14 & 15 \\
    \text{Row 4} & 16 & 17 & 18 & 19 & 20 \\
    \text{Row 5} & 21 & 22 & 23 & 24 & 25 \\
    \text{Row 6} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
    \text{Row 7} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\end{array}
\]

Continue the pattern.

a. What is the last number in Row 6? ______
b. What is the last number in Row 10? ______
c. In which row is 60 the last number? ______

\textbf{Figure 3.02 Problem 2, Pilot Study A}

Pilot A, Problem 3, shown in Figure 3.03, suggested by the work of Rivera (2014), Radford (2012), Richardson (1999), and Moss & London McNab (2011), presents an horizontal arrangement of collections of two-dimensional shapes (circles, squares, and triangles). Same shapes are same color. From Figures 1 through 4, only the number of squares varies, and is equal to the figure number. Numbers of other shapes remain unchanged. Questions are posed about the number of each shape in each of the
figures 6 and 10. Unlike Problems 1 and 2, Question d focuses on the total number of shapes in specified figures.

![Image of figures 1 to 4 with question prompts]

Figure 3.03 Problem 3, Pilot Study A

The function table format of Problem 4, shown in Figure 3.04, was suggested by the work of Sorkin (2011) and Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon, (2015). The function rule is: Input (☐) plus 3 equals Output (∆). To complete the table, participants have to determine two missing outputs and two missing inputs. The latter requires application of the inverse relationship, i.e., subtract 3 from the output to produce the input.
Pilot Study A was administered in a private room at an urban charter school to three students individually: one female in Grade 1, one male in Grade 2, and one male in Grade 3. Each participant was given a copy of the assessment, one problem at a time, and all questions were read aloud.

The three formats, number pattern, shape pattern, and function table, were particularly challenging for the participants in Grades 1 and 2. They did not provide comments useful for revising the problems. All questions in Problem 3 were particularly difficult for the Grade 1 student, who was unable to complete Item d. All three participants were not familiar with the function table format in Problem 4 and did not know how to proceed. The researcher explained the function rule and how the number in the first column relates to the number in the second column. That still remained complex for all participants.
Based on Pilot Study A results, the decision was made to not use function tables in the final assessment because of their complexity, and to use shapes versus numbers in the patterns.

**Pilot Study B.** Based on the results of Pilot Study A, the decision was made to design a second pilot study that (a) uses growing patterns of shapes; (b) employs a vertical presentation of shapes in figures to enable the number of figures per problem to be increased more easily, and (c) increases the number of problems to enable presentation of different types of linear functions. Because the term “figure” appeared to cause some concern among participants in Pilot Study A, and because the term “design” has shown to be more familiar to students than the term “figure” (Friel & Markworth, 2009), the decision was made to use “Design” with a number under each figure.

Furthermore, results from Pilot Study A indicated the need for the researcher to provide assistance to gain greater insight into participants’ thinking. Therefore, guiding questions that lead participants to analyze specific aspects of a pattern were incorporated into the Pilot Study B interview process. Among the types of assistance were questions of a more general nature, as for example, “What part of the pattern changes?” and “What part of the pattern stays the same?” Other questions focused on particular aspects of a pattern, such as, “Count the number of shapes in (the miscounted figure) again,” and on relationships between consecutive designs in the pattern, as for example, “How many shapes are in Design 1? Design 2?”

As well as questions to assist solvers, “prompts” (gestures and comments) were provided to encourage participants to continue problem solving. Prompts provided by the researcher included friendly nods or supportive comments, such as “You are doing
great,” “Nice,” and “Tell me more.”

The Pilot Study B assessment contained eight items. Item 1 is shown in Figure 3.05. All Design figures in the pattern contain one blue circle. Beginning with Design 2, with one red square, the number of red squares increases by the figure number. Questions lead participants to figure out the total number of shapes in Designs 10. Pilot Study 2 items are displayed in Appendix I.

![Diagram of the pattern with shapes labeled in different designs](image)

**Figure 3.05 Problem 1, Pilot Study B**

The eight functions used in Pilot Study B to generate the total number of shapes are identified by problem number below. In those functions, $d$ designates the design number and $T$ represents the total number of shapes in that design. Problem 3 is presented in Figure 3.06 and Problem 4 is presented Figure 3.07. Both show the same function, $T = d + 2$ using two different representations with two different shapes/colors: blue circles and red squares for Problem 4 (Figure 3.06) and three different shapes/colors (blue circles, red squares, and yellow triangles) for Problem 5 (Figure 3.07). The functions for
all other problems vary as shown in the list below. Problems 7 and 8 also use 3 different shapes/colors. The number of shapes/colors also are indicated in the list.

List of number of shapes/colors and functions for Pilot Study B problems

Problem 1: (2 shapes/colors) $T = d$, as shown in Figure 3.5.
Problem 2: (2 shapes/colors) $T = d + 1$
Problem 3: (2 shapes/colors) $T = d + 2$, as shown in Figure 3.6.
Problem 4: (3 shapes/colors) $T = d + 2$, as shown in Figure 3.7.
Problem 5: (2 shapes/colors) $T = 2d$
Problem 6: (2 shapes/colors) $T = 2d – 1$
Problem 7: (3 shapes/colors) $T = 2d + 1$
Problem 8: (3 shapes/colors) $T = 2d$

![Diagram showing designs with different shapes](image)

*Figure 3.06 Problem 3: (2 shapes/colors) $T = d + 2$, Pilot Study B*
Figure 3.07 Problem 4: (3 shapes/colors) $T = d + 2$, Pilot Study B

Pilot Study B was administered individually to two males, one who had recently completed Grade 1 and one who had recently completed Grade 2. The assessments were conducted at a small table in the home of one of the participants. Participants were each given a copy of the assessment, one problem at a time, with questions intended to be read aloud to them by the interviewer. Results follow.

The format of the pages was such that both participants preferred to read the problems aloud, rather than having them read aloud to them. As with Pilot Study A, participants continued to have difficulty and needed assistance with knowing the meaning of the phrase, “The pattern continues.” The Grade 1 participant talked about the previous patterns which all included the same shape/color figures. He needed ongoing guidance to attend to the given pattern and not refer back to the previous problem. He was unable to answer questions about extending the pattern without the prompt, “Tell me what you see,” and with assistive questions, such as, “How is the number of shapes in Design 2 different from the number of shapes in Design 1?” The Grade 2 participant made several
counting errors, but when provided with the recommendation to “Count again.” was successful with all eight items.

Difficulties participants experienced during Pilot Study B were imagining the pattern to continue beyond the page. The use of the same colors and shapes in the patterns was confusing. Participants thought that values of shapes in one problem carried into subsequent problems.

**Pilot Study C.** Findings from Pilot Study B that (a) continuation of the pattern was difficult for participants to envision, (b) the use of the same colors and shapes in several patterns created confusion, and (c) the questions about specific shapes/colors prior to the question about the total number, seemed to distract the participants.

Therefore, for Pilot Study C, problems were reformatted to fit on two adjoining pages, landscape style, showing labeled spaces for missing figures. Colors and shapes in patterns were varied from one problem to the next to avoid giving the impression that values of colors and shapes needed to be recalled. The number of questions per problem was reduced to focus on the total number of shapes in a figure and not on the cardinality of subsets of shapes. The descriptor “Design” was changed back to “Figure.” When comparing Pilot Study A, using the term “figure” with Pilot Study B, using the term “design,” the researcher found that participants’ focus on the number of shapes in a figure was more evident in Pilot Study A than B.

For Pilot Study C, the number of questions per problem was limited to two, both focusing on extending and generalizing the patterns: “How many shapes are in Figure 6?” and “How many shapes are in Figure 10?” Items contained exactly two different types of shapes, each of a different color.
Problem 2, shown in Figure 3.08, displays one green triangle in each figure. Only the number of yellow circles changes in the growing pattern, from 1 to 2 to 3, and so on. Problem 6, shown in Figure 3.09, displays changes in both the number of purple circles (1 to 2 to 3, and so on) and the number of green triangles (2 to 3 to 4, and so on).

\[ \text{Problem 2, Figure 3.08, Pilot Study C} \]

\[ \text{Problem 6, Figure 3.09, Pilot Study C} \]

Functions by problem number are presented below. In each function, \( f \) designates the figure number and \( T \) represents the Total number of shapes in that figure.

Problem 1: \( T = f \)

Problem 2: \( T = f + 1 \)

Problem 3: \( T = f + 2 \)

Problem 4: \( T = f - 1 \)
Problem 5: $T = 2f$
Problem 6: $T = 2f + 1$
Problem 7: $T = 2f + 2$
Problem 8: $T = 2f - 1$

Pilot Study C was administered to three Grade 3 students, one female and two males. The interviews were conducted one-on-one in a private room in the students’ elementary school in an urban area. Each participant was given a copy of the 8-item assessment, one problem at a time, with the oral instruction, “Tell me what you see.”

Results showed that the combination of two shapes and colors within a pattern was distracting for all three participants. When asked, “How is Figure 2 different from Figure 1?,” participants described the different colors in the figures rather than the total number of shapes in the figures. Directing participants’ attention to the total number of shapes in a figure was only partially successful.

Based on the findings of Pilot Study C, the following modification was made to the design of the final version of the assessment instrument. Colors and shapes in patterns were changed so that only one color and one shape was used in each problem. Questions omitted any reference to the different colors or shapes, and narrowed the focus on the total number of shapes in a specified figure.

**Population and Sample**

Participants for this study were drawn from a K-5 elementary school in a K-12 urban school district in the southwestern part of the United States. In 2013, of the 904 students enrolled in the school, 84% were Latino, 9% were African American, 4% were Caucasian, and the remaining 3% were not declared. As identified by the federal free-
and-reduced-lunch standard, 100% of the student population was identified as “economically disadvantaged.”

To be considered as participants in the study, two requirements had to be met. To eliminate the influence of previous school experiences, students had to have attended the same school for at least one year prior to the assessment. To eliminate language fluency as a factor in solution success, all participants had to be fluent English speakers. From students at each Grade level who met these two criteria, Statements of Informed Consent were obtained. See Appendix B for a copy of the informed consent form. Only students with signed parental permission forms were included in the study.

When the number of participants meeting the above criteria reached 20 per grade level, with a close to equal number of males and females, the sample was determined to be complete. Of the total of 60 participants in the sample, 30 were female and 30 were male. The number of participants by gender and grade level were: Grade 1: 10 females and 10 males; Grade 2: 10 females and 9 males; and Grade 3: 10 females and 11 males. The Growing Patterns and Functions Assessment (GPFA) instrument was administered one-on-one to participants in the order in which their Informed Consent forms were submitted.

Study Instruments

Two instruments were designed for the study: 1) Growing Patterns and Functions Assessment (GPFA), and 2) the GPFA Interview Protocol.

Growing Patterns and Functions Assessment

The Growing Patterns and Functions Assessment was designed to assess participants’ abilities to solve problems and generalize growing patterns of shapes (linear
functions) without assistance, and when assistance was needed, the nature of the assistance, and in both situations, the nature of the reasoning methods employed to solve the problems. All problems incorporated different numbers of the same shape. All shapes were the same color. Each problem showed five figures.

**Construction of the Growing Patterns and Functions Assessment (GPFA) Instrument**

Based on results of the pilot studies, each of the eight problems in the *GPFA* is displayed in landscape orientation on a piece of paper 22” by 8½.” Under each of the five figures is the word “Figure” followed by a number indicating its position, from left to right in the pattern. To the right of Figure 5 are spaces for Figures 6 through 10 (which are labeled below the space for the missing figures).

All problems have the same directions and questions:

“‘The pattern continues.’

“In Figure 6, how many shapes altogether? ” ____

“In Figure 10, how many shapes altogether? ” ____

The complete *GPFA* assessment is in Appendix C. Reductions of the pages can be seen on the following pages.

**Problem Types.** In the *GPFA*, Type 1 problems are of the form \( y = mx^f + b \) where \( m = 1 \) and \( b \) is 0, +1, +2, or -1. In Type 2 problems, \( m = 2 \) and \( b \) varies as with Type 1 problems. Each of the types of problems is shown below along with a description. In the description, \( y = mx^f + b \) is shown as \( T = mf + b \) where \( T \) is the total number of shapes and \( f \) is the figure number.

**Type 1 Problems.** Problems 1-4 of the assessment are shown and described in the order in which they appear in the *GPFA*. Problem 1 is shown in Figure 3.10. The pattern
displays a growing number of circles, beginning with one circle in Figure 1, two circles in Figure 2, three circles in Figure 3, and so on. The total number of shapes for each figure is equal to the figure number \((T = f)\). Therefore, Figure 6 will have 6 circles and Figure 10 will have 10 circles. The two questions are the same for all problems. They are enlarged here.

**Figure 3.10 Problem 1 GPFA**

*Problem 2:* As can be seen in Figure 3.11, Problem 2 presents a growing number of small squares beginning with two squares in Figure 1, three squares in Figure 2, four squares in Figure 3, and so on. The total number of shapes is equal to the figure number plus one \((T = f + 1)\). Therefore, Figure 6 will have 7 squares and Figure 10 will have 11 squares.
Problem 3: As can be seen below in Figure 3.12, Problem 3 presents a growing number of triangles, beginning with three triangles in Figure 1, four triangles in Figure 2, five triangles in Figure 3, and so on. The total number of shapes is equal to the figure number plus two \((T = f + 2)\). Therefore, Figure 6 will have 8 triangles and Figure 10 will have 12 triangles.

Problem 4: As can be seen in Figure 3.13, Problem 4 displays a growing number of circles, with zero circles in Figure 1, one circle in Figure 2, two circles in Figure 3, and so on. The total number of shapes is equal to the figure number minus one \((T = f - 1)\). Therefore, Figure 6 will have 5 circles and Figure 10 will have 9 circles.
Type 2 Problems. Type 2 problems present linear functions of the form, $T = mf + b$, in which $m = 2$, and $b = 0, 1, 2$ or $-1$. Problems 5-8 of the assessment are of this type.

Problem 5: As can be seen below in Figure 3.14, Problem 5 presents a growing number of triangles, beginning with two triangles in Figure 1, four triangles in Figure 2, six triangles in Figure 3, and so on. The total number of shapes is equal to the product of 2 and the figure number ($T = 2f$). Therefore, Figure 6 will have 12 triangles and Figure 10 will have 20 triangles.

Problem 6: As can be seen in Figure 3.15, Problem 6 displays a growing number of small squares, beginning with three squares in Figure 1, five squares in Figure 2, seven squares in Figure 3, and so on. The total number of shapes is equal to twice the figure
number plus one \((T = 2f + 1)\). Therefore, Figure 6 will have 13 squares and Figure 10 will have 21 squares.

**Figure 3.15 Problem 6 GPFA**

**Problem 7:** As can be seen in Figure 3.16, Problem 7 presents a growing number of squares, beginning with four squares in Figure 1, six squares in Figure 2, eight squares in Figure 3, and so on. The total number of shapes is the sum of twice the figure number plus 2 \((T = 2f + 2)\). Therefore, Figure 6 will have 14 squares and Figure 10 will have 22 squares.

**Figure 3.16 Problem 7 GPFA**

**Problem 8:** As can be seen in Figure 3.17, Problem 8 displays a growing number of pentagons, beginning with one pentagon in Figure 1, three pentagons in Figure 2, five pentagons in Figure 3, and so on. The total number of shapes is equal to one less than the
product of 2 and the figure number \((T = 2f-1)\). Therefore, Figure 6 will have 11 pentagons and Figure 10 will have 19 pentagons.

Figure 3.17 Problem 8 GPFA

**Scoring problems with no assistance.** Scores for problem, problem type, and question were based on the number of questions answered correctly without assistance. Each question is scored 1 point for a correct answer and 0 points for an incorrect answer. Each of the eight problems has two questions, so each problem has a maximum score of 2 and the maximum problem score is 8 problems x 2 points or 16 points. Four problems comprise each problem type, so the maximum problem type score is 8 points. Scores on the entire GPFA may range from 0-16 points.

**Instrument 2: GPFA Interview Protocol**

On the basis of information gained from the Pilot Studies, an interview protocol was developed to accompany the GPFA. The interview protocol includes procedures for greeting participants, seating them, introductory questions, how to introduce the GPFA, suggested prompts, general and specific assistances, wording to use to determine reasoning methods participants employ to solve the problems, and directions for scoring the interviews. The complete interview protocol may be found in Appendix B.
**Assistance.** A predetermined set of assistive questions was designed to gain greater insight into participants’ solution process and the types of reasoning they used. For each of the eight problems, there were two forms of assistance, general and specific. Once a question was answered correctly, assistance ceased, and assistance scores were tabulated for that question. Two general and two specific forms of assistance could be offered for each question, for a maximum of four forms of assistance per problem.

**General assistance.** Guiding questions or suggestions that focus on the overall pattern and how it is changing were designated as general forms of assistance, with codes H1.1 through H1.3:

- **H1.1** Tell me how the pattern changes.
- **H1.2** What part of the pattern changes?
- **H1.3** What part of the pattern stays the same? (if applicable)

**Specific assistance.** Guiding questions or suggestions that focus on one figure or on relationships between consecutive figures in a pattern, are designated as specific forms of assistance, with codes H2.1 through H2.3:

- **H2.1** (Point to figure.) Count the number of (shape miscounted) again.
- **H2.2** How many shapes are in Figure 1? Figure 2?
- **H2.3** How is the number of shapes in Figure x different from the number of shapes in Figure y? (Appropriate figure numbers were used.)

If a participant did not answer the first question, “How many shapes are in Figure 6?” it was still necessary to ask the second question, “How many shapes are in Figure 10?”
**Scoring problems solved with assistance.** A maximum of two general forms of assistance and two specific forms of assistance could be offered for each question in a problem. General forms of assistance score 1 point each and specific forms of assistance score 2 points each, for a maximum of 6 assistance points per question, or 12 points per problem. The total assistance score on the *GPFA* of 8 problems ranges from 0 to 96 points. Higher scores indicate greater need for assistance.

**Scoring reasoning methods used in the solution of problems.** Two methods employed by participants to solve problems were tabulated: recursion and covariation. When participants counted by ones or twos, added or subtracted from the previous number of shapes, they used a *recursive* method, and received a code of R. Example: “I skip counted by 2.”

When participants stated how the figure number and the number of shapes are related, they used a *covariation* method, receiving a code of C. Example: “I used the figure number and added 1 to the figure number.”

After participants completed problems 3 and 7, they were directed to, “Tell me how you figured it out.” If they did not relate the figure number to the number of shapes, they were queried further: “The figure number and the number of shapes are related–like relatives. Can you tell me how they are related?” Because the participant had already responded to the number of shapes in Figures 6 and 10, these follow-up questions were not scored for RQ 1 or 2, but were used in the interpretation of reasoning method for RQ3. The note taker and researcher recorded the codes and tabulated them for analyses. The total score for reasoning methods was 8, one point per problem, which may have
been split by reasoning method (e.g., 4 covariation and 4 recursive or 1 covariation and 7 recursive.)

**Gestures of support.** Prompts were used to provide encouragement to continue thinking, talking about, and solving the problems. Examples of prompts are: “Keep going.” “Good.” “Yes.” “Tell me more.” “Okay.” “And?” “Tell me what you see.” Prompts also include nods or gestures of support. Prompts were not scored.

**Observation tabulations.** Observational notes recorded on Individual Participant Assessment Sheets by both the note taker and the interviewer, were examined. Notes included: 1) participants’ problem scores, assistance scores, descriptions of how participants solved each problem, and reasoning methods they employed, and comments. The full observation protocol is provided in Appendix E.

**Study Questions and Methods of Analysis**

**Research Question 1 (RQ 1): Does the Growing Patterns and Functions Assessment (GPFA) score vary by**

a. Problem?
b. Problem Type?
c. Question?
d. Grade Level?
e. Gender?

To address this first research question, participants’ solution scores for problems, problem types, and questions with no assistance were used to analyze performance by grade level and gender. A repeated measures analysis of variance (ANOVA) was employed to examine the influence of effects of problem, problem type, question, grade
level, and gender on problem solution scores.

Research Question 2 (RQ2). Does the assistance score vary by
  a. Problem?
  b. Problem Type?
  c. Question?
  d. Grade Level?
  e. Gender?

To address the second research question, assistance scores were employed. A repeated measures analysis of variance (ANOVA) was used to examine the influence of effects of problem, problem type, question, grade level, and gender on assistance scores.

Research Question 3 (RQ3). Do reasoning methods (recursive reasoning and covariational reasoning), vary by
  a. Problem?
  b. Problem Type?
  c. Grade Level?
  d. Gender?

To address the third research question, to determine which reasoning method (recursive or covariational) participants employed, the researcher and note taker carefully listened as participants “thought aloud” while solving the problems. Frequency of use of the two reasoning methods was tabulated for each of the four factors. Those numbers were converted to percents of the maximum.
Administering the Growing Patterns and Functions Assessment

Procedures

The note taker was provided with instructions (one-on-one tutorial) on types of observations, behaviors, and the amount and nature of assistance, to record. Interview Protocols can be found in Appendix D. The tutorial was administered to the note taker by the researcher at the school site, one day prior to the start of the study. The form on which the note taker and interviewer documented observations is included in Appendix E.

Interview location and arrangement. One-on-one participant interviews were conducted by the researcher in a private room at the participants’ school. A pencil was placed next to the GPFA on the table, within easy reach of the participant. The researcher was seated at the table at a right angle to each participant. The note taker sat across from the researcher with a clear view of the participants and their written work. Interview duration ranged from 15 to 30 minutes.

Interview. At the beginning of each assessment interview, participants were shown where to sit. The researcher introduced herself and the note taker, and described their jobs. To introduce participants to the problem format and to the “think aloud” process that would be used during the interview, the researcher began each interview by asking participant to describe activities that transpired since arriving at school that morning, beginning with what happened first, then second, and so on. The researcher then asked each participant to talk about each problem in the same way: “What did you do/think first?” “Next?”

The Project Timeline displays the schedule for the development of the assessment instrument and protocols, conduct of the three pilot studies, confirmation of site and
student participation with the school district and administrator at the school site, distribution of family permission forms, recruitment and training of note taker, administration of student interviews, data analyses, and preparation of the dissertation.
Timeline for Instrument Development,
Recruitment of Participants, Data Collection and Analyses

Table 3.01

Timeline for Instrument Development, Recruitment of Participants, Data Collection and Analyses

<table>
<thead>
<tr>
<th>Time Period</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>August 2, 2013</td>
<td>Assessment instruments and protocols were approved by the University’s Research Compliance Board.</td>
</tr>
<tr>
<td>March 2013</td>
<td>Pilot Study I conducted.</td>
</tr>
<tr>
<td>June 2013</td>
<td>Pilot Study II conducted.</td>
</tr>
<tr>
<td>October 2013</td>
<td>Met with school administrator to obtain site and student participation confirmation.</td>
</tr>
<tr>
<td>October 25, 2013</td>
<td>Elementary school site approved by district administration.</td>
</tr>
<tr>
<td>November, 2013</td>
<td>Family permission forms distributed.</td>
</tr>
<tr>
<td>November 2013</td>
<td>Pilot Study III conducted.</td>
</tr>
<tr>
<td>November 2013</td>
<td>Note taker recruited and trained to observe and record types of participant behaviors during solution process, and how to code these for analyses.</td>
</tr>
<tr>
<td>Nov –Dec 2013</td>
<td>All participant interviews conducted in 3 school weeks.</td>
</tr>
<tr>
<td>Dec 2013-Oct 2016</td>
<td>Data Analyses performed and dissertation prepared</td>
</tr>
</tbody>
</table>
Chapter Summary

In this chapter, the development of the assessment instruments were described beginning with three pilot studies which informed the construction, design, contents, and scoring of the *Growing Patterns and Functions Assessment (GPFA)*. The GPFA instruments, interview protocols, assistance, reasoning methods, and scoring procedures were presented. Selection and demographics of the sample were described next, followed by the research questions and procedures for administering the assessment. The timeline of activities, concludes the chapter.

In Chapter 4, results of the data analyses for each research question are presented. Problem, problem type, question, Grade level, and gender are analyzed to identify the effect they have on each other in relation to solution success without and with assistance, and the reasoning methods employed by participants.
CHAPTER FOUR

DATA ANALYSIS

The goal of this study was to gain insight into the functional reasoning talents of students in Grades 1, 2, and 3 prior to instruction. Individual interviews were conducted with 60 participants as they solved problems posed in the *Growing Patterns and Functions Assessment (GPFA)* instrument. The GPFA interview protocol providing a predetermined sequence of assistive questions, and were used with participants who had difficulty solving the problems.

In this chapter, types of data collected during participant performance are described, followed by results of the analyses for each of the three research questions.

**Types of Data**

**Problem Score.** Each of the eight GPFA items has two questions, one relating to Figure 6 and one about Figure 10. One point is awarded for each question answered correctly without assistance. The maximum problem score is 2 points. The maximum score for the eight problems in the GPFA is 16 points (8 problems x 2 points).

**Problem Type Score.** The GPFA contains two types of problems: Type 1 includes Problems 1-4, and Type 2 includes Problems 5-8. Thus the maximum score for each Problem Type is 8 (4 problems x 2 points/problem).

**Question Score.** Each GPFA problem consists of two questions about the number of shapes in Figures 6 and 10. Each question answered correctly is scored 1 point.

**Assistance Score.** Assistance was provided for participants having difficulty with each question in each problem. Assistance questions were scored as follows: 1 point for each of the two general questions, and 2 points for each of the two specific questions, for
a maximum of 6 points per question, or 12 points per problem. The maximum GPFA Assistance Score for the eight problems is 96 (8 problems x 12 points/problem).

Reasoning Methods. Two methods of reasoning employed during the solution process were documented: covariational reasoning and recursive reasoning. One point per problem was assigned to the method of reasoning for each problem.

Research Results

The three research questions parallel each other in that they seek to determine the influence of problem, problem type, question, grade level, and gender on problem success, without and with assistance. Those results follow.

Research Question 1. Does the GPFA problem score vary by:

a. Problem?
b. Problem Type?
c. Question?
d. Grade Level?
e. Gender?

Analyses of Solution Success without Assistance

Since means of problem scores and question scores were proportional to the standard deviations (SD), a transformation of the data was conducted. When means are proportional to the SD, the recommended transformation is to add 0.5 to each score, compute the square root of the result, and analyze those data. To answer Research Question 1 (RQ 1) and analyze the transformed problem score data, a repeated measures analysis of variance (ANOVA) was conducted. Table 4.01 presents means of scores without assistance by problem, problem type, question, grade, and gender.
In Table 4.01, Question 1 refers to Figure 6 and Question 2 refers to Figure 10 in the GPFA. As can be seen in the table, for Problem Type 1, Problem 1, Question 1, Grade 1, the mean score for males is 0.64. For Problem Type 1, Problem 1, Question 2, Grade 2, the mean score for females is 0.90.

The within-subjects effects were problem, problem type, and question; whereas the between-subjects effects were grade level and gender.
<table>
<thead>
<tr>
<th>Grade</th>
<th>Problem Type</th>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Q 1</td>
<td>Q 2</td>
<td>Q 1</td>
</tr>
<tr>
<td>Grade 1</td>
<td>Male</td>
<td>0.64</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>Female</td>
<td>0.56</td>
<td>1.00</td>
</tr>
<tr>
<td>Grade 2</td>
<td>Male</td>
<td>0.78</td>
<td>0.89</td>
</tr>
<tr>
<td></td>
<td>Female</td>
<td>0.70</td>
<td>0.90</td>
</tr>
<tr>
<td>Grade 3</td>
<td>Male</td>
<td>0.73</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td>Female</td>
<td>0.80</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 4.01

Means of Transformed Scores (RQ 1) without assistance by Problem, Problem Type, Question, Grade, and Gender
**Problem effect.** The maximum score for each problem is 2 points. The means of problem scores for all grade levels and genders, displayed in Figure 4.01, ranged from 0.78 (Type 2, Problem 6) through 1.75 (Type 1, Problem 4). The means of scores on problems are statistically different from each other. The effect for problem is $F(3, 162) = 48.35, p < .001$, with $\eta^2 = .472$, which is a very large within-subjects effect (Olejnik & Algina, 2000).

**Problem type effect.** As can be seen in Figure 4.01, the mean of problem scores for Problem Type 1 is 1.53 and Type 2 is 1.09. The effect for problem type is significant, $F(1, 54) = 51.13, p < .001$, and $\eta^2 = .486$. Based on Cohen’s criteria (Olejnik & Algina, 2000), this is a very large within-subjects effect.

![Means of Problem Scores by Problem Number and Type](image)

*Figure 4.01. Means of Problem Scores by Problem Number and Problem Type*

The data shown in Figure 4.01 are reproduced in Figure 4.02 to illustrate similar trends in changes in problem scores by problem type. As can be seen in Figure 4.02, problems are paired by the value of $b$ in $T = mf + b$ where $f$ is the figure number and $b$ is
the value added or subtracted. The first problem pair displays means of scores for Problems 1 and 5: 1.62 and 1.27, respectively. In both problems, the value of $b$ is +0 (Problem 1: $T = 1f +0$ and Problem 5: $T = 2f +0$). Dotted lines show similar trends of scores for all pairs of Type 1 and Type 2 problems. Both trend lines show a drop in means of scores from the first problem pair with $b$ values of +0, to the next problem pair with $b$ values of +1. Scores then rise.

![Means of Problems Scores by Problem Pairs and Types](image)

*Figure 4.02. Trends in Problem Scores by Problem Pairs and Problem Types*

**Question score effect.** Question score refers to the two questions in each problem that request the number of shapes in Figure 6 (Question 1) and Figure 10 (Question 2). The maximum question score is 1. The effect for question was significant: $F(1, 54) = 8.08, p < .006, \eta^2 = .130$. This is a medium within-subjects’ effect (Olejnik & Algina, 2000). In the growing patterns in the GPFA, the means of question scores for
Figure 10 (0.75) were greater than for Figure 6 (0.56). This result will be considered more fully in the discussion section in Chapter 5.

**Grade level effects.** To gain insight into variations of student performance on the GPFA by grade level, the mean of problem and question scores without assistance were analyzed using the same repeated measures ANOVA. Means of problem scores varied by grade level, with Grade 2 the highest at 1.62, and Grade 1 the lowest at 1.12. The Grade 3 mean was 1.34. The effect for grade level was significant, $F(2, 54) = 3.29, p < .05$, with $\eta^2 = 0.109$, which is a small between-subjects effect.

**Gender effect.** The means of problem scores by gender were 1.08 for males and 1.02 for females. The difference between those means was not statistically significant: $F(1, 54) = 0.57, p < .46$.

In summary, analyses by grade level and gender separately showed that grade level influenced performance and gender did not.

**Research Question 2.** Does the assistance score vary by:

a. Problem?

b. Problem Type?

c. Question?

d. Grade Level?

e. Gender?
Analyses of Assistance Scores

For each GPFA item, two types of assistance were provided: 1) general assistance, that focused on the overall pattern, and 2), specific assistance, that focused on how the pattern is changing. Assistance scores for the entire GPFA ranged from 0 (no assistance) to 96 points (full assistance) with a maximum score of 6 points per question or 12 points for each of the eight problems. Prior to analyses of the assistance scores, a reliability analysis of those scores was conducted. Cronbach’s alpha reliability was 0.70, which demonstrated an acceptable level of reliability. A repeated measures analysis of variance (ANOVA) was then conducted on the assistance score data. The between-subjects effects were grade and gender. The within-subjects effects were problem type, problem, and question.

Table 4.02 displays assistance score data. As can be seen in the table, the assistance score for Type 1, Problem 1, Question 2 (Q2), Grade 1 males is 0.91. The assistance score for Type 2, Problem 6, Question 1 (Q1), Grade 3 females is 3.80.
Table 4.02

Means of Scores (RQ 2) with Assistance by Problem, Problem Type, Question, Grade, and Gender

<table>
<thead>
<tr>
<th>Problem Type</th>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>Q1 0.73 Q2 0.45</td>
<td>Q1 1.18 Q2 0.73</td>
</tr>
<tr>
<td>Problem 2</td>
<td>Q1 1.09 Q2 0.91</td>
<td>Q1 1.18 Q2 0.73</td>
</tr>
<tr>
<td>Problem 3</td>
<td>Q1 2.00 Q2 1.64</td>
<td>Q1 2.00 Q2 1.64</td>
</tr>
<tr>
<td>Problem 4</td>
<td>Q1 0.73 Q2 0.45</td>
<td>Q1 1.18 Q2 0.73</td>
</tr>
<tr>
<td>Grade 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>1.09</td>
<td>2.82</td>
</tr>
<tr>
<td>Female</td>
<td>1.00</td>
<td>4.33</td>
</tr>
<tr>
<td>Grade 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>1.00</td>
<td>0.67</td>
</tr>
<tr>
<td>Female</td>
<td>0.90</td>
<td>1.90</td>
</tr>
<tr>
<td>Grade 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>0.73</td>
<td>1.18</td>
</tr>
<tr>
<td>Female</td>
<td>0.70</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Note: Q1 and Q2 represent different questions or variables.
Problem effect. As can be seen by the means of the eight assistance scores displayed in Figure 4.03, the need for assistance varied by problem. The greatest amount of assistance was needed for Problem 6 (5.33); and the least amount of assistance was needed for Problem 4 (0.48). These assistance scores varied significantly, $F(3, 162) = 11.37, p < .001$ with $\eta^2 = 0.174$, which is a large within-subjects effect (Olejnik & Algina, 2000).

Problem type effect. Also to be seen in Figure 4.03, participants’ needs for assistance varied by problem type. Type 1 problems are displayed as solid bars. Type 2 problems are displayed with striped bars. Participants required significantly more assistance for Type 2 (mean: 4.09) than for Type 1 problems (mean: 1.63), $F(1,54) = 28.42, p < .001$, with $\eta^2 = 0.35$. This is a large within subjects’ effect size.

![Assistance Scores by Problem Number and Problem Type](image)

Figure 4.03. Means of assistance scores by problem number and problem type

Figure 4.04 displays means of assistance scores by problem pairs and problem type. Problems are paired by the value of $b$ in $T = mf + b$, where $f$ is the figure number and $b$ is the value added or subtracted. Problem type is represented by the value of $m$. 98
Although the bar graph clearly displays the greater need for assistance for Type 2 problems (striped bars), the dotted trend lines show similarities in participants’ needs for assistance for both Type 1 and Type 2 problems. The trend lines show an increase in means of scores for assistance from the first problem pair to the next problem pair followed by continual decreases to the final problem pair (with the lowest scores).

**Figure 4.04.** Trends in assistance scores by problem pairs and problem types

*Question effect.* The difference in mean scores by question was significant: $F(1, 54) = 54.31, p < .001$, with $\eta^2 = 0.501$. Participants required significantly more assistance for Question 1 about Figure 6 with a mean assistance score of 1.82 than for Question 2 about Figure 10, with a mean assistance score of 1.06.

*Grade level effect.* The effect for grade level was not significant: $F(2, 54) = 1.18, p < .32$. The means of assistance scores for Grades 1, 2, and 3 were 1.67, 1.15, and 1.50, respectively.
**Gender effect.** The effect for gender was not significant: \( F(1, 54) = 1.49, p < .23. \)
The means of assistance scores were 1.27 and 1.61 for males and females, respectively.

The following sections describe data collected about reasoning methods employed by participants during the solution of GPFA problems.

**Research Question 3.** Does reasoning method vary by:

a. Problem?

b. Problem Type?

c. Grade Level?

d. Gender?

The researcher and note taker identified participants’ use of covariational or recursive reasoning methods based on their verbal descriptions and gestures as they determined the number of shapes in Figures 6 and 10. Data (one point for each method observed) were analyzed to determine the percentage of covariational and recursive reasoning by problem, problem type, grade level, and gender.

**Covariational reasoning.** When the figure number was used to determine the number of shapes in Figures 6 and 10, that method was identified as using covariational reasoning.

The greatest use of covariational reasoning (55%) was for Problem 1 in which the number of shapes is equal to the figure number. Examples of participants’ comments demonstrated the use of covariational reasoning by problem follow:

*Problem 1.* Participant A (Grade 1): “I looked at the bottom number.”

Participant B (Grade 3): “Same number of figure is how many shapes.”
Participant C (Grade 3): “Because it says figure 6, figure 10. So that tells the number.”

*Problem 2.* Participant D (Grade 2): “It's getting bigger so I might need to use plus or times. I think plus (pointing to figure number).”

*Problem 5.* Participant E (Grade 3): Naming the figure number and then saying, “I used the figure number and multiplied it. 2 times 6 (for Figure 6) and multiply 2 by 10 (for Figure 10). I know my 2's alot.”

Participant F (Grade 3): “It would have 12 because it is a double. 6 + 6 and 10 + 10.”

*Problem 6.* Participant G (Grade 2): “These have doubles so 6 and 6 is 12. 10 and 10 is 20.”

Participant H (Grade 3): responded using multiplication for both Figures 6 and 10, “I multiplied 2 times 6. I multiplied 2 by 10. I know my 2's alot.”

**Recursive reasoning.** Participants who counted, skip counted, or added based on the previous number of shapes, and not based on the figure number were identified as using recursive reasoning.

*Problem 2.* Participant I (Grade 3): “They started at 2. Then I counted to 3-4-5-6, ‘count normal’.”

*Problem 4.* Participant J (Grade 1): Drew the shapes and then counted to identify the number of shapes in Figures 6 and 10, without assistance.

Participant K (Grade 2): “I went from 0 to 4, then 5-6-7-8-9.”

*Problem 6.* Most Grades 2 and 3 participants used the terms, “skip counting” and “odd numbers.”
Participant L (Grade 3): “Numbers are counting in odds: 3, 5, 7…”

**Problem 8.** Participant M (Grade 2): Recording the numbers “skipped” while skip counting.

![Figure 4.05. Grade 2 participant’s recording for Problem 8 (GPFA, 2016)](image)

**Analyses of Reasoning Methods**

Numbers of problems solved using each reasoning method were analyzed in three ways: (a) reasoning method used without assistance, (b) reasoning method used with assistance, and (c) a combination of (a) and (b). Participant comments can be found in Appendix G.

In Figure 4.06, the percent of problems solved using covariational reasoning is represented by the striped bars. The percent of problems solved using recursive reasoning are represented by the solid bars. The pair of bars on the left display the percent of problems solved correctly with no assistance. The pair of bars on the right show the percent of problems solved with assistance. The percent of problems solved with no assistance and the use of recursive reasoning (78%) far exceeds the percent of problems solved using covariational reasoning (22%). Likewise, the percent of problems solved
with assistance and using recursive reasoning (86%) far exceeds the percent of problems solved using covariational reasoning (14%).

![Problems Solved by Reasoning Methods](image)

*Figure 4.06. Use of recursive and covariational reasoning by need for assistance*

**Covariational Reasoning.** Percentages of problems solved using covariational reasoning were analyzed by problem, problem type, grade level, and gender.

*Problem effect.* As can be seen in Figure 4.07, participants’ use of covariational reasoning decreased by problem number from a high of 55% for Problem 1 to a low of 2% for Problem 8.

*Problem type.* Also to be noted in Figure 4.07, the use of covariational reasoning by problem type decreased substantially from 35% for Type 1 Problems (1-4) to 6% for Type 2 Problems (5-8).
Grade level effect. Percentages of participants employing covariational reasoning rose from 8% in Grade 1, to 16% in Grade 2, and to 35% in Grade 3.

Gender effect. The use of covariational reasoning by gender revealed only slight differences. Only 22% of females and 20% of males used covariational reasoning.

Recursive Reasoning. Percentages of problems solved using recursive reasoning were analyzed by problem, problem type, grade level, and gender.

Problem effect. Figure 4.08 shows the percent of participants who solved each problem using recursive reasoning. As can be seen in the bar graph, the use of recursive reasoning increased steadily from a low of 45% for Problem 1 to a high of 98% for Problems 7 and 8. This increasing use of recursive reasoning parallels increases in complexity of functions by problem and problem type.
**Problem type effect.** As can be seen in Figure 4.08, the use of recursive reasoning differed by problem type. For Type 1 problems (Problems 1-4), the percent of participants who used recursive reasoning was 65%. For Type 2 (Problems 5-8), 94% of participants used recursive reasoning.

![Use of Recursive Reasoning](Figure 4.08. Use of recursive reasoning by problem and problem type)

The researcher and note taker observed that participants employed only three recursive reasoning methods: count by ones, skip count, and add 1 or 2 to the number of shapes in the previous figure. Percentages of use by each method by problem number are shown in Figure 4.09.
For Type 1 problems, the prominent recursive method was counting by ones, as shown by the striped bars in Figure 4.09. For Type 2 problems, skip counting, as shown by the solid bars in Figure 4.09 was the primary recursive method, followed by counting by ones. Of note, skip counting was not useful for Type 1 problems, because the number of shapes in the figures increase by increments of 1. The use of adding from one figure to the next, shown by checkered bars was the least used recursive method.

**Grade level effect.** Use of recursive reasoning decreased by grade level from 92% for Grade 1, to 78% for Grade 2, to 65% for Grade 3. Data showed that use of specific recursive methods and grade level were related.

**Gender effect.** Participants’ use of recursive reasoning methods were tallied for females and males. Data revealed minimal differences by gender. Females employed
recursive reasoning for 78% of the problems and males used recursive reasoning for 81% of the problems.

Chapter Summary

In Chapter 4, the data collected and the analyses performed to answer each research question, were presented. Each research question examined the effect of problem, problem type, question, grade level and gender on (1) solution success, (2) amount of assistance needed, and (3) reasoning methods employed.

Research Question 1 was designed to determine factors that affect solution success on the GPFA without assistance. Results showed that solution success varied significantly by problem, problem type, and question. Participants’ scores on the GPFA were significantly different from each other. The most challenging problem was Problem 6. The least challenging was Problem 4. Participants’ scores for Type 1 problems were significantly greater than for Type 2 problems. However, question scores were significantly greater for GPFA Question 2 (Figure 10 in the far position) than for Question 1 (Figure 6 in the near position). Means of problem scores varied significantly by grade level. Differences by gender were not significant.

Research Question 2 was designed to determine the amount and nature of assistance needed by participants who were not able to solve the GPFA problems independently. Analyses showed that the need for assistance varied significantly by problem, problem type, and question. The need for assistance differed significantly by problem. Problem 6 needed the greatest amount of assistance. Problem 4 needed the least amount of assistance. For Type 1 problems (1-4) participants needed less assistance than they did for Type 2 problems (5-8). Regarding assistance by question, participants needed
more assistance to answer *GPFA* Question 1 (Figure 6 in the near position) than for Question 2 (Figure 10 in the far position). The need for assistance by gender was not significant.

Research Question 3 was designed to assess the reasoning methods participants used to solve the problems. Results showed that reasoning methods were related to problem and problem type. The use of covariational reasoning decreased by problem number and problem type, while the use of recursive reasoning increased by problem number and problem type. Grade level was also related to the use of the two types of reasoning methods. The percentage of participants employing a covariational strategy increased by grade level from Grade 1 to Grade 3, whereas participants’ use of recursive reasoning decreased by grade level from Grade 3 to Grade 1. Differences in reasoning methods by gender were minimal.

In Chapter 5, discussion of results and conclusions drawn from the study are presented. This is followed by limitations of the study, recommendations for teaching practice and recommendations for further research.
CHAPTER FIVE

DISCUSSIONS AND RECOMMENDATIONS

The current study was conducted to investigate the functional reasoning abilities of Grades 1-3 students prior to instruction. Of the study population of 60 participants, 20 were in Grade 1, 19 were in Grade 2, and 21 were in Grade 3. All participants were enrolled in the same Grades K-5 elementary school in an urban area in Arizona. There were 30 males and 30 females.

To assess functional reasoning abilities, the *Growing Patterns and Functions Assessment (GPFA)*, an eight-item assessment instrument, was developed along with an interview protocol. The *GPFA* requires identification and extension of growing patterns of geometric shapes of increasing complexity. Two types of growing patterns, referred to as Problem Types, were used. Problem Types differed by the nature of the function. The one-on-one Interview Protocol procedure included methods of assistance for participants who had difficulty during the solution process. Those methods took the form of guiding questions (tabulated) and prompts of encouragement (not tabulated). For each participant, solution-success scores were recorded by problem, problem type, question, grade level, and gender without assistance, and then with assistance. Reasoning methods employed by participants were also examined.

In this chapter, results of the analyses and conclusions are presented for each of the three research questions. Limitations of the study, recommendations for teaching practice, and recommendations for future research follow.
Discussion and Conclusions

Research Question 1

Research Question 1 was designed to gain insight into the functional reasoning abilities of participants, and to determine how solution success without assistance varied by five factors: problem, problem type, question, grade level, and gender. No participant scored 100% on all problems.

Of the five factors analyzed, gender was the only one that did not significantly influence performance on the solution of problems without assistance. That information was of considerable interest to the researcher since the tasks in the GPFA require reasoning about changes in position and the configurations of shapes in the figures, all spatial ideas. Spatial reasoning differences by gender have been studied by many researchers (Casey, 2002, 1996; Casey, Kersh, & Mercer Young, 2004; Casey, Pezaris, Anderson, & Bassi, 2004; Gong, He, & Evans, 2011; Gurian, 2011; Gurian & Stevens 2005; Gurian, Stevens, & King, 2010; Halpern, Benbow, Geary, Gur, Hyde, & Gernsbacher, 2007; Keller, & Menon, 2009; Pomerantz, Altermatt, & Saxon, 2002; Sax, 2007, 2011). All of these studies showed that females were less successful with spatial reasoning tasks than were males. Recent neuroimaging studies have provided considerable data that suggest a strong correlation between gender and cognitive differences, such as spatial reasoning (Gong, He, Evans, & 2011). Although, the GPFA findings did not reveal significant gender differences, research on the human brain and cognitive differences between genders, suggest that future studies take gender into account when examining cognitive performance.
Among the other four factors, Problem and Problem Type had the greatest influence on participants’ performances, with about the same degree of influence. With regard to scores, problem by problem, all were significantly different from each other \((p < .001)\). Means of scores ranged from 0.78 points (Problem 6) to 1.75 points (Problem 4). Note that in conjunction with their positions in the assessment, problems increased in complexity by the value of \(b\) in the function, \(y = mf+b\), where \(f\) is the figure number, and \(b = 0, +1, +2,\) or \(-1\). However, problem scores did not directly relate to arithmetic complexity. This finding is counter to the research of McEldoon & Rittle-Johnson (2010), who commented on the notable effect of arithmetic complexity on performance with function problems. This finding is re-examined in the discussion of Problem Type scores.

Problem Type scores (Type 1 versus Type 2) were significantly different; Type 2 scores were lower than Type 1 scores. The difference in scores by Type may be attributed to the value of \(m\) in the function, \(y = mf+b\), in Type 1 and Type 2 problems. This finding is not surprising, and is in concert with findings of McEldoon & Rittle-Johnson (2010), who found that problems with increasing complexity produce decreases in problem success.

When examining problems within each type, the first problem produced higher scores than did the second problem. However, from the second to the third to the fourth problems within each type, scores increased steadily. This kind of increase in success scores was exhibited in studies by Tsankova (2003) and Sorkin (2011). In Tsankova’s study, success rates increased with subsequent problems of the same form. She posited that as participants “gained experience solving the same type of problem in a different context, they recognized the sameness of structure and applied the same strategies” (p.
The phenomenon of scores increasing as complexity increases will be addressed further in discussions about Research Questions 2 and 3.

With regard to significant differences between means of scores for Questions 1 and 2 in each problem, participants were reliably more successful answering Question 2 about Figure 10 (far position) than Question 1 about Figure 6 (near position). This question effect is counter to results of studies by Kim & Sloane (2010), Moss & London McNab (2011), and Radford (2014), who found that the level of difficulty increases “by distance.” Near positions are easier for students to solve than far or remote positions. GPFA results for question effect are in agreement with the research of Tsankova (2003) and Sorkin (2011), whose explanation of higher scores on later more complex items may be the result of “learning” from the previous items.

Although GPFA problem scores varied significantly by grade level, the finding that Grade 3 problem scores were significantly lower than Grade 2 scores is counter to results of studies by Ginsburg and Baroody (2003), McEldoon & Rittle-Johnson (2010), and Tsankova (2003). In McEldoon & Rittle-Johnson’s 2010 study of Grades 2 – 6 students’ functional reasoning, Grade 3 students outscored Grade 2 students significantly in their varied levels of applying, recognizing, generating, and using function rules.

One explanation for the discrepancy in grade level scores on the GPFA may be found in the performance of participants on the Galileo mathematics pretest (Assessment Technology, Incorporated, 2010) that was administered in their school about two months prior to administration of the GPFA. Galileo scores include data on the percent of students who meet or exceed standards in mathematics at each grade level. On the Galileo, GPFA participants in Grade 3 (66%) had lower mathematics scores than did
students in Grade 2 (78%) and Grade 1 (68%). Clearly, this discrepancy needs further study.

**Research Question 2**

The second research question was designed to gain insight into the functional reasoning abilities of participants who received assistance during problem solution, and to determine how the amount of assistance varied by problem, problem type, question, grade level, and gender. Results of statistical analyses of assistance scores showed that there was no significant difference in scores by grade level or by gender. This grade level finding is not supported by Tsankova’s (2003) research that showed that, with increasing grade level, students needed less assistance. Unfortunately, other researchers who did provide assistance during problem solutions, addressed only one grade level (Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon, 2015; Kim & Sloane, 2010; Radford, 2014; Sorkin, 2011; & Warren, Miller, & Cooper, 2011, 2013). When more than one grade level was involved, researchers did not document the nature of or the amount of assistance provided (McEldoon & Rittle-Johnson, 2010).

Means of assistance scores were significantly different by Problem Type. Participants needed less assistance for Type 1 problems than for Type 2 Problems. The need for assistance may be related to the arithmetic complexity of the Type 2 problems. The need for additional assistance for more complex problems has been also identified in the research of Kim & Sloane (2010); Warren, Miller, & Cooper (2011, 2013); and McEldoon & Rittle-Johnson (2010).

With regard to Questions 1 and 2, participants required reliably more assistance for Question 1 (Figure 6) than for Question 2 (Figure 10). This finding continues to
strengthen the theory that participants “learned” from their experience with Question 1, and is in concert with the work of other researchers who examined the influence of feedback on learning (Grant & Dweck, 2003, Tsankova, 2003). In Teacher Feedback Strategies In Primary Classrooms–New Evidence (Hargreaves, McCallum, & Gipps, 2004), the researchers state, “Feedback can be the vital link between the teacher’s assessment of a child and the action following that assessment which then has a formative effect on the child’s learning” (p. 21). Whether a participant’s response to Question 1 on the GPFA was correct or required assistance, the feedback received from Question 1 (e.g., a nod of approval or guiding questions) may have provided the information the participant needed to solve Question 2 more readily.

Questions raised by this study that have implications for teaching and assessment include: (a) What is the optimal time to wait for responses from students before intervening? (b) What gestures, facial expressions, body language, physical and vocal cues are students giving that indicate they are confused, or confident that their thinking is on the right track?

**Research Question 3**

The third research question was designed to determine how reasoning methods vary by problem, problem type, grade level, and gender, without or with assistance. For problems solved correctly with no assistance, 22% of the problems were solved using covariational reasoning and 78% were solved using recursive reasoning. With regard to problems solved correctly with assistance, 14% were solved using covariational reasoning and 86% were solved using recursive reasoning. The nature of the assistance may have promoted one reasoning method as compared to the other. Nevertheless,
without or with assistance, the lack of use of covariational reasoning in the early grades at pretest is consistent with the research of Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon (2015), McEldoon & Rittle-Johnson (2010), Moss & London McNab (2011), Radford (2010), and Sorkin (2011).

Variations in the use of covariational reasoning by grade level increased from 8% in Grade 1, to 16% in Grade 2, and to 35% in Grade 3. This increase may be attributed to textbook programs used by participants. One of the programs was *Investigations in Number, Data, and Space* (Russell, Economopoulos, Cochran, Murray, Hollister, Bastable, Bloomfield, Horowitz, & Schifter, 2012). In Grade 1, for example, *Investigations* includes explorations with a jar to be filled with pennies. On Day 1, there is one penny in the jar. Each day 3 more pennies are placed in the jar. Students investigate how many pennies will be in the jar on Day 7? In Grade 2, *Investigations* introduces function tables to organize data and to uncover a rule that describes how one quantity changes in relation to another. It is interesting, however, that Grade 3 participants’ used covariational reasoning more than twice as frequently as did Grade 2 participants. However, Grade 3 problem success scores were reliably less than those of Grade 2 participants.

Differences in reasoning by problem and problem type were dramatic. The use of recursive reasoning increased with successive problems as covariational reasoning decreased. Further, the use of recursive reasoning was much greater for Type 2 than Type 1 problems. Participants had the greatest difficulty with Problem 6, which may be attributed to the recursive reasoning method that 97% of participants employed: 27% counted by 1s, 48% skip counted by 2s, and 11% added 2 to the previous number of
shapes in the pattern. For those using skip counting, this appeared to be their first encounter with “skip counting” with odd numbers, as evidenced by their comments. Moreover, their instructional programs presented skip counting always beginning with zero.

The dominant reasoning method employed by both females and males on the *GPFA* was recursive reasoning. Differences in results by gender were negligible. Although the *GPFA* results did not reveal a relationship between spatial reasoning and reasoning methods employed by females and males, this is counter to results of studies by Casey, 2002; Casey, Pezaris, Anderson, Bassi, 2004; Gong, He, & Evans, 2011; Gurian 2011; and Sax, 2011, who found that females’ performance on spatial tasks was reliably lower than males. Further research is needed to compare the reasoning methods based on gender.

**Summary of Results and Conclusions**

Problem and Problem Type had the greatest influence on participants’ performances on the *Growing Patterns and Functions Assessment (GPFA)*. Other than the first problem within each Type, problem scores increased as arithmetic complexity increased. This may be attributed to a “learning” effect from solving similar problems. Likewise for question scores, participants’ performance improved from Question 1 (Figure 6 in the near position) to Question 2 (Figure 10 in the far position). As well as the likely “learning” from previous problems or questions, the recursive reasoning methods used (counting, skip counting or adding 1 or 2 to the number of shapes in the previous figure) by the majority of participants enabled them to determine the number of shapes in Figures 6 and 10.
Grade level had a significant effect on both problem scores and reasoning methods, but not on assistance scores. The problem scores for participants in Grade 3 were significantly lower than those in Grade 2. This result cannot be explained by any of the literature examined, but perhaps by participants’ performance on the city-wide assessment administered. With regard to reasoning method, interestingly, Grade 3 participants used covariational reasoning twice as often as did participants in Grade 2 and four times as often as participants in Grade 1.

With regard to gender, there were no significant differences by problem, problem type, or question scores, for either performance or assistance scores and there were minimal differences in their uses of reasoning methods.

**Limitations of the Study**

Limitations of the study focus on three categories: the sample, the assessment instruments, and the administration of the assessment.

*Sample.* The sample for the study was drawn from only one elementary school in an urban area in Arizona with a population that may not be representative of the majority of schools in the United States. For example, 100% of the student body was eligible for free/reduced price lunch, as compared to 52% nationally (National Center for Education Statistics, 2016). Of the study sample, 84% were Latino. Participants in other communities and regions with greater ethnic diversity may have performed differently on the *GPFA*.

The same two textbook programs were used in the participants’ school. However, there was no information about the instructional practices or other topics presented to
participants that may have influenced their performance on the GPFA. This information would have been helpful in interpreting results.

Assessment Instruments. The Growing Patterns and Functions Assessment (GPFA) instrument used to assess the functional reasoning abilities of participants in Grades 1, 2, and 3, contained increasingly more complex growing patterns. The sequence of problems was carefully designed to ensure increasing arithmetic complexity from problems 1 to 8. This sequence may have produced a “learning effect.” The wording of the GPFA may have influenced participants performance, as well. The term “altogether” may have created a misunderstanding in which participants thought that they were to count all of the shapes that precede the specified figure.

Administration. The researcher was the sole administrator of the assessment and the assistance provided. Although a note taker was present for all interviews to document participants’ responses, the assistance provided, and the reasoning methods employed, it was the researcher who determined when assistance should be provided and the nature of that assistance. The researcher’s choices and timing of assistance offered, and the nature of that assistance may have influenced participants’ performance.

Recommendations for Future Research

Based on the limitations of the study described above, recommendations for future research address the sample, the assessment instruments, and the administration of the GPFA.

1. To validate the results, the study should be replicated with greater numbers of participants selected from populations that are more representative of the national
population. A sample with greater variation in socio-economic status, ethnicity, and types of instructional programs, could confirm (or refute) results.

2. To gain greater insight into participants’ experiences with growing patterns, information could be gathered through surveys or interviews with classroom teachers. Of interest is the nature of the mathematics programs and supplementary materials used by those teachers, and their in-class lessons and homework assignments.

3. Solution success increased with subsequent problems in the GPFA. This may reflect a “learning effect.” A new study with a mix of problem types and problems, not sequenced by difficulty, may produce different results.

4. With regard to the design of the GPFA, the number of figures in a pattern may have affected success and the need for assistance. A new study might present different numbers of completed figures in a pattern (4 or 6 versus 5 in the present study), in order to examine how the number of figures provided influences performance.

5. Certain terms in the questions of the GPFA caused confusion among participants. In particular, the use of the term, “altogether” in both questions in each problem stumped several participants. They thought that it signaled the addition of all preceding shapes in figures. Consider wording, such as: “How many shapes will be in Figure 6?” or “Please describe the 10th figure.” This may reveal the nature of the composition of the figures to which students are attending.

6. To assess the effect of computational facility on performance, administer the GPFA both at the beginning of and later in the school year. Compare student performance on the GPFA to academic performance in school mathematics.
7. To eliminate potential bias in interpreting the types of assistance provided, interviews could be video-taped and then analyzed by more than the administrator and note taker. This may produce different, and perhaps, more detailed information.

**Recommendations for Teaching Practice**

Based on observations made by the researcher and the note taker, and student actions observed, several recommendations are made to enable teachers to gain greater insight into what their students know.

*Teaching covariational reasoning.* Although participants in the GPFA study did not employ covariational reasoning for many of the problems, teaching studies conducted by Blanton, Stephens, Knuth, Gardiner, Isler & Jee-Seon (2015) and Moss & London McNab, (2011) demonstrated that young students are capable of learning to employ covariational reasoning to solve problems. If they learn it, they will use it. Since covariational reasoning is central to the study of algebra and development of algebraic reasoning abilities, curriculum developers and teachers must attend to this need.

*Talking math and probing questions.* The “learning” that was observed in the present study, through the presentation of problems ordered by complexity with easily identified differences between consecutive problems along with probing questions that provided varying degrees of assistance, may be useful to teachers and curriculum developers in the design and conduct of lessons. Requiring students to describe what they are doing (the sequence of their actions) provides a grand way to gain insight into what students know and their depths of understanding. The National Board for Professional Teaching Standards for Early Childhood Education (2012) affirms, “When appropriate,
Teachers strive to elicit what a child knows by prompting, probing, and rephrasing” (p. 71).

Teachers are encouraged to design and provide appropriate questions to help students to focus on relevant aspects of problems. Questions should not “give away” the answers. Questions from the present study are of that type: *How does the pattern change? What part of the pattern changes? How is the number of shapes in this figure, different from the number of shapes in the next figure?*...

*Wait time.* Teachers need to pause after questions to provide students with time to process their thoughts. In the present study, no assistance was provided until 15 seconds had elapsed.
REFERENCES


Anderson, (2014). An 82% failure rate on high school algebra exams?
http://www.forbes.com/sites/stuartanderson/2014/06/30/an-82-percent-failure-rate-on-high-school-algebra-exams/


Burns, M. (2010). Meeting students where they are: Snapshots of student misunderstandings. Educational Leadership. 67. 5 pp. 18-22.


http://commoncoretools.files.wordpress.com/2011/05/ccss_progression_cc_oa_k5_2011_05_302.pdf


Robb, L. (2014). Teach kids to build their own prior knowledge. MiddleWeb · 02/12/2014.


APPENDIX A

IRB APPROVAL
To: Carole Greenes  
Discovery

From: Mark Roosa, Chair  
Soc Beh IRB

Date: 08/02/2013
Committee Action: Exemption Granted
IRB Action Date: 08/02/2013
IRB Protocol #: 1308009465
Study Title: Growing Patterns and Functions Assessment (GPFA)

The above-referenced protocol is considered exempt after review by the Institutional Review Board pursuant to Federal regulations, 45 CFR Part 46.101(b)(1).

This part of the federal regulations requires that the information be recorded by investigators in such a manner that subjects cannot be identified, directly or through identifiers linked to the subjects. It is necessary that the information obtained not be such that if disclosed outside the research, it could reasonably place the subjects at risk of criminal or civil liability, or be damaging to the subjects' financial standing, employability, or reputation.

You should retain a copy of this letter for your records.
Dear Parent or Guardian,

I am conducting a study about young children’s understanding of an important idea in mathematics in order to discover better ways to support early learning. I am inviting your child to participate in this study. Participation means that your child will be interviewed by me while solving math problems. The interview will take place during school time and in the school. It will take about 30 minutes.

Your child’s participation in this study is voluntary. Results of the interview will not affect your child’s grade in school. When results are used in the study report, your child’s name will not be used. At all times, I will keep your child’s identity private.

Your child’s participation will benefit the learning of mathematics by other young students in our state and nation.

If you have any questions concerning the research study, please contact me or my advisor.

Thank you in advance for your help.

Mary Cavanagh
Executive Director, PRIME Center
(480) 727-0907
mcavanagh@asu.edu

Carole Greenes
Advisor, Carole Greenes
Associate Vice Provost for STEM Education
Professor, Mathematics Education
(480) 727-0902
cgreenes@asu.edu

If you have any questions about your child’s rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Office of Research Integrity and Assurance, at (480) 965-6788.

By signing below, you are giving consent for your child (Print Child’s Full Name)

Child’s age _____ Grade _____ Birthmonth ____________

Parent or Guardian Signature                  Printed Name                  Date

A copy of your signed permission will be sent to you.
Email address where you would like your copy sent. ______________________________

Or

Home mailing address ____________________________
APPENDIX C

GROWING PATTERNS AND FUNCTIONS ASSESSMENT INSTRUMENT

PROBLEMS 1-8
Problem 1

Figure 1 | Figure 2 | Figure 3 | Figure 4 | Figure 5

The pattern continues.

GPFA
The two previous pages for Problem 1 were attached to create a 22” x 8½” sheet as shown below. The wide display helps participants more clearly understand that the pattern continues by showing the missing spaces and labels for each of the Figures 6 through Figure 10. Problems 2 - 8 are presented with the two attached pages as shown below. The remaining seven Problems will display the left side of the pattern. The right sides are all the same as the previous page.

Problem 1:

Problem 1 displays a growing number of circles, starting with one circle in Figure 1, two circles in Figure 2, three circles in Figure 3 and so on. The total number of shapes is equal to the figure number. Therefore, Figure 6 has six circles and Figure 10 has ten circles.

Problem 2:

Problem 2 displays a growing number of small squares, starting with two squares in Figure 1, three squares in Figure 2, four squares in Figure 3, and so on. The total number of shapes is equal to the figure number plus one. Therefore, Figure 6 has seven squares and Figure 10 has eleven squares.

In Figure 6, how many shapes altogether? ____

In Figure 10, how many shapes altogether? ____
The pattern continues.
The pattern continues.

Problem 3

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5
Problem 4

Figure 1 | Figure 2 | Figure 3 | Figure 4 | Figure 5

The pattern continues.

GPFA
The pattern continues.

Figure 1

Problem 5

Figure 2

Figure 3

Figure 4

Figure 5

GPFA
The pattern continues.
Problem 7

The pattern continues.
Problem 8

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

The pattern continues.

GPFA
APPENDIX D

INTERVIEW PROTOCOLS
Interview Protocols

An interview protocol was developed, based on information gained from the pilot studies. It consists of logistics of setting up the interview room, the method to greet the student, placement of the test, offering the option to use colored pencils provided and the administration of the assessment. The protocol includes a sequence of guiding questions to give in order to assist the student in the solution of the problems. The guiding questions follow a progression of general questions providing simple assistance to specific questions providing major assistance.

Directions for Interviewer and Note taker

Both the researcher/interviewer, described in this section as “interviewer” and the note taker will use a separate GPFA Student Interview Notes sheet shown in Appendix F for each student. The interviewer will record student’s name, age, month of birth from data given on the permission form and the ethnicity of the child based on observation. She will keep a running record of assistance, circling the hint type and number on the form. When the same hint is given twice for a question, note taker will mark the numeral 2 above the circled hint type/number. Prompts will also be noted by item. Note taker will record total points for each problem, problem type and overall score following the test. The note taker will take notes of student responses, describe circumstances around drawings, note time elapsed for problem solution, and observations of the student that include attitudes, solution methods, solution strategies and difficulties. Note taker will also record indications of a lack of understanding of a direction or question (e.g., “The pattern continues;” “Tell me how the pattern changes;” “What part of the pattern stays the same?”), miscounting the number of individual or combined shapes and the questions
asked by subject during the solution process and the subject’s suggestions of unique ways to solve the problem. To check for reliability of scoring, both the interviewer and note taker will keep track of and compare scoring of assistance given for 15 students.

**Welcome the student:**

Wording for the interviewer is italicized without quotation marks in this section.

- Welcome the student and seat him/her at the table at a right angle to interviewer. The note taker is seated at the table to the left of the student.
- Interviewer introduces herself and the note taker to the student.
- Ask the following question to establish a friendly atmosphere and focus the student on reporting the solution process in order.

  - *When you got to school this morning, what did you do first? (pause) What did you do next? And then?*

**Conduct the interview/assessment**

- Place the assessment on the table in front of the student and hand him/her a pencil.

  - *I will show you some patterns. Your job is to figure out how many shapes are in some parts of the pattern. And, I want you to talk about what you are doing.*

  - *There are pencils on the table for you to use. You can use them at any time.*

  - As you turn to each new pattern, say the following: *Here’s a pattern. Start with Figure 1. Now look at all of the shapes.*

- *How many shapes are there in Figure 6?*

- *How about Figure 10?*
Assistance will not be given until the student demonstrates the need and asks for assistance or 20 seconds passes and the student makes no indication of consideration of the problem.

The Scoring Sheet on the next page describes scoring for assistance Types.

**Prompts**

Prompts are reassurances given freely and are not scored. They include:

- Head nods
- Smiles
- Keep going.
- Good.
- Yes.
- Okay.
- And...?
- Tell me what you see.

**Assistance and Scoring**

**General Assistance and Codes, H1.1 through H1.3** (1 point for each).

Maximum of 4 points (2 for Figure 6 and 2 for Figure 10), then go to specific assistance if needed.

- H1.1 Tell me how the pattern changes.
- H1.2 What part of the pattern changes?
- H1.3 What part of the pattern stays the same? (if applicable)
**Assistance: Specific and Codes (2 points for each) as needed.** Maximum of 8 points (4 for Figure 6 and 4 for Figure 10).

- **H2.1** (Point to figure.) *Count the number of (shape miscounted) again.*

- **H2.2** *How many shapes are in Figure 1? Figure 2? How is the number of shapes in Figure 2 different from the number of shapes in Figure 1?* (Change figure numbers as needed to compare.)

If a student cannot answer “How many are in Figure 6?”, it is still necessary to ask “How many are in Figure 10?”

**Post-Questions Request**

After the student has finished an item, say, “*Tell me how you figured it out.*”

After questions 2 and 7, “*The figure number and the number of shapes are related—like relatives. Can you tell me how they are related?*”
APPENDIX E

PARTICIPANT ASSESSMENT FORM FOR NOTE TAKER AND INTERVIEWER
<table>
<thead>
<tr>
<th>Student Name</th>
<th>Grade Level</th>
<th>Age</th>
<th>Birth Month</th>
<th>Gender</th>
</tr>
</thead>
</table>

**Pattern 1: $T = F$ Correct Solution: Yes No Problem Score:**

- Figure 6: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)
- Figure 10: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)

**Pattern 2: $T = F + 1$ Correct Solution: Yes No Problem Score:**

- Figure 6: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)
- Figure 10: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)

**Pattern 3: $T = F + 2$ Correct Solution: Yes No Problem Score:**

- Figure 6: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)
- Figure 10: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)

**Pattern 4: $T = F - 1$ Correct Solution: Yes No Problem Score:**

- Figure 6: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)
- Figure 10: $H_1 (1,2,3)$ (1,2,3) $H_2 (1,2)$ (1,2)

**Problem Type 1 Score:** (Sum of Problem Scores 1, 3, 4)
Pattern 5: $T = 2F$  Correct Solution: Yes  No  Problem Score: ____

Figure 6  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Figure 10  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Pattern 6: $T = 2F + 1$  Correct Solution: Yes  No  Problem Score: ____

Figure 6  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Figure 10  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Pattern 7: $T = 2F + 2$  Correct Solution: Yes  No  Problem Score: ____

Figure 6  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Figure 10  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Pattern 8: $T = 2F - 1$  Correct Solution: Yes  No  Problem Score: ____

Figure 6  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Figure 10  H1 (1,2,3)  (1,2,3)  H2 (1,2,)  (1,2)

Problem Type 2 Score ____  (Sum of Problem Scores 5, 6, 7, 8)  Total Score ____  (Sum of all problem scores 1-8)
APPENDIX F

DOCUMENTATION OF PARTICIPANTS’ COMMENTS
Examples of Participants’ Covariational Reasoning

For covariational reasoning, the figure number is used to determine the number of shapes in Figures 6 and 10. For example, for both figures, one participant said, “I used the figure number and multiplied it. Two times 6. I know my 2’s a lot.” Others doubled the figure number or “added it to itself.” Examples of covariational reasoning follow.

*Grade 1 Participants*

Problem 1: “I looked at the bottom number.”

Problem 1: “Cause right there is a 10.” Pointing to the figure number.

Problem 2: “I think in my brain a lot, very different. Figure number is number of shapes and one more.”

*Grade 2 Participants*

Problem 2: “They added 1 more than the figure number.”

Problem 4: “Because they are taking one away from the figure number.”

Problem 5: “These have doubles so 6 [Figure Number] and 6 is 12.”

*Grade 3 Participants*

Problem 1: “Same number of figure [number] is how many shapes.”

Problem 2: “It's getting bigger so I might need to use plus or times, I think plus.” She recognized that the figure number is one less than the number of shapes.

Problem 4: If there are 10 [shapes] in [pointing to the label Figure 10] 10 then I take one away.”

Problem 5: “I multiplied 2 by [figure] 10.”

Problem 6: “It would have 12 because it is a double.”
Examples of Participants’ Recursive Reasoning

For recursive reasoning, participants indicated that they were “moving” from figure to figure, never referring to the figure number.

Grade 1 participants

Problem 1: Participants counted the number of shapes beginning with Figure 1 while pointing, “1, 2, 3,…6.” They continued through Figure 10, “7, 8, 9,…10.”

Problem 2. Participants counted by ones, starting with 2 (Figure 1): “2, 3, 4,…6” They continued through Figure 10, “7, 8, 9,…11.”

Problem 6: “Climbing up by 2 more, it's kinda like a mountain.”

Problem 6: “I don't like this one. I hope it's not going to be harder, now this is way different, so we are skipping?”

Problem 7: “Are we doing the skipping? I skip 19 and get 20.”

Problem 7: “This one is even more trickier.”

Problem 8: “Cuz I'm the smartest one in my class.”

Grade 2 participants

Problem 2: “I see you added 1 more, you add 1 more to 6 then have 7.”

Problem 4: Participants started at Figure 1 and said, “Zero,” they then counted by ones to 9.
Problem 8: Participant recorded each number “skipped” between the numbers counted in the figure below.

Grade 3 participants

Problem 2: “They started at 2. Then I counted to 3-4-5-6, ‘count normal’.” Many of the participants referred to test items as “they,” perhaps referring to the test author.

Problem 5: “I counted by 2's. It goes 2 more upper.”

Problem 6: “Numbers are counting in odds: 3, 5, 7…”

Problem 6: “It's doing hard math. I need to count by my fingers, 6+6 is 12 + 1 more is 13.”

Problem 8: “Skip by odd numbers: 3, 5, 7…”

Problem 8: “I less, I think I could make an educated guess, this one is kinda harder.”

Problem 8: “Odd number counting, count by odd numbers again.”
APPENDIX G

PILOT STUDY A
Pattern Continuation

1.

<table>
<thead>
<tr>
<th>Row 1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>2 2</td>
</tr>
<tr>
<td>Row 3</td>
<td>2 2 2</td>
</tr>
<tr>
<td>Row 4</td>
<td>2 2 2 2</td>
</tr>
</tbody>
</table>

Continue the pattern.

a. What is the sum of the 2s in Row 5? ________

b. What is the sum of the 2s in Row 10? ________

c. How many 2s are in Row 100? ________
Rule: Continue the pattern.

a. What is the last number in Row 6? ______
b. What is the last number in Row 10? ______
c. What is the first number in Row 11? ______
d. In which row is 60 the last number? ______
e. What is the color of row with the number 85? ________________
Pattern Continuation

7.

Rule: Continue the pattern.

a. How many \(\bigcirc\) are in Figure 5? ____
   How many \(\bigcirc\) are in Figure 6? ____
   How many \(\bigcirc\) are in Figure 10? ____

b. How many \(\Box\) are in Figure 5? ____
   How many \(\Box\) are in Figure 6? ____
   How many \(\Box\) are in Figure 10? ____

c. How many \(\blacklozenge\) are in Figure 10? ____

d. There are 4 shapes in Figure 1.
   How many shapes are in Figure 2? ____
   How many shapes are in Figure 3? ____
   How many shapes are in Figure 10? ____

Pilot Study A
APPENDIX H

PILOT STUDY B
This is a pattern. It keeps growing.

Research Questions:

1. Can students draw or predict Design 6? Do they get the correct number of each shape/color? Do they follow the same arrangement?
2. Can students predict Design 10?
3. Does asking about the parts of the pattern (e.g., 1 Color A and 4 Color) aid students in figuring out the total number of figures? Note that this may be a stinky question. May want to throw it out if it means needing a greater number of kids. Would it be most relevant with a particular grade?

Pilot Study B
This is a pattern. It keeps growing.

Pattern 1

How many blue circles are in Design 6? ___

How many red squares are in Design 6? ___

How many red squares are in Design 10? ___

How many shapes are in Design 10 altogether? ___

Pilot Study B
This is a pattern. It keeps growing.

Pattern 2

How many blue circles are in Design 6? ____

How many red squares are in Design 6? ____

How many blue circles are in Design 10? ____

How many shapes are in Design 10 altogether? ____

Pilot Study B
This is a pattern. It keeps growing.

Pattern 3

How many blue circles are in Design 6? ____

How many red squares are in Design 6? ____

How many red squares are in Design 10? ____

How many shapes are in Design 10 altogether? ____

Pilot Study B
This is a pattern. It keeps growing.

Pattern 4

How many blue squares are in Design 6? ____

How many red squares are in Design 6? ____

How many yellow triangles are in Design 6? ____

How many of each shape are in Design 10? ______  ______  ______

How many shapes are in Design 10 altogether? ______
This is a pattern. It keeps growing.

Pattern 5

Design 1  Design 2  Design 3  Design 4  Design 5

How many blue circles are in Design 6? ____

How many red squares are in Design 6? ____

How many of each shape are in Design 10? ____  ____

How many shapes are in Design 10 altogether? ____

Pilot Study B
This is a pattern. It keeps growing.

Pilot Study B
This is a pattern. It keeps growing.

Pattern 7

- How many □ are in Design 6? ____
- How many ● are in Design 6? ____
- How many △ are in Design 6? ____

How many of each shape are in Design 10? ____  ____  ____

How many shapes are in Design 10 altogether? ____

Pilot Study B
This is a pattern. It keeps growing.

Pattern 8

Design 1     Design 2     Design 3     Design 4     Design 5

How many □ are in Design 6? ____

How many □ are in Design 6? ____

How many △ are in Design 6? ____

How many of each shape are in Design 10? ____ □  ____ □  ____ △

How many shapes are in Design 10 altogether? ____

Pilot Study B
## Algebraic Thinking with Growing Patterns

<table>
<thead>
<tr>
<th>Pattern #</th>
<th>Variables</th>
<th>Design* #1</th>
<th>D#2</th>
<th>D#3</th>
<th>D#4</th>
<th>D#5</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Color A varies &amp; Color B is constant</td>
<td>Circle &amp; Square</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Color A varies &amp; Color B is constant</td>
<td>Circle &amp; Square</td>
<td>BA</td>
<td>BA</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Color A varies &amp; Color B is constant</td>
<td>Circle &amp; Square</td>
<td>A</td>
<td>BA</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3 colors A &amp; C are constant, B varies</td>
<td>Circle Square &amp; Triangle</td>
<td>ABC</td>
<td>ABC</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Both colors vary</td>
<td>Circle &amp; Square</td>
<td>AB</td>
<td>AB</td>
<td>AB</td>
<td>AB</td>
<td>AB</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Both colors vary</td>
<td>Circle &amp; Square</td>
<td>A</td>
<td>BA</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>A &amp; B vary C is constant</td>
<td>Circle Square &amp; Triangle</td>
<td>BAC</td>
<td>BAC</td>
<td>BA</td>
<td>BA</td>
<td>BA</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>All 3 colors vary</td>
<td>Circle Square &amp; Triangle</td>
<td>A</td>
<td>AB</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Pilot Study B
Pattern 1:
\[ T = F \]
Correct Solution: Yes   No
Problem Score: ______

Problem 2:
\[ T = F + 1 \]
Correct Solution: Yes   No
Problem Score: ______

Problem 3:
\[ T = F + 2 \]
Correct Solution: Yes   No
Problem Score: ______

Problem 4:
\[ T = F - 1 \]
Correct Solution: Yes   No
Problem Score: ______

In Figure 6, how many shapes altogether? ____

In Figure 10, how many shapes altogether? ____
The two sections (actual sizes: 8 ½ “ 11” in landscape orientation) shown on the previous page for Problem 1 were attached to create a 22” x 8½” sheet as shown on the following page. The wide display helps participants more clearly understand that the pattern continues by showing the missing spaces and labels for each of the Figures 6 through Figure 10. Problems 2 - 8 are presented with the two attached pages as shown below. The remaining seven Problems will display the left side of the pattern only. The right sides are all the same as the example displayed at the bottom of previous page.
Pattern 1: \( T = F \) 
Correct Solution: Yes   No
Problem Score: ______

Pattern 2: \( T = F + 1 \) 
Correct Solution: Yes   No
Problem Score: ______

Pattern 3: \( T = F + 2 \) 
Correct Solution: Yes   No
Problem Score: ______

Pattern 4: \( T = F - 1 \) 
Correct Solution: Yes   No
Problem Score: ______
Pattern 1: \( T = F \)  
Correct Solution: Yes   No  
Problem Score: ______

Pattern 2: \( T = F + 1 \)  
Correct Solution: Yes   No  
Problem Score: ______

Pattern 3: \( T = F + 2 \)  
Correct Solution: Yes   No  
Problem Score: ______

Pattern 4: \( T = F - 1 \)  
Correct Solution: Yes   No  
Problem Score: ______

The pattern continues.
Pattern 1:
T = F
Correct Solution: Yes   No
Problem Score: ______

Pattern 2:
T = F + 1
Correct Solution: Yes   No
Problem Score: ______

Pattern 3:
T = F + 2
Correct Solution: Yes   No
Problem Score: ______

Pattern 4:
T = F - 1
Correct Solution: Yes   No
Problem Score: ______
Pilot Study C
Problem 6

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

The pattern continues.
Pattern 5:
\( T = 2 \)
\( F \) = Correct Solution

Problem Score: ______

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

The pattern continues.

Pattern 6:
\( T = 2 \)
\( F + 1 \) = Correct Solution

Problem Score: ______

Figure 6

Figure 7

Figure 8

Figure 9

Figure 10

The pattern continues.

Pattern 7:
\( T = 2 \)
\( F + 2 \) = Correct Solution

Problem Score: ______

Figure 11

Figure 12

Figure 13

Figure 14

Figure 15

The pattern continues.

Pattern 8:
\( T = 2 \)
\( F - 1 \) = Correct Solution

Problem Score: ______

Figure 16

Figure 17

Figure 18

Figure 19

Figure 20

The pattern continues.
Appendix D
Assessment Instrument
Continued

Problem 8
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5

The pattern continues.