Nonlinear Phase Based Control to Generate and Assist Oscillatory Motion with Wearable Robotics

by

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ABSTRACT

Wearable robotics is a growing sector in the robotics industry, they can increase the productivity of workers and soldiers and can restore some of the lost function to people with disabilities. Wearable robots should be comfortable, easy to use, and intuitive. Robust control methods are needed for wearable robots that assist periodic motion.

This dissertation studies a phase based oscillator constructed with a second order dynamic system and a forcing function based on the phase angle of the system. This produces a bounded control signal that can alter the damping and stiffens properties of the dynamic system. It is shown analytically and experimentally that it is stable and robust. It can handle perturbations remarkably well. The forcing function uses the states of the system to produces stable oscillations. Also, this work shows the use of the phase based oscillator in wearable robots to assist periodic human motion focusing on assisting the hip motion. One of the main problems to assist periodic motion properly is to determine the frequency of the signal. The phase oscillator eliminates this problem because the signal always has the correct frequency. The input requires the position and velocity of the system. Additionally, the simplicity of the controller allows for simple implementation.
To my parents, without whom this would not have been possible.
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TABLE OF CONTENTS

<p>| LIST OF TABLES | .......................................................... | viii |
| LIST OF FIGURES | ......................................................... | x |
| 1 INTRODUCTION | ......................................................... | 1 |
| 1.1 Amputees in the USA | .......................................................... | 2 |
| 1.2 Wearable Robotics for Motion Assistance | .................................................. | 4 |
| 1.3 Wearable Robots Requirements | ................................................. | 5 |
| 1.4 Text Organization | ..................................................... | 6 |
| 1.5 Contributions of this Dissertation | .................................................. | 8 |
| 2 RELATED WORK | ......................................................... | 10 |
| 2.1 Phase use in Prosthesis and Orthoses Control | ............................................. | 10 |
| 2.2 Sensors in Prosthesis and Orthoses control | ............................................ | 17 |
| 2.2.1 Control | ....................................................... | 17 |
| 2.2.2 Mechanical Signal as Control Input | ............................................... | 19 |
| 2.2.3 Electromyography | .................................................. | 20 |
| 2.2.4 Electroencephalography | ............................................... | 23 |
| 2.2.5 Sonography | ....................................................... | 25 |
| 2.2.6 Mechanomyography | .................................................. | 26 |
| 2.2.7 Myokinematic | ....................................................... | 27 |
| 2.2.8 Force myography | .................................................. | 28 |
| 3 PHASE BASED OSCILLATOR: CONTROLLING OSCILLATIONS WITH A PHASE BASED FORCING FUNCTION | ........................................ | 32 |
| 3.1 Oscillators and Limit Cycles | .................................................. | 32 |
| 3.2 Phase Based Oscillator Controller | .................................................. | 33 |
| 3.3 Existence of a Limit Cycle | .................................................. | 35 |</p>
<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.4</td>
<td>38</td>
</tr>
<tr>
<td>Stability of the Limit Cycle</td>
<td>38</td>
</tr>
<tr>
<td>3.5</td>
<td>40</td>
</tr>
<tr>
<td>Solution</td>
<td>40</td>
</tr>
<tr>
<td>3.6</td>
<td>44</td>
</tr>
<tr>
<td>Purely Sinusoidal Case Analysis</td>
<td>44</td>
</tr>
<tr>
<td>3.6.1</td>
<td>45</td>
</tr>
<tr>
<td>Stability of the Limit Cycle Using Polar Coordinates</td>
<td>45</td>
</tr>
<tr>
<td>3.6.2</td>
<td>47</td>
</tr>
<tr>
<td>Comparison with the Time Based Oscillator</td>
<td>47</td>
</tr>
<tr>
<td>3.6.3</td>
<td>48</td>
</tr>
<tr>
<td>Simulations of the Phase Oscillator and the Time Based Oscillator</td>
<td>48</td>
</tr>
<tr>
<td>3.7</td>
<td>50</td>
</tr>
<tr>
<td>Analysis of the Effects of the Forcing Function</td>
<td>50</td>
</tr>
<tr>
<td>3.7.1</td>
<td>51</td>
</tr>
<tr>
<td>Sine of Phase Angle as a Forcing Function</td>
<td>51</td>
</tr>
<tr>
<td>3.7.2</td>
<td>54</td>
</tr>
<tr>
<td>Cosine of Phase Angle as a Forcing Function</td>
<td>54</td>
</tr>
<tr>
<td>3.7.3</td>
<td>57</td>
</tr>
<tr>
<td>Sine + Cosine of Phase Angle as Forcing Function</td>
<td>57</td>
</tr>
<tr>
<td>3.8</td>
<td>59</td>
</tr>
<tr>
<td>Use as Controller</td>
<td>59</td>
</tr>
<tr>
<td>3.8.1</td>
<td>60</td>
</tr>
<tr>
<td>Comparison with Linear Controller</td>
<td>60</td>
</tr>
<tr>
<td>3.9</td>
<td>62</td>
</tr>
<tr>
<td>Adding Offset to the Oscillations</td>
<td>62</td>
</tr>
<tr>
<td>3.10</td>
<td>63</td>
</tr>
<tr>
<td>Summary</td>
<td>63</td>
</tr>
<tr>
<td>4</td>
<td>64</td>
</tr>
<tr>
<td>SIMULATION AND EXPERIMENTAL VALIDATION OF THE PHASE BASED OSCILLATOR CONTROLLER</td>
<td>64</td>
</tr>
<tr>
<td>4.1</td>
<td>64</td>
</tr>
<tr>
<td>Simulation Examples</td>
<td>64</td>
</tr>
<tr>
<td>4.2</td>
<td>68</td>
</tr>
<tr>
<td>Oscillating the Angular Velocity of a DC Motor</td>
<td>68</td>
</tr>
<tr>
<td>4.3</td>
<td>83</td>
</tr>
<tr>
<td>Oscillating a Pendulum</td>
<td>83</td>
</tr>
<tr>
<td>4.4</td>
<td>97</td>
</tr>
<tr>
<td>Transitory Response During Perturbation Recovery</td>
<td>97</td>
</tr>
<tr>
<td>4.4.1</td>
<td>104</td>
</tr>
<tr>
<td>Simulation of Perturbation Recovery</td>
<td>104</td>
</tr>
<tr>
<td>4.4.2</td>
<td>109</td>
</tr>
<tr>
<td>Real data of Perturbation Recovery</td>
<td>109</td>
</tr>
<tr>
<td>5</td>
<td>114</td>
</tr>
<tr>
<td>UNCERTAINTY ANALYSIS IN THE PHASE BASED OSCILLATOR</td>
<td>114</td>
</tr>
</tbody>
</table>
5.1 Bounds for Perturbation and Uncertainty .................................. 115
  5.1.1 Linear System with Stochastic Perturbation ....................... 115
  5.1.2 Extension to System with Phase Oscillator Controller ...... 117
5.2 Handling Perturbations and Uncertainty with Lyapunov Redesign . . . 122
  5.2.1 Simulation Example: Using the Controller Obtained by Lyapunov
       Redesign on a Pendulum ........................................ 126
  5.2.2 Implementation: Reducing the Trajectory Variability Around
       the Limit Cycle .............................................. 129
  5.2.3 Using the Lyapunov Redesign Controller with Nonlinear
       Pendulum Models ........................................... 132
5.3 Robust Stability Analysis of the Phase Based Oscillator .......... 136
  5.3.1 Small Gain Theorem ........................................ 136
  5.3.2 Uncertainty Models ........................................ 137
  5.3.3 Stability Analysis of a Nonlinear Pendulum with the Phase
       Based Forcing Function Using the Small Gain Theorem ...... 141
6 ASSISTING OSCILLATORY MOTION USING THE PHASE BASED
   FORCING FUNCTION ............................................. 144
  6.1 Periodic Motion .............................................. 144
  6.2 Assisting Periodic Motion .................................... 146
    6.2.1 Phase Angle Error .................................... 148
    6.2.2 Assisting Periodic Motion with Constant Values of c and d .. 156
    6.2.3 Considering the Neural Delay ................................ 172
    6.2.4 Assisting Periodic Motion Adapting the Values of c and d .. 174
  6.3 Implementation to Assist Hip Motion .............................. 177
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Projected Number of Persons with Major Limb Loss in 2050 [1]</td>
</tr>
<tr>
<td>2.1</td>
<td>Summary of Trajectory Generation Methods for Motion Assistance with Exoskeletons</td>
</tr>
<tr>
<td>2.2</td>
<td>State of the Art in Powered Hand Prostheses</td>
</tr>
<tr>
<td>2.3</td>
<td>Summary of Control Signals for Prostheses, Orthoses, and Exoskeletons</td>
</tr>
<tr>
<td>3.1</td>
<td>Necessary and sufficient conditions for stability of limit cycles in nonlinear systems</td>
</tr>
<tr>
<td>3.2</td>
<td>Summary of the Three Cases Simulated and Analyzed Using the Time Based Oscillator and the Phase Based Oscillator, and the Values of the Coefficients for Each Case</td>
</tr>
<tr>
<td>4.1</td>
<td>Comparison of the Transitory Responses of the Time Based Oscillator and the Phase Based Oscillator</td>
</tr>
<tr>
<td>5.1</td>
<td>Stability Bounds for the Phase Based Oscillator for Each Uncertainty Model, Using the Small Gain Theorem</td>
</tr>
<tr>
<td>6.1</td>
<td>Technical Specifications of the Wearable Hip Robot</td>
</tr>
<tr>
<td>6.2</td>
<td>Summary of Values of $c_a$ and $d_a$ and the Respective Shift Angle Equivalence that Were Tested</td>
</tr>
<tr>
<td>6.3</td>
<td>Average and Maximum Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 1</td>
</tr>
<tr>
<td>6.4</td>
<td>Average Ground Reaction Force Along Each Axis for Both Legs During Each Test SESSION 1</td>
</tr>
<tr>
<td>6.5</td>
<td>Average and Maximum Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 2</td>
</tr>
</tbody>
</table>
6.6  Average Ground Reaction Force Along Each Axis for Both Legs During Each Test SESSION 2. ................................. 190

6.7  Average and Maximum Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 3. . 194

6.8  Average and Maximum Heart Rate in Beats per Minute Gathered After Steady State was Achieved for Each Test SESSION 3. ................. 194

6.9  Average Ground Reaction Force Along Each Axis for Both Legs During Each Test SESSION 3. ................................. 199

A.1  Anatomical Movements Involving the Muscles of the Leg [2]. ........... 229

A.2  Anatomical Movements Involving the Muscles of the Forearm.[2] ...... 232
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Age Distribution of Major Limb Amputees in the USA in the Year 2005 by Etiology,[1]</td>
<td>3</td>
</tr>
<tr>
<td>2.1 Phase Portrait of the System. To the Left of the Vertical Axis at 0, the Hopper is Touching the Ground and the Leg Spring is Compressed. To the Right of the Vertical axis at 0, the Hopper is in the Flight Phase [3].</td>
<td>15</td>
</tr>
<tr>
<td>2.2 3D representation of Myoelectric control for Upper Limb Prostheses. Two Examples of Commercially Available Prostheses are Indicated in the Diagram as Well as Two Examples from Research [4].</td>
<td>19</td>
</tr>
<tr>
<td>3.1 Graphic Representation of the Definition of the Phase Angle with a Vectorial Representation of an External Stimulus. The Perturbation Affects the Phase of the Oscillatory System with the Component that is Tangential to the Phase Plot, $F_t$, and the Magnitude of the Oscillations Perturbed by $F_r$ [5].</td>
<td>33</td>
</tr>
<tr>
<td>3.2 Block Diagram of the System with the Phase Oscillator Controller.</td>
<td>33</td>
</tr>
<tr>
<td>3.3 Resulting Orbit for Eq. 3.16, with a Forcing Function Based on $\sin \phi + \cos \phi$. The Constants Used to Generate the Solutions were $\omega_n = 5$, $\zeta = 0.5$, $c = 35$, $d = -24$.</td>
<td>37</td>
</tr>
<tr>
<td>3.4 Resulting Orbit for Eq. 3.20 with a Forcing Function Based on $\sin \phi$. The Constants Used to Generate the Solutions were $\omega_n = 1$, $\zeta = 0.5$, $c = 5$.</td>
<td>38</td>
</tr>
<tr>
<td>3.5 Graphic Representation of the Definition of the Phase Angle in Polar Coordinates</td>
<td>45</td>
</tr>
</tbody>
</table>
3.6 Output of the System for the 3 Cases Using the Phase Based Oscillator and the Time Based One. The Output of the Nonlinear Phase Based Oscillator is Shown in Blue. The Output of the Time Based Oscillator is Shown in Red. The Constants Used to Generate the Solutions were $\omega_n = 5, \, \zeta = 0.5$.

3.7 Output of the System When Applying a Step Perturbation with 10 Seconds Duration. The Constants Used to Generate the Solutions Were $\omega_n = 5, \, \zeta = 0.5$. The Output of the Nonlinear Phase Based Oscillator is Shown in Blue. The Output of the Time Based Oscillator is Shown in Red.

3.8 Output of the Nonlinear Phase Based Oscillator with Gaussian Noise Added to the Values of $\dot{x}$, $\omega_n = 5, \, \zeta = 0.5$. The Top Plot Shows $\dot{x}$. The Plot on the Bottom Shows the Output $x$.

3.9 Results of Simulation Using the Sine of the Phase Angle as a Forcing Function. The First Plot Shows the Values of $c$ vs Time. The Second Plot Shows the Output of the Dynamic System. The Oscillation is Damped for the Negative Value of $c$ and the Amplitude is Different for Both Positive Values of $c$. The Last Plot has the Value of the Sine of the Phase Angle. The Constants Used to Generate the Solutions Were $\omega_n = 1, \, \zeta = 0.5$. 
3.10 Phase Portrait of the Oscillator Using $\sin \phi$ as a Forcing Function. In Blue are Two Solutions, one With Initial Conditions $y_1 = 3$, $y_2 = 0$ and the Second Case $y_1 = -6$, $y_2 = 0$. A Stable Limit Cycle is Shown in Bold Blue. The Start of the Trajectories is Marked With a Square and the end With a Diamond. The Constants Used to Generate the Solutions Were: $\omega_n = 1$, $\zeta = .5$, $c = 5$.

3.11 Results of Simulation Using the Cosine of the Phase Angle as a Forcing Function. The First Plot Shows the Values of $d$ vs Time. The Second Plot Shows the Output of the Dynamic System (Position). The Oscillation Decays and the Frequency Changes With $d$. The Last Plot Has the Value of the Cosine of the Phase Angle. The Constants Used to Generate the Solutions Were $\omega = \omega_n = 1$, $\zeta = 0.5$.

3.12 Phase Portrait of the Oscillator Using $\cos \phi$ as Forcing Function. In Blue are Shown Solutions for Initial Conditions $y_1 = 3$ and $y_1 = -6$; $y_2 = 0$. The System with the Cosine Function Does Not Generate a Limit Cycle. The Start of the Trajectories is Marked with a Square and the end with a Diamond. The Constants Used to Generate the Solutions Were $d = -5$, $\zeta = 0.5$, $\omega = \omega_n = 1$. 
3.13 Results of Simulation Using the Cosine and Sine of the Phase Angle as a Forcing Function. The First Plot Shows the Output of the Dynamic System (Position). The Second Plot Shows the Value of $c$ vs Time. The Third Plot Shows $\sin \phi$ vs Time. The Fourth Plot Shows the Value of $d$ vs Time; and the Last Plot Shows $\cos \phi$ vs Time. The Oscillation Decays for the Negative Value of $c$. The Constants Used to Generate the Solutions Were $\omega_n = 1$, $\zeta = 0.5$. ........................................... 58

3.14 Phase Portrait of the Oscillator Using $\sin \phi + \cos \phi$ as a Forcing Function. In Blue are Two Solutions, One With Initial Conditions $y_1 = 3$, $y_2 = 0$ and the Second Case $y_1 = -6$, $y_2 = 0$. A Stable Limit Cycle is Formed in Bold Blue. The Start of the Trajectories is Marked with a Square and the End with a Diamond. The Constants Used to Generate the Solutions Were $c = 5$, $d = -2$, $\zeta = 0.5$, $\omega_n = 1$. ..................... 59

3.15 System response with the two controllers. The first plot shows the response using the phase based oscillator controller; the second plot shows the comparison of the two systems; and the third plot shows the response using the PID controller.................................. 61

3.16 Close Up of the System Response with the Two Controllers. The First Plot Shows the Response Using the Phase Based Oscillator Controller; the Second Plot Shows the Comparison of the Two Systems; and the Third Plot Shows the Response Using the PID Controller. .......... 61

3.17 Simulation Output of the Phase Based Oscillator Adding Offset. The System Used for the Simulation Was $\omega_n = \sqrt{k/m}$, $\zeta = 0.111$, Using an Offset $x_d = 1$. .................................................. 62
4.1 Response of the Phase Based Oscillator During a Change of Amplitude
Maintaining the Frequency Constant at $\omega_n = 2 \text{ rad/s}$. The Plot on Top
Show the Response in the Phase Plane, in Red is Shown the First 40 s of
the Simulation Where the Values of $c$, and $d$ are Maintained Constant
to Obtain $A = 1.5\text{ rad}$, $\omega = 2\text{ rad/s}$. In Blue is Shown the Response
After the Values of $c$ and $d$ are Modified to Obtain Final Conditions
$A = 2\text{ rad}$, $\omega = 2\text{ rad/s}$. The Plot on the Bottom Shows the System
Response in the Position vs Time Plane. ................................. 65

4.2 Response of the Phase Based Oscillator when the Frequency is Changed
and the Amplitude is Maintained Constant at $A = 1.5 \text{ rad}$. The Plot on
Top Show the Response in the Phase Plane, in Red is Shown the First
50 s of the Simulation Where the Values of $c$, and $d$ are Maintained
Constant to Obtain $A = 1.5\text{ rad}$, $\omega = 0.7\text{ rad/s}$. In Blue is Shown
the Response After the Values of $c$ and $d$ are Modified to Generate
$A = 1.5\text{ rad}$, $\omega = 1.2\text{ rad/s}$. In Cyan is Shown the Response After the
Values of $c$ and $d$ are Modified to Obtain Final Conditions $A = 1.5\text{ rad}$,
$\omega = 1.7\text{ rad/s}$. The Plot on the Bottom Shows the System Response in
the Position vs Time Plane. .................................................. 66
4.3 Response of the Phase Based Oscillator for Three Different Conditions. The Plot on Top Show the Response in the Phase Plane, in Red is Shown the First 50 s of the Simulation Where the Values of \( c \), and \( d \) are Maintained Constant to Obtain \( A = 1.5\text{rad}, \omega = 0.8\text{rad/s} \). In Blue is Shown the Response After the Values of \( c \) and \( d \) are Modified to Generate \( A = 1\text{rad}, \omega = 1.5\text{rad/s} \). In Cyan is Shown the Response After the Values of \( c \) and \( d \) are Modified to Obtain Final Conditions \( A = 1.5\text{rad}, \omega = 2\text{rad/s} \). The Plot on the Bottom Shows the System Response in the Position vs Time Plane.

4.4 System Output \( \omega_M \) vs Time with Constant Values \( c = 1, d = 0, \) and \( \omega = 1 \). The Output Has a Frequency \( \omega \approx 69 \text{ rad/s} \).

4.5 Motor Output \( \omega_M \) vs Time with Constant Values \( c = 1, d = 0, \) and \( \omega = 69 \). The Output Has a Frequency \( \omega \approx 78 \text{ rad/s} \).

4.6 Motor Response for a Unitary Step Input \( \omega_M \) vs Time. The Overshoot is Approximately 1%.

4.7 Motor Angular Velocity \( \omega_M \) During a Test with \( c = 1.373, \) and \( d = -0.417 \). The Frequency of the Oscillations is \( \omega = 97 \text{ rad/s} \). The Amplitude of the Oscillations is \( A = 153 \text{ rad/s} \).

4.8 Motor Angular Acceleration \( \dot{\omega}_M \) During a Test with \( c = 1.373, \) and \( d = -0.417 \). The Frequency of the Oscillations is \( \omega = 97 \text{ rad/s} \). The Amplitude of the Oscillations is \( A = 1.6 \times 10^4 \text{ rad/s}^2 \).
4.9 System Output in the Phase Plane During a Test with $c = 1.373$, and $d = -0.417$. The Frequency of the Oscillations is $\omega = 97$ rad/s. The Amplitude of the Oscillations is $A = 153$ rad/s. The Trajectory Starts at $(0,0)$.

4.10 On Top, Magnitude of the Forcing Function Output vs Time During the Test with $c = 1.373$, and $d = -0.417$. On bottom, Magnitude of $c$ and $d$ vs Time.

4.11 Motor Angular Velocity $\omega_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.37$, $d_1 = -0.41$. The Final Values are $c_2 = 1.11$, $d_2 = -0.03$. The Initial Amplitude is $A_1 = 153$ rad/s, $A_2 = 148$ rad/s. The Initial Frequency is $\omega_1 = 103$ rad/s, and the Final Value is $\omega_2 = 81.6$ rad/s.

4.12 Motor Angular Acceleration $\dot{\omega}_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.37$, $d_1 = -0.41$. The Final Values are $c_2 = 1.11$, $d_2 = -0.03$. The Initial Amplitude is $A_1 = 1.6 \times 10^4$ rad/s, $A_2 = 1.25 \times 10^4$ rad/s. The Initial Frequency is $\omega_1 = 103$ rad/s, and the Final Value is $\omega_n = 81.6$ rad/s.

4.13 System Output in the Phase Plane During the Frequency Change Test. The Initial Values are $c_1 = 1.37$, $d_1 = -0.41$. The Final Values are $c_2 = 1.11$, $d_2 = -0.03$. The Initial Amplitude is $A_1 = 153$ rad/s, the Final Amplitude is $A_2 = 148$ rad/s. The Initial Frequency is $\omega_1 = 103$ rad/s, and the Final Value is $\omega_2 = 81.6$ rad/s.
4.14 On Top, Magnitude of the Forcing Function Output vs Time During the Test. On Bottom, Magnitude of $c$ and $d$ vs time. With $c_1 = 1.37$, $d_1 = -0.41$, $c_2 = 1.11$, and $d_2 = -0.03$.

4.15 Motor Angular Velocity $\omega_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.09$, $d_1 = -0.3$. The Final Values are $c_2 = 0.732$, $d_2 = -0.022$. The Initial Amplitude is $A_1 = 146$ rad/s, $A_2 = 96.5$ rad/s. The Initial Frequency is $\omega_1 = 80.55$ rad/s, and the Final Value is $\omega_n = 79.53$ rad/s.

4.16 Motor Angular Acceleration $\dot{\omega}_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.09$, $d_1 = -0.3$. The Final Values are $c_2 = 0.732$, $d_2 = -0.022$. The Initial Amplitude is $A_1 = 1.3 \times 10^4$ rad/s, $A_2 = 0.8 \times 10^4$ rad/s. The Initial Frequency is $\omega_1 = 80.55$ rad/s, and the Final Value is $\omega_n = 79.53$ rad/s.

4.17 System Output in the Phase Plane During the Amplitude Change Test. The Initial Values are $c_1 = 1.09$, $d_1 = -0.3$. The Final Values are $c_2 = 0.732$, $d_2 = -0.022$. The Initial Amplitude is $A_1 = 146$ rad/s, the Final Amplitude is $A_2 = 96.5$ rad/s. The Initial Frequency is $\omega_1 = 80.55$ rad/s, and the Final Value is $\omega_2 = 79.53$ rad/s.

4.18 On Top, Magnitude of the Forcing Function Output vs Time During the Test. On Bottom, Magnitude of $c$ and $d$ vs Time. With $c_1 = 1.09$, $d_1 = -0.3$, $c_2 = 0.732$, and $d_2 = -0.022$.

4.20 Step Test Response of the Pendulum $\theta$ vs Time. The Overshoot is Approximately 80%.

4.21 System Output $\theta$ vs Time, with Constant Values $c = 2$, $d = 0$, and $\omega = 4.88$ rad/s. The Output Has an Amplitude of $A = 1.33$ rad.

4.22 System Velocity $\dot{\theta}$ vs Time, with Constant Values $c = 2$, $d = 0$, and $\omega = 4.88$ rad/s.

4.23 Pendulum Position $\theta$ During a Test with $c = 1.5411$, and $d = -0.552$. The Frequency of the Oscillations is $\omega = 4.92$ rad/s. The Amplitude of the Oscillations is $A = 1.018$ rad.

4.24 Pendulum Velocity $\dot{\theta}$ During a Test with $c = 1.5411$, and $d = -0.552$.

4.25 System Output in the Phase Plane During a Test with $c = 1.5411$, and $d = -0.552$. The Frequency of the Oscillations is $\omega = 4.92$ rad/s. The Amplitude of the Oscillations is $A = 1.018$ rad. The trajectory Starts at $(0,0)$.

4.26 On Top, Magnitude of the Forcing Function Output vs Time During the Test with $c = 1.5411$, and $d = -0.552$. On Bottom, Magnitude of $c$ and $d$ vs Time.

4.27 Pendulum Angular Position $\theta$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.849$, $d_1 = -5.6744$. The Final Values are $c_2 = 1.5411$, $d_2 = -0.5521$. The Initial Amplitude is $A_1 = 1.07$ rad, $A_2 = 1.06$ rad. The Initial Frequency is $\omega_1 = 6.04$ rad/s, and the Final Value is $\omega_2 = 4.79$ rad/s.
<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.28</td>
<td>Pendulum Angular Velocity $\dot{\theta}$ vs Time for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.849$, $d_1 = -5.6744$. The Final Values are $c_2 = 1.5411$, $d_2 = -0.5521$. The Initial Amplitude is $A_1 = 1.07$ rad, $A_2 = 1.06$ rad. The Initial Frequency is $\omega_1 = 6.04$ rad/s, and the Final Value is $\omega_2 = 4.79$ rad/s.</td>
</tr>
<tr>
<td>4.29</td>
<td>System Output in the Phase Plane During the Frequency Change Test. The Initial Values are $c_1 = 1.849$, $d_1 = -5.6744$. The Final Values are $c_2 = 1.5411$, $d_2 = -0.5521$. The Initial Amplitude is $A_1 = 1.07$ rad, $A_2 = 1.06$ rad. The Initial Frequency is $\omega_1 = 6.04$ rad/s, and the Final Value is $\omega_2 = 4.79$ rad/s.</td>
</tr>
<tr>
<td>4.30</td>
<td>On Top, Magnitude of the Forcing Function Output vs Time During the Test. On Bottom, Magnitude of $c$ and $d$ vs Time. With $c_1 = 1.849$, $d_1 = -5.6744$, $c_2 = 1.5411$, and $d_2 = -0.5521$</td>
</tr>
<tr>
<td>4.31</td>
<td>Pendulum Angular Position $\theta$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.84$, $d_1 = -0.6625$. The Final Values are $c_2 = 1.23$, $d_2 = -0.4416$. The Initial Amplitude is $A_1 = 1.31$ rad, $A_2 = 0.69$ rad. The Initial Frequency is $\omega_1 = 4.8$ rad/s, and the Final Value is $\omega_2 = 4.9$ rad/s.</td>
</tr>
<tr>
<td>4.32</td>
<td>Pendulum Angular Velocity $\dot{\theta}$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.84$, $d_1 = -0.6625$. The Final Values are $c_2 = 1.23$, $d_2 = -0.4416$. The Initial Amplitude is $A_1 = 1.31$ rad, $A_2 = 0.69$ rad. The Initial Frequency is $\omega_1 = 4.8$ rad/s, and the Final Value is $\omega_2 = 4.9$ rad/s.</td>
</tr>
</tbody>
</table>
4.33 System Output in the Phase Plane. The Initial Values are \( c_1 = 1.84, \)
\( d_1 = -0.6625 \). The Final Values are \( c_2 = 1.23, d_2 = -0.4416 \). The Ini-
tial Amplitude is \( A_1 = 1.31 \text{ rad}, A_2 = 0.69 \text{ rad} \). The Initial Frequency
is \( \omega_1 = 4.8 \text{ rad/s}, \) and the Final Value is \( \omega_2 = 4.9 \text{ rad/s} \). .......... 95

4.34 On Top, Magnitude of the Forcing Function Output vs Time During
the Test. On Bottom, Magnitude of \( c \) and \( d \) vs time. With \( c_1 = 1.84, \)
\( d_1 = -0.6625, c_2 = 1.23, \) and \( d_2 = -0.4416 \) ......................... 96

4.35 Shorter Recovery Time for the Exponential Decay of the Time Based
Oscillator and the Phase Based Oscillator. When the Time Was Smaller
for the Time Based Oscillator it is Shown as Negative. The Values Used
to Generate the Plot Were: \( \zeta = 0.06782, \omega_n = 4.88 \text{ rad/s}, \alpha = 0.01. \) ... 103

4.36 System Response for Initial Conditions Not on the Limit Cycle. In
This Case, the Initial Conditions are \( \theta_0 = -0.1 \text{ rad}, \dot{\theta}_0 = 0 \text{ rad/s}. \) In
Red, the Response Using the Phase Oscillator Controller and in Blue the Response Using the Equivalent Time Based Forcing Function. The
Plot on Top Shows the Response in the Phase Plane. The Plot in the
Middle Shows the Angular Position vs Time. The Plot at the Bottom Shows Power in the System \( P = \tau \omega. \) ......................... 105
4.37 System Response for Initial Conditions Not on the Limit Cycle. In This Case, the Initial Conditions are $\theta_0 = 3 \text{ rad}$, $\dot{\theta}_0 = 0 \text{ rad/s}$. In Red, the Response Using the Phase Oscillator Controller and in Blue the Response Using the Equivalent Time Based Forcing Function. The Plot on Top Shows the Response in the Phase Plane. The Plot in the Middle Shows the Angular Position vs Time. The Plot at the Bottom Shows Power in the System $P = \tau \omega$. ................................. 106

4.38 System Response for Initial Conditions Not on the Limit Cycle. In This Case, the Initial Conditions are $\theta_0 = 0 \text{ rad}$, $\dot{\theta}_0 = 4 \text{ rad/s}$. In Red, the Response Using the Phase Oscillator Controller and in Blue the Response Using the Equivalent Time Based Forcing Function. The Plot on Top Shows the Response in the Phase Plane. The Plot in the Middle Shows the Angular Position vs Time. The Plot at the Bottom Shows Power in the System $P = \tau \omega$. ................................. 107

4.39 System Response for Initial Conditions Not on the Limit Cycle. In This Case, the Initial Conditions are $\theta_0 = 0 \text{ rad}$, $\dot{\theta}_0 = 2 \text{ rad/s}$. In Red, the Response Using the Phase Oscillator Controller and in Blue the Response Using the Equivalent Time Based Forcing Function. The Plot on Top Shows the Response on the Phase Plane. The Plot in the Middle Shows the Angular Position vs Time. The Plot at the Bottom Shows Power in the System $P = \tau \omega$. ................................. 108

4.40 Pendulum Angular Position $\theta$ vs Time Using the Phase Based Forcing Function. $c = 1.849$, $d = -5.6744$, $\omega = 6$. A External Force is Introduced at $t = 21 \text{ s}$, and $t = 40 \text{ s}$. ................................. 110
4.41 Pendulum Angular Position $\theta$ vs Time Using the Time Based Forcing Function. A External Force is Introduced at $t = 20$ s, and $t = 40$ s. 

4.42 Pendulum Angular Position $\theta$ vs Time Using the Phase Based Forcing Function. $c = 1.849$, $d = -5.6744$, $\omega = 6$. Trajectory Perturbed from $t = 17s$ to $t = 20s$. 

4.43 Pendulum Angular Position $\theta$ vs Time Using the Time Based Forcing Function. Trajectory Perturbed from $t = 17s$ to $t = 19s$. 

5.1 Bound for $E \left\{ h_1^2(t) + 2h_1(t)(\omega_n^2 - \omega^2) + h_2^2(t)[\omega^2 + 2\omega^2\sqrt{c^2 + d^2}] + \omega^2h_2(t) \right\}$ that Guarantee the Stability of the System. $\omega_n = 4.88$ rad/s, $\zeta = 0.06782$. 

5.2 System Response in the Phase Plane. In Red the System Without Noise and Without Non Linearities. In Blue the System with Added Noise $\Delta u$ and Non Linearity $f(x, t)$. In Green the Response of the System with Noise $\Delta u$ and Non Linearity $f(x, t)$ and the Controller $\nu^*$. 

5.3 Close View of the System Response in the Phase Plane. In Red the System Without Noise and Without Non Linearities. In Blue the System with Added Noise $\Delta u$ and Non Linearity $f(x, t)$. In Green the Response of the System with Noise $\Delta u$ and Non Linearity $f(x, t)$ and the Controller $\nu^*$. 

5.4 System Response in the Phase Plane Without Controller $\nu^*$. The Trajectory Starts at $(0, 0)$. The System was Run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5$ rad/s.
5.5  System Response in the Phase Plane with Controller $\nu^*$. The Trajectory Starts at $(0,0)$. The System was Run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5 \text{ rad/s}$. ................................. 130

5.6  Close View of the System Response in the Phase Plane Without Controller $\nu^*$. The Trajectory Starts at $(0,0)$. The System was run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5 \text{ rad/s}$. ................................. 131

5.7  Close View of the System Response in the Phase Plane with Controller $\nu^*$. The Trajectory Starts at $(0,0)$. The System was Run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5 \text{ rad/s}$. ................................. 131

5.8  Nonlinear Model Response in the Phase Plane. In Green, the System with $\Delta = 0$ and $\nu^* = 0$. In Blue, the System with Added Noise $\Delta u$. In Red, the Response of the System with Noise $\Delta u$ and the Controller $\nu^*$. $c = 1.849$, $d = -0.6625$, $\omega = 5$, $\omega_n = 4.88$, and $\zeta = 0.0678$. ............... 133

5.9  Close View of the Nonlinear Model Response in the Phase Plane. In green, the System with $\Delta = 0$ and $\nu^* = 0$. In Blue, the System with Added Noise $\Delta u$. In red, the Response of the System with Noise $\Delta u$ and the Controller $\nu^*$. $c = 1.849$, $d = -0.6625$, $\omega = 5$, $\omega_n = 4.88$, and $\zeta = 0.0678$. .................................................. 133
5.10 Simplified Nonlinear Model Response in the Phase Plane. In green, the System with $\Delta = 0$ and $\nu^* = 0$. In Blue, the System with Added Noise $\Delta u$. In Red, the Response of the System with Noise $\Delta u$ and the Controller $\nu^*$. $c = 1.849$, $d = -0.6625$, $\omega = 5$, $\omega_n = 4.88$, and $\zeta = 0.0678$.

5.11 Close View of the Simplified Nonlinear Model Response in the Phase Plane. In green, the System with $\Delta = 0$ and $\nu^* = 0$. In Blue, the System with Added Noise $\Delta u$. In Red, the Response of the System with Noise $\Delta u$ and the Controller $\nu^*$. $c = 1.849$, $d = -0.6625$, $\omega = 5$, $\omega_n = 4.88$, and $\zeta = 0.0678$.

5.12 Block Diagram of a Feedback System for the Small Gain Theorem.

5.13 Block Diagram of the Model of Additive Uncertainty.

5.14 Block Diagram of the Model of Feedback Uncertainty.

5.15 Block Diagram of the Model of Multiplicative Uncertainty.

5.16 Block Diagram of the Model of Divisive Uncertainty.

5.17 DC Gain of the Nonlinear Model of the Pendulum vs Amplitude and Frequency of the Input. The Parameters Used for the Simulation were: $\zeta = 0.5$, $\omega_n = 4.66\text{ rad/s}$.

6.1 Two Limit Cycles are Shown, One With $\omega x$ on the Horizontal Axis and the Second One with $\tilde{\omega} x$ on the Horizontal Axis. The Systems Match when $x = 0$.

6.2 Graphic Representation of the Error Caused by the Difference Between $\omega$ and $\tilde{\omega}$. The Error is Defined at $\psi$, the System’s Phase Angle is $\phi$ and the Approximated Phase Angle is $\tilde{\phi}$.
6.3 The Position $x = 2 \sin(2t) + \cos(4t+1)$ is Shown in Black. The Velocity $\dot{x} = 4 \cos(2t) - \sin(4t+1)$ is Shown in Dotted Black Line. The Sine of the System was Plotted Using $\sin \phi = \frac{\dot{x}}{\sqrt{x^2 + \omega^2 x^2}}$. The Blue Line Shows the Output for $\omega = \omega$ ($\tilde{\omega} = 2$ rad/s). The Red Line Shows the Output for $\omega = 0.5\omega$ ($\tilde{\omega} = 1$ rad/s). The Green Line Shows the Output for $\omega = 2\omega$ ($\tilde{\omega} = 4$ rad/s). ........................................... 151

6.4 The Position $x = 2 \sin(2t) + \cos(4t+1)$ is Shown in Black. The Velocity $\dot{x} = 4 \cos(2t) - \sin(4t+1)$ is Shown in Dotted Black Line. The Cosine of the System was Plotted Using $\cos \phi = \frac{x}{\sqrt{x^2 + \omega^2 x^2}}$. The Blue Line Shows the Output for $\omega = \omega$ ($\tilde{\omega} = 2$ rad/s). The Red Line Shows the Output for $\omega = 0.5\omega$ ($\tilde{\omega} = 1$ rad/s). The Green Line Shows the Output for $\omega = 2\omega$ ($\tilde{\omega} = 4$ rad/s). ........................................... 152

6.5 On Left, Plot of the Position vs Time. In Blue $x = 2 \sin(2t) + \cos(4t+1)$ (No Offset). In Red $x = 2 + 2 \sin(2t) + \cos(4t + 1)$. Right, Plot of the Limit Cycle. In Blue the Limit Cycle with No Offset, and in Red the Limit Cycle With an Offset. ........................................... 153

6.6 Sine of the Phase Angle $\sin \phi = \frac{\dot{x}^2}{\sqrt{x^2 + \omega^2 x^2}}$. In blue, $\sin \phi$ is Shown Calculated with No Offset in the Position Signal, $x = 2 \sin(2t) + \cos(4t+1)$. In Red, $\sin \phi$ is Shown Calculated with Offset in the Position Signal $x = 2 + 2 \sin(2t) + \cos(4t + 1)$. In Solid Black is Shown the Position and in Dotted Black is Shown the Velocity. ........................................... 153
6.7 Cosine of the Phase Angle \( \cos \phi = \frac{\omega x}{\sqrt{x^2 + \omega^2 x^2}} \). In Blue, \( \cos \phi \) is Shown Calculated with No Offset in the Position Signal, \( x = 2 \sin(2t) + \cos(4t + 1) \). In red, \( \sin \phi \) is Shown Calculated with Offset in the Position Signal \( x = 2 + 2 \sin(2t) + \cos(4t + 1) \). In Solid Black is Shown the Position and in Dotted Black is Shown the Velocity. ......................... 154

6.8 Graphic Representation of the Error Caused by the Difference in Offset in \( x \). The Error is Defined at \( \psi \), the System’s Phase Angle is \( \phi \) and the Phase Angle with the Error is \( \tilde{\phi} \). ......................................................... 155

6.9 Inverse Tangent. ................................................................. 156

6.10 On Left, Joint Trajectories of the Hip, Knee and Ankle During Normal Gait Cycle in the Sagittal Plane. On right, Sagittal Plane Internal Joint Moments (Nm/Kg) During a Single Gait Cycle of Right Hip (Extensor Moment Positive), Knee (Extensor Moment Positive), and Ankle (Plantarflexor Moment Positive) [6]. ........................................... 157

6.11 Truncated Fourier Series Representation of the Knee Trajectory in the Sagittal Plane During Normal Gait. Angle vs Time. \( \theta(t) = 0.266 + 0.2722 \cos(2\pi t + 1.7787) + 0.2181 \cos(4\pi t - 2.29) + 0.0483 \sin(6\pi t - 1.39181) \). 159

6.12 Gait Cycle Phases Shown. 1 Initial Contact. 2 Opposite Toe Off. 3 Heel Rise. 4 Toe Off. 5 Tibia Vertical. Diagram Modified From [6] .... 159

6.13 Limit Cycle of the Forth Order Fourier Series Representation of the Knee Trajectory in the Sagittal Plane During Normal Gait. Degrees vs Degrees/s ......................................................... 160

xxvi
6.14 Representation Over Time of the Knee Torque, Shown in Solid Blue; the Power, Shown in Dotted Blue; the Angle $\theta$ in Solid Black, and the Angular Velocity $\dot{\theta}$ in Dotted Black. Torque and Power Scale on the Left. Angle and Angular Velocity Scale on the Right. 

6.15 On Top, Plot of the Simulated torque Exerted by the Human $E_{human}$ at the Knee During a Gait Cycle in the Sagittal Plane. On Bottom, Plot of the Power Required From the Human to Flex the Knee $P_{human}$. On Both, the Black Line Shows the Values with No Assistance Provided, in Blue When $c_a = d_a = 3.1651$, in Red $c_a = 4.4763$, $d_a = 0$, and in Green $c_a = 0$, $d_a = 4.4763$. 

6.16 Results of the Simulation Assistance of the Knee Motion Using the Forcing Function $E_{assist} = \frac{c_a \ddot{x} + d_a \dddot{x}}{ \sqrt{\dot{x}^2 + \ddot{x}^2}}$. On Top, Torque and Power Using $c_a = d_a = 3.1651$, in the Middle $c_a = 4.4763$, $d_a = 0$, on Bottom $c_a = 0$, $d_a = 4.4763$. Each Case was Simulated Using $\ddot{\omega} = \omega$, $\dddot{\omega} = 0.5\omega$, and $\dddot{\omega} = 2\omega$, Shown in Blue, Red and Green. 

6.17 Truncated Fourier Series Representation of the Hip Trajectory in the Sagittal Plane During Normal Gait. Angle vs Time. $\theta(t) = 0.12268 + 0.3675 \sin(2\pi t + 2.03) + 0.04 \sin(4\pi t + 4.76) + 0.02 \sin(6\pi t + 0.24)$. 

6.18 Gait Cycle Phases Shown for the Dark Leg. 1 Initial Contact. 2 Heel Rise. 3 Toe Off. 4 Feet Adjacent. Diagram Modified from [6]. 

6.20 Representation Over Time of the Hip Torque (Nm/Kg), Shown in Solid Blue; the Power (W/Kg), Shown in Dotted Blue; the Angle $\theta$ (rad) in Solid Black and the Angular Velocity $\dot{\theta}$ (rad/s) in Dotted Black. Torque and Power Scale on the Left. Angle and Angular Velocity Scale on the Right.  ................................................................. 168

6.21 On Top, Plot of the Simulated Torque Exerted by the Human $E_{human}$ on the Hip During a Gait Cycle in the Sagittal Plane. On Bottom, Plot of the Power Required From the Human to Flex the Hip $P_{human}$. On Both, the Black Line Shows the Values with No Assistance Provided, in Blue When $c_a = d_a = 8.1305$ Nm, in Red $c_a = 2.9764$ Nm, $d_a = 11.1081$ Nm, and in Green $c_a = 11.1081$ Nm, $d_a = 2.9764$ Nm.  ........................................ 169

6.22 Results of the Simulation Assistance of the Hip Motion Using the Forcing Function $E_{assist} = \frac{c_a \dot{x} + d_a \ddot{x}}{\sqrt{x^2 + \omega^2 \dot{x}^2}}$. On Top, Torque and Power Using $c_a = d_a = 8.1305$ Nm, in the Middle $c_a = 2.9764$ Nm, $d_a = 11.1081$ Nm, on Bottom $c_a = 11.1081$ Nm, $d_a = 2.9764$ Nm. Each Case was Simulated Using $\ddot{\omega} = \omega$, $\tilde{\omega} = 0.5\omega$, and $\tilde{\omega} = 2\omega$, Shown in Blue, Red and Green.  ................................................................. 170

6.23 Linear Frequency Sweep Simulation, In Black is Shown the Torque $E_{human}$ When No Assistance is Provided, in Blue for $c_a = 2.9764$ Nm, $d_a = 11.1081$ Nm; and in Red for $c_a = 11.1081$ Nm, $d_a = 2.9764$ Nm. The Initial Frequency is $\omega_0 = 3.76$ rad/s, and the Final Frequency is $\omega_f = 18.84$ rad/s.  ................................................................. 171

6.24 Block Diagram of the Proposed Model to Represent How the Phase Based Oscillator Can Be Used to Assist the Human.  ........................................ 172
6.25 Simulated Hip Torque $E_{human}$ During Normal Gait in the Sagittal Plane. In Black it is Shown the Torque When No Assistance is Provided. In Blue is Shown the Torque When Assistance is Provided. ......173

6.26 Simulated Hip Power $P_{human}$ During Normal Gait. In Black the Torque When No Assistance is Used, and in Blue with Assistance. $(\tau \times \dot{\theta})$ ......173

6.27 Magnitude of $c$ and $d$ Over One Cycle for the Truncated Fourier Series Representation of the Hip Trajectory.................................175

6.28 On Top, Plot of the Simulated Torque Exerted by the Human $E_{human}$ on the Hip During a Gait Cycle in the Sagittal Plane. On Bottom, Plot of the Power Required From the Human to Flex the Hip $P_{human}$. The Simulation was Done With $c_a = 0.2c$, $d_a = 0.2d$ from Equation 6.50, and for the Constant Case $c_a = 1.45188$, $d = 4.10974$ .............176

6.29 HESA Hip Robot Developed at the Human-Machine Integration Laboratory. .................................................................177

6.30 Test Subject Walking on Instrumented Treadmill Wearing the HESA Robot During One of the Tests. .....................................................179

6.31 Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved. The Sampling Frequency was 4 Hz. SESSION 1 181

6.32 Average (Blue) and Maximum (Red) Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 1. .........................................................183

6.33 Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right Leg. SESSION 1. 185
6.34 Average (Red) and Maximum (Green) Oxygen Consumption \((VO_2)\) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test.
SESSION 2. ................................................................. 188

6.35 Average (Blue) and Maximum (Red) Oxygen Consumption \((VO_2)\) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test.
SESSION 2. ................................................................. 189

6.36 Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Red, Left Leg. In Green, Right Leg. SESSION 2. ......... 191

6.37 Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right leg. SESSION 2. .......... 192

6.38 Average (Blue) and Maximum (Orange) Oxygen Consumption \((VO_2)\) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test. SESSION 3. ................................................................. 195

6.39 Average (Blue) and Maximum (Orange) Oxygen Consumption \((VO_2)\) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test. SESSION 3. ................................................................. 196

6.40 Average (Blue) and Maximum (Orange) Heart Rate in Beats per Minute Gathered After Steady State was Achieved for Each Test. SESSION 3. 197

6.41 Average (Blue) and Maximum (Orange) Heart Rate in Beats per Minute Gathered After Steady State was Achieved for Each Test. SESSION 3. 198

6.42 Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right Leg. SESSION 3. ............ 200

6.43 Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right Leg. SESSION 3. ............ 201

xxx
A.1 Diagram of Hill’s Functional Model of the Skeletal Muscles. ............... 221
A.2 Force-Velocity Relation for Four Activation During the Shortening of the Muscles. This Plot Follows the Hill Model [8]. ......................... 223
A.3 Force-Velocity-Length Relationship in the Muscular Activity. The Resultant Force Depends also on the Length of the Muscle; it also Depends on the Cross Sectional Area Given that the Volume is Constant.
[8] ................................................................. 223
A.4 Relationship Between the Ankle Joint Angle and the Physiological Cross Sectional Area (PCSA) of the Gastrocnemius Medialis When the Muscle is at Rest [9]. ............................... 224
A.5 Schematic of the Functioning Principle of the Force Myography. Transverse Section of Right Leg Showing that the Resultant Forces on the Surface is Affected by the Internal Pressures in Each Muscle. .............. 224
A.6 FSR Mounting System Showing the Acrylic Plate Used as the Main Support and the Pad Used to Distribute the Load on the Sensor Area. 225
A.7 Picture of the Location of the Four FSRs on the Proximal Forearm, and the Elastic Armband Used for the Tests. FSR4 is on the Back of the Forearm in this View. ................................. 226
A.8 Signal Flow Diagram for the System that Uses the Fours FSRs in a Full Wheatstone Bridge Configuration. ............................... 226
A.9 Full Wheatstone Bridge Using FSRs. ........................................ 226
A.10 Voltage Divider Used in for Every FSR in the Decoding System. ....... 227
A.11 Signal Flow in the System that Uses 4 FSRs Independently. ............ 228
A.12 Muscles of the Leg [2]. .................................................. 229
<table>
<thead>
<tr>
<th>Number</th>
<th>Figure Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.13</td>
<td>HERMES Ankle from SpringActive Inc.</td>
<td>231</td>
</tr>
<tr>
<td>A.14</td>
<td>Block Diagram of the FMG System to Control a Robotic Ankle.</td>
<td>231</td>
</tr>
<tr>
<td>A.16</td>
<td>Supine Wrist</td>
<td>234</td>
</tr>
<tr>
<td>A.17</td>
<td>Neutral Wrist</td>
<td>234</td>
</tr>
<tr>
<td>A.18</td>
<td>Output of Bridge During Calibration and Test.</td>
<td>234</td>
</tr>
<tr>
<td>A.19</td>
<td>Output of Bridge During Calibration and Test with Wrist Rotation. The Plot Show Four Cycles Opening and Closing the Hand, then the Wrist is Rotated and the Hand is Opened and Closed. It can Be Seen that the Range of Operation Changes When the Wrist Rotates.</td>
<td>235</td>
</tr>
<tr>
<td>A.20</td>
<td>On Top, Results of a Tracking Test Using a Sinusoidal Reference. The Reference is Shown in Blue, the Output from the Wheatstone Bridge Signal in Green. The Correlation Reference-Output was $R = 0.8764$. On the Bottom, the Results for the Tracking Test Using a Square Signal Reference. The Correlation Reference-Output was $R = 0.74428$. Both Test Were Done With the Wrist on Neutral Position.</td>
<td>236</td>
</tr>
<tr>
<td>A.21</td>
<td>Virtual Hand Controlled Using the Four FSR’s in Full Wheatstone Bridge Configuration Shown in Four Different States.</td>
<td>237</td>
</tr>
<tr>
<td>A.22</td>
<td>Hand Positions for Classification Using KNN. The Numbers on the Upper Left Corner in Each Picture are the State Number Assigned in the Algorithm.</td>
<td>237</td>
</tr>
</tbody>
</table>
A.23 Data from FSRs During a Test Where the Hand was Positioned in Each of the 12 Defined States. In Light Green is the Value of the State vs Time. ......................................................... 238

A.24 Off-line Percentage Accuracy for the Same Data Sample Using the KNN Algorithm to Classify the State of the Hand vs Number of K Neighbors Used in the Test. ............................................ 239

A.25 Confusion Matrix of One of the Test Done with KNN Algorithm to Classify the 12 Hand States Using Independent FSRs Readings Using K=1. Accuracy = 96 %. ................................................. 239
Chapter 1

INTRODUCTION

Being independent is important for the humans well being. The loss of motor abilities like walking and grasping highly diminish the independence of a person. Robots can be used to restore some of the lost autonomy. In the case of amputees, the use of a prosthetic device can restore much of the abilities that were lost [11]. Although, in recent years the technology in prostheses and orthoses has developed greatly, there is still much more to do. Currently, most of the people that have lost a limb, commonly use a non smart passive prosthetic device; in the case of hand amputees they use a mechanical, body powered one; and foot amputees just use a support prosthetic device. Robotic devices can improve this situation, and it has been shown that prostheses that use microcontrollers improve significantly the abilities to perform common activities [11]. Closely related is the assistance of motion to the human to improve their performance or make tasks easier with the use of exoskeletons and wearable robotics. These devices allow healthy subjects, or people with diminished capabilities, to improve their performance of physical activities. The signals used to control them are the same ones that are used to control prosthetic devices. The research in this area is focusing on making it easier for amputees to perform basic activities such as descending stairs, walking on uneven surfaces, holding an object, press buttons, etc; to improve the quality of life of the subject, and to improve the performance of healthy people. Control and sensors are major research opportunities in the medical robotics field. It is required to improve the control of these devices and the human-machine interface. The mechanical and electronic design is helping to create exoskeletons, prostheses, and orthoses with multiple degrees of freedom that
potentially could provide a more natural and intuitive performance, but the proper use of these devices is still limited by the control algorithms.

1.1 Amputees in the USA

The target of amputations, regardless the cause, is to remove tissue that has been affected by a disease [12]. Amputation above and below the elbow; and above and below the knee are defined as major limb loss. The number of people with missing limbs in the country is growing each year. According to [1], the number of persons living without a limb in 2005 was close to 1.6 million. This number is approximately 0.5% of the population estimated to be close to 300 million that year [13], that means one in 190 Americans lives without a limb. The number of amputees is projected to be 3.6 million by 2050 [1]. Currently, 185,000 patients undergo a major limb amputation yearly [1].

The lack of a limb can be classified in these different etiologies: cancer, trauma, dysvascular disease and other; where the other category represents less than 3 % of the hospitalizations due to amputation, and includes amputations due to complications of procedures, internal derangement of joints, birth defects, etc[1]. From the total amputations, 38 % are related to dysvascular disease with diabetes mellitus and 45 % due trauma. An important amount of trauma related amputations (66 %) occurs in individuals younger than 45 years old [1]. Among the US service personnel, 76% of the amputation cases were major limb loss due to cancer or trauma [14]. After the age of 45, the distribution changes, the amputations related to dysvascular disease increase significantly because of the aging of the population and the increase of diabetes mellitus [1]. A plot with the distribution in the USA population in the year 2005, by age ranges is shown in figure 1.1. Additionally, people that undergo an amputation related to vascular disease are in high risk of needing a re-amputation. Of the lower
limb amputees with this kind of disease, 26% re-ingress to a hospital for a subsequent amputation in less than 12 months [1, 12], and more than 33% die in that period [12]. Another important topic is the preservation of limb length because a longer residual limb produces better ambulatory function [12]. In the case of lower limb amputation, the trans-femoral and trans-tibial levels induce fewer re-amputations (better healing) than the foot and ankle [12].

A further problem that amputees suffer, is pain related to the procedure that can be residual limb pain and phantom pain. Phantom pain occurs within nearly 85% of the amputees population, while residual limb pain close to 76% on long standing amputees [15]. Also, the intensity of the pain can be related to the emotional state of the patient, while depressive symptoms indicate more pain [15]. This issue can be reduced using prostheses that provide feedback to the user as shown in [16], and

Figure 1.1: Age Distribution of Major Limb Amputees in the USA in the Year 2005 by Etiology.[1]
Table 1.1: Projected Number of Persons with Major Limb Loss in 2050 [1].

<table>
<thead>
<tr>
<th>Etiology (cause)</th>
<th>Projected number</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>3,627,000</td>
<td>100.00%</td>
</tr>
<tr>
<td>Cancer</td>
<td>29,000</td>
<td>0.79%</td>
</tr>
<tr>
<td>Trauma</td>
<td>1,326,000</td>
<td>36.55%</td>
</tr>
<tr>
<td>Dysvascular disease without diabetes</td>
<td>605,000</td>
<td>16.68%</td>
</tr>
<tr>
<td>Dysvascular disease with diabetes</td>
<td>1,667,000</td>
<td>45.96%</td>
</tr>
</tbody>
</table>

providing the amputees with prostheses that increase their quality of life.

Table 1.1 shows the projected numbers of people that will be living in the USA in 2050 with a major limb loss. As it can be seen, the numbers are very significant and although not every amputee will require a prosthetic device, it is important to perform research in the field to improve the quality of life of this fragment of the American society.

1.2 Wearable Robotics for Motion Assistance

Wearable robots are machine that interact directly with the human, designed with the function of the human body in mind, to couple and transmit energy and information between the person and the robots working in synchrony [17, 18]. The function of these devices is to amplify the capabilities of the human[19], assist impaired people [18], or use as controllers for teleoperation [17]. When used for motion assistance, the wearable robots should reduce the amount of energy a human spends to do a task by allowing the individual to do it faster and/or for prolonged time. This process involves the generation of trajectories for the joints of the robots. The trajectories are position, velocity, and acceleration of the joint over time, and are important to properly assist the motions of the human. This has become a research opportunity
since there is not yet an optimal algorithm to do it. The trajectory affects the subjects comfort, safety and performance [18]. Although, off line trajectory planning has been used in exoskeletons, on-line trajectory control is needed to operate these robots in unknown environments.

There is a important number of industries where wearable robots for motion assistance can be used to make soldiers run or walk faster for longer periods, and to help workers with highly physically demanding jobs (nurses, manufacturing plant operators, merchandise handlers). The robots can be used by people to do heavy tasks increasing the amplitude and/or frequency of the movements, or just providing energy in synchrony with the individual. To work with the human, a good human machine interface is needed [20]. The quality of the flow of information and energy depends on the interface. This is closely related to the prostheses control because both require the reading of signals from the human to convert them to movement. Currently the most used signal is electromyography, but this technique has some disadvantages such as change in characteristics when the user sweats, and requires precise electrode location. New approaches have shown promising results like the use of arbitrary mapping functions with EMG reported in [21], where the human can quickly learn to control an external device. However it is important to explore alternative solutions like force myography.

1.3 Wearable Robots Requirements

Prostheses and exoskeletons are wearable robots with different functions, however they have important similarities in the sensors and control systems that are required for proper operation. Robotic prostheses are the result of the search to provide amputees more independence and try to restore more of the functions of the lost limb, and definitively are going to be the most used kind of prostheses in the future.
Passive prosthesis can not restore the full functionality of the lost limb [14]. The mechanical design of robotic devices has improved tremendously in the last decade, and will continue to do so. However, the capacity of the current human-machine interfaces still needs improvements to allow the amputee to use these prosthetics according to their full capabilities.

Wearable robots are machines that should be, reliable, functional, operational, and cost effective. Moreover, since they become part of the person, they have to be easy to use, intuitive, comfortable, and have a good appearance. In the case of prosthetic devices, these features affect patient’s quality of life. According to [22], the most important attributes for the users are cost, durability, comfort and appearance, but if a prosthesis has these characteristics and is not capable of demonstrably enhancing the patients life it will not be useful. To determine that a given control method works, tests to measure the times to complete a defined task, or to determine the number of successful attempts of a task for a completion metric.

With this in mind, it is easy to see that there are still a lot of research opportunities in the wearable robotics field, and how important it is given it can have a large impact to society. Better exoskeletons can improve productivity in industry, and better protheses and orthoses can improve the quality of life for the users.

1.4 Text Organization

The goals of this dissertation are to study a new method to control wearable robots based on a phase based oscillator to assist human periodic motion and to study force myography to acquire signals from the muscle activity and transform them to commands for external machines. This document is organized as follows:

Chapter 2. Related work. This section covers the background research in the different ways to control oscillatory robots to assist the human and to control robotic
prostheses. It also covers the state of the art of the methods to acquire signals from
the human to be used for motion control. The methods covered are electromyography,
electroencephalography, mechanomyography, sonography, myokinemetric, and force-
myography. For each method research papers were reviewed to analyze how they are
being used.

Chapter 3. Phase based oscillator: controlling oscillations with a phase based
forcing function. This chapter covers the theory of the phase based oscillator; shows
the existence and stability of the limit cycle, the solution using Fourier expansion of
periodic signals, and the analysis of the effect of the proposed phase based forcing
function on the system.

Chapter 4. Simulations and experimental validation of the phase based oscillator
controller. The phase based oscillator controller was used to oscillate the angular
velocity of a DC motor and the angular position of a pendulum. Data from simulations
and laboratory test is shown in this chapter for three cases: change of frequency,
change of amplitude, and perturbation response. Also, the transitory response of
the system was analyzed, simulated, and tested after a perturbation using the phase
based oscillator and the time based oscillator.

Chapter 5. Analysis of uncertainty and perturbations on the phase based os-
cillators. First, stability bounds are found for the phased based oscillator adding
perturbation modeled as stochastic process. Then, the Lyapunov redesign method is
used to find a controller to deal with noise and uncertainty in the system. Lastly, the
small gain theorem is used to evaluate the stability of the system. The methods show
the system is stable and bounds can be found where the system is guaranteed to be
stable even with uncertainty/perturbations present.

Chapter 6. Assisting an arbitrary limit cycle using the phase based forcing func-
tion. Analysis of the error in the phase based forcing function when used to assist
periodic motion. Model of the interaction human-assisting device. Testing using a wearable robot assisting the motion of the hip with the non linear phase based forcing function.

Chapter 7: Conclusions. This section summarizes the work and importance of this dissertation. It list the contributions, and covers recommendations for future research.

Appendix A. Force myography to control robotic devices. This section explains the principles of force-myography, the type of signal that is generated and the sensors required. Use of force myography to control robotic devices. The first section covers the use of force myography to control a robot ankle. The second section is on the use of force myography from the forearm to control robotic hands.

1.5 Contributions of this Dissertation

• Phase based oscillator.
  Show relationship with a time based system.
  Prove the existence of stable limit cycle using a Lyapunov function.
  Prove stability with polar coordinates.

• Solution using the Fourier series representation of a periodic trajectory to calculate the best values of $c$ and $d$.

• Analysis of the behavior of the Phase Based Oscillator.
  with $c$ constant, $d = 0$.
  with $d$ constant, $c = 0$.
  with $c$ constant, $d$ constant.

• Phase based oscillator implementation and testing.
Oscillating the angular velocity of a DC motor.

Oscillating a physical pendulum.

- Analysis and comparison of the transition after perturbation of the phase based oscillator and the time based oscillator.

- Uncertainty analysis on the Phase Based Oscillator.
  
  Stability bounds for stochastic perturbations.

  Uncertainty / noise reduction using Lyapunov redesign.

  Stability bounds using Small Gain Theorem.

- Assistance model using the nonlinear phase based forcing function.

- Simulation of knee and hip motion assistance.

- Test of the hip motion assistance approach with the phase based forcing function.

- Also, contributions complementary to this dissertation.

  Built force myography system for robotic ankle control.

  Proportional control of a robotic ankle using force myography.

  Proportional control of a virtual hand using force myography.

  Hand kinematics decoding using force myography.
Chapter 2

RELATED WORK

There are still many challenges in the development of truly effective wearable robots that can improve the physical performance of a human for example mechanical design, interface design, data acquisition, power supply portability, controllability, and identifying model parameters of the human [17, 20, 23, 24]. This chapter covers the related work done in recent years about the methods that use the phase of the system to control devices, and the signals from the human used for this purpose.

2.1 Phase use in Prosthesis and Orthoses Control

In wearable robotic systems, it is important to assist the user at the correct time and add energy to the oscillatory behavior to maintain a limit cycle for activities such as walking and running, given that these tasks exhibit cyclic behavior [25]. Oscillatory locomotion is ideally represented as a limit cycle in biology [26] because the mechanical motion of humans and animals are governed by continuous equations of motion [25]. The stiffness component of human joints can be virtually altered to control behavior with a controller adding or removing energy from the system [27]. We are using the phase oscillator controller to avoid finite state machines, and control motions in a continuous method. We assist the motion of the user based on a continuous function.

The most common approach to obtain the phase of the system, is to estimate it using one or several estimators with an intrinsic oscillatory model of the form:

\[ \hat{\theta} = \nu \omega + \eta \]  \hspace{1cm} (2.1)
where $\hat{\dot{\theta}}$ is the estimated value of $\dot{\theta}$, $\omega$ is the angular velocity of the system, $\nu$ is a learning factor, and $\eta$ is the noise.

Tilton et al. [26] approach the estimation of the gait cycle using a periodic version of particle filtering. They model the gait cycle as a stochastic process and for a solution they implement a particle filter based on a coupled oscillator. $\dot{\hat{\theta}}_i = \omega_i + \sigma_B^i \dot{B}^i_t + U^i_t$, where, $\theta$ is the stochastic variable, $\omega$ is the firing frequency, $\dot{B}^i_t$ is the white noise processes, and $U^i_t$ is the control input for the $i^{th}$ particle. The values of $\omega$ are drawn from a uniform distribution, and $U^i_t$ is updated with an innovation process. They used 1000 particles for their tests. It shows good results but, as a particle filter, it requires several sample periods to converge to the real value of the variable, in their words "the quantitative performance of the filter is poorer during the slow speed condition." Therefore, the disadvantages of this approach are the slow initial response since the particle filter needs to have information from several cycles to produce a good estimation, and the computational requirements.

Revzen et al. develop the theoretical frame used to estimate the phase of multiple synchronized oscillators and use it in biological processes including walking. They define the process as stochastic with an underlying oscillatory behavior. They accomplish a good estimation of the phase and reduce the noise from the sensor at the same time transforming the data using single value decomposition and principal component analysis to improve the estimate [25, 28].

Seo et al. [29] uses an adaptive frequency oscillator to assist the hip during gait. They use a series of adaptive frequency oscillators to estimate the state of the phase angle and then, use a multidimensional table to select what kind of torque assistance they provide given the estimated value. Their work is based on the work of Righetti et al. [5], but they expand the approach using multiple oscillators. They show that this system can reduce the metabolic cost during walking, however since it is a learning
approach, it means that at the beginning of the process or after a sudden change, the estimated value of the gait phase and the real angle are not close to the actual value. This approach needs several sampling periods to converge to the real value.

A. Jan Ijspeert et al. [5, 30, 31] also uses a phase based oscillator to provide assistance to periodic motions. In their case, they synchronize adaptive oscillators with the external signal, and then estimate position and velocity. The estimated states are used to compute control signals that are then used to force the system to follow a behavior [5, 30, 31]. Their oscillator has the form:

\[
\begin{align*}
\dot{x} &= (\mu - r^2)x - \omega \dot{x} + \epsilon F \\
\ddot{x} &= (\mu - r^2)\dot{x} + \omega x \\
r &= \sqrt{x^2 + \dot{x}^2}
\end{align*}
\] (2.2)

Setting \( x = r \cos \phi \), and \( \dot{x} = r \sin \phi \), the oscillator becomes

\[
\begin{align*}
\dot{r} &= (\mu - r^2)r + \epsilon F \cos \phi \\
\dot{\phi} &= \omega - \frac{\epsilon}{r} F \sin \phi
\end{align*}
\] (2.3)

where \( F \) is the external periodic signal that they want to learn, and \( \mu \) controls the amplitude of the oscillations. The system is based in the entrainment or phase lock of the external signal and the oscillator. \( \omega \) of the oscillator will converge to the frequency of the external signal. \( \epsilon \geq 0 \) is a parameter representing the magnitude of the external signal \( F \), also acting as a learning parameter. This means that the external periodic signal \( F \) has a frequency \( \omega_F \) and the frequency of the oscillator \( \omega \) will change in the direction of \( \omega_F \). The rate of change of \( \omega \) is given by:

\[
\dot{\omega} = -\epsilon F \frac{\dot{x}}{x^2 + \dot{x}^2}
\] (2.4)

They use a sum of several oscillators to assist the motion of the elbow in [31], and to assist walking in [30]. Lenzi et al. [32] also use the same approach using an adaptive
oscillator to estimate the duration of the gait cycle and assist the hip during walking. The amount of torque provided by the exoskeleton is obtained from reference tables. They report reductions in the peaks of the hip position, velocity, and muscular activity measured using EMG.

In [33], Yifan Li et al. estimate the phase during walking with a robotic ankle calculating the cross correlation between past measurements and a learned model of the gait position. They use two force sensors, one at the heel and one at the tip of the foot. They show good estimation results, however, all estimating processes require several sampling periods; therefore the time to converge to a good estimation takes several gait cycles.

Gregg et al. propose the use of the center of pressure, that is the point where the reaction force against the ground is concentrated, as a phase variable to control a prosthetic leg, introducing a virtual kinematic constraint to drive the system to the desired position [34, 35]. In [34], they use this method to control a transfemoral prosthesis during the stance phase of gait. In [35] they use impedance control as a virtual constraint to apply the method to the whole gait cycle and not just stance. The constraint relates the position of the center of pressure with the desired position of the ankle.

Asano et. [36] used the energy equation of the system as a control law to move the legs of a biped robot, adding a virtual gravity force towards the desired walking direction. The method produces a defined limit cycle, but according to them, is not robust enough against disturbances and initial conditions.

In [19], Miranda et al., use sliding control to assist the motion of the elbow. As a feedback signal, they use the force between the human and the exoskeleton. Using sliding control gives better results than a feedback linear controller since the stiffness of the soft tissue of the human has a large variability and the sliding equations allow
a wider range of stiffness values where the controller has a good performance.

Nagarajan et al. [37] use the integral admittance of the system to shape the torque trajectory of the exoskeleton to assist motion. Admittance is the relationship \( \dot{\theta}/\tau \), where \( \dot{\theta} \) is the angular velocity of the system, and \( \tau \) is the torque. The integral admittance is then defined as the relationship \( \theta/\tau \), where \( \theta \) is the angular position. The exoskeleton assists motion when it increases the admittance of the whole dynamic system human + exoskeleton. Their control law has the form\[
\tau_{exo}(t) = (I_{exo} - I^d_{exo})\ddot{\theta}_{exo}(t) + (b_{exo} - b^d_{exo})\dot{\theta}_{exo}(t) + (k_{exo} - k^d_{exo})\theta_{exo}(t),
\]
where \( I_{exo}, b_{exo}, \) and \( k_{exo} \) are the inertia, damping coefficient, and stiffness of the exoskeleton respectively. \( I^d_{exo}, b^d_{exo}, \) and \( k^d_{exo} \) are the desired values of the same coefficients, and these are calculated to have a positive admittance. They show motion amplification and human torque reduction. In [18], there is a review of trajectory generation for exoskeletons, covering on-line and off-line methods. Table 2.1 shows a summary of the methods presented in this work designed to assist motion.

In a more closely related work at ASU, [38], presents the idea of using an algorithm based on phase plane invariants to control a robotic ankle. In [3], it is proved by examples that adding the sine of the phase angle to an inertial system, produces an oscillator, therefore a stable limit cycle. These results then are used in [39], where a version of the phase oscillator controller is used to add energy during running with a pogo suit. They used only the sine of the phase angle, producing a stable limit cycle with a small amplitude for negative values of \( x \) because the mechanical characteristics of the system change when the human is in the air as shown in Fig. 2.1. The same method is used in [40] to enhance running using a jet pack, and in [41] to control a robotic hopper. A deeper analysis of the use of the sine of the phase to produce an oscillator is done in [42]; it also covers possible applications of the phase oscillator like energy harvesting and motion assistance.
**Figure 2.1:** Phase Portrait of the System. To the Left of the Vertical Axis at 0, the Hopper is Touching the Ground and the Leg Spring is Compressed. To the Right of the Vertical axis at 0, the Hopper is in the Flight Phase [3].

<table>
<thead>
<tr>
<th>Method</th>
<th>Input signal</th>
<th>Optimization method</th>
<th>Year</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fourier series + online learning</td>
<td>Force</td>
<td>Fuzzy adaptation algorithm</td>
<td>2009</td>
<td>[18, 43]</td>
</tr>
<tr>
<td>Cubic spline + online learning using neural networks</td>
<td>Position</td>
<td>Quasi linear parameter varying</td>
<td>2009</td>
<td>[18, 44]</td>
</tr>
<tr>
<td>Complementary Limb motion estimation</td>
<td>EMG</td>
<td>Statistical regression</td>
<td>2009</td>
<td>[18, 45]</td>
</tr>
<tr>
<td>Cubic spline + online learning using NOSC</td>
<td>Hybrid</td>
<td>Levenberg Marquardt</td>
<td>2011</td>
<td>[18, 46]</td>
</tr>
<tr>
<td>Space discretization permits + online learning</td>
<td>EEG + Vision</td>
<td>Greedy algorithm</td>
<td>2011</td>
<td>[18, 47]</td>
</tr>
<tr>
<td>-----------------------------------------------</td>
<td>--------------</td>
<td>------------------</td>
<td>------</td>
<td>---------</td>
</tr>
<tr>
<td>Patient + online learning</td>
<td>Hybrid</td>
<td>Multi layer perceptron</td>
<td>2011</td>
<td>[18, 48]</td>
</tr>
<tr>
<td>Online learning</td>
<td>EMG</td>
<td>Fuzzy neuro modifier</td>
<td>2011</td>
<td>[18, 49]</td>
</tr>
<tr>
<td>Sum of adaptive frequency oscillator</td>
<td>Acceleration</td>
<td>-</td>
<td>2011</td>
<td>[30, 31]</td>
</tr>
<tr>
<td>Double S Velocity profile + online learning</td>
<td>EMG</td>
<td>Fuzzy</td>
<td>2013</td>
<td>[18, 50]</td>
</tr>
<tr>
<td>Adaptive oscillator + reference tables</td>
<td>Heel strike and tow off switches</td>
<td>-</td>
<td>2013</td>
<td>[32]</td>
</tr>
<tr>
<td>Sliding control</td>
<td>Force</td>
<td>Feedback linearization</td>
<td>2013</td>
<td>[19]</td>
</tr>
<tr>
<td>Linear control of integral admittance</td>
<td>Angular position and torque</td>
<td>-</td>
<td>2015</td>
<td>[37]</td>
</tr>
<tr>
<td>Sum of adaptive frequency oscillator + reference tables</td>
<td>Angular position</td>
<td>-</td>
<td>2015</td>
<td>[29]</td>
</tr>
<tr>
<td>Phase oscillator controller</td>
<td>Accelerometers</td>
<td>-</td>
<td>2009</td>
<td>[3, 38–42]</td>
</tr>
</tbody>
</table>

16
Control of force and position are essential for the human dexterity. Precise control of exoskeletons, prostheses, and orthoses is fundamental to provide users with a device that is easy to use; therefore it is essential in the development of new assisting devices, to complement the mechanical and electronic design [51]. To control these kind of mechatronic devices, it is needed to extract neuromotor information from the human, and this is still an important area for future research [20, 23, 51]. Although lower in magnitude, the same bio-signals that can be read from a non amputated individual can be obtained from the remaining muscles in an amputee. A summary of the current work done in the prosthetic control field with the most commonly used biometric sensors is given below.

2.2.1 Control

There are three types of control used in artificial limbs, ON/OFF, multilevel and proportional. These three can be used for a single output, in mutex configuration (multi-output but actuating just one at the time) and simultaneous multi-output configuration. Although sometimes the literature uses different terms to refer to the kind of control that is being implemented. The control of a prosthetic device includes eight layers according to [4], these are: 1 input signal, 2 signal conditioning, 3 feature extraction/parameter estimation, 4 control channel decoding (split of input signals), 5 motor function determination (mapping of the coded input signals to available motor functions, 6 actuator function selection , 7 motor control, 8 actuation and sensing. In this dissertation layers from 1 to 5 are covered, determining a signal from the human, processing it and, generating a signal ready for a motor controller.

ON/OFF control refers to the devices that can be just in one of two states, for
example, open/closed, advancing/stopped, up/down, etcetera, and it is used in the most basic mechatronic prostheses. Implemented in two different ways, the activation signal can be a square wave or a ramp, that allows a little more control to the user. Multilevel activation refers to devices that can be in multiple states but can not move between states in a continuous manner, for example robotic hands that open/close in stages. Usually this type of control is achieved using pattern recognition methods. These machine learning techniques are used to identify patterns of activity across multiple electrodes that correspond to a particular movement. These techniques have the potential to enable prosthetic control of upwards of 10 DOFs, and require feature extraction from the signals read from the human to have a better performance. The features considered can range from straightforward quantities such as the mean absolute value of the signal and the number of zero crossings to more complicated values such as the coefficients used to decompose the EMG signal using wavelets. There is a lot of research being done using classification techniques, and even when some works have shown very good classification accuracy there is still a need to develop new ways to avoid the classification approach. When the variable of interest is sectioned in finite states, as classification does, we are introducing limitations to the use of the device. In [22] they used electromyography data to classify 10 movements. The classifier trained from this data had a mean classification accuracy of 88 % [22].

Proportional control refers to the control of an output signal in a continuous manner. Proportional control can give the users a more comfortable and natural interaction with the devices. Proportional control is mostly used in walking tasks, although usually is combined with finite state machines that section the processes in gait stages [52]. Fig. 2.2 shows that, from the devices evaluated in [4] almost all of them do not use proportional control. There is still research needed in the field to use proportional control reliably, and ideally use it for multiple independent outputs.
that would allow the control multiple degrees of freedom.

Figure 2.2: 3D representation of Myoelectric control for Upper Limb Prostheses. Two Examples of Commercially Available Prostheses are Indicated in the Diagram as Well as Two Examples from Research [4].

2.2.2 Mechanical Signal as Control Input

Mechanical signals not directly related to muscular activity can be used to control wearable robots. In the work presented by Huo et al. [53], they use the reaction force between the human and a upper limb exoskeleton as an input signal. They measure the force around the arm and forearm, and move the exoskeleton in the direction of the reaction force. The goal is to have a very small reaction; therefore most of the work will be done by the exoskeleton and the human will only guide the motion.

There are several research groups that use mechanical signals as input to control robotic ankles [52]. In the work done by Holgate et al. [38] the acceleration, velocity and position of the leg are used as control input to determine the position and torque
of a robotic ankle. In similar work by Kerestes et al. [39], they use an accelerometer to control a pogo suit to add energy to the human during walking and running. In [3, 42], they also use accelerometers to determine the kinematics of the system and assist the hip during walking and running. Some systems use switches on the foot to detect the heel strike and toe off, and control the device using these as markers for predetermined gait patterns [52, 54–56].

2.2.3 Electromyography

Electromyography (EMG) is the reading of the voltage activity in the body and can be done superficially and internally [14], and it is directly related to neurological volition [57]. The majority of the commercially available prosthetic devices that can be found today are controlled using this type of signal [58], and lot of research work is done using EMG [14, 21–23, 58–66]. It is being used for all types of control and is the most commonly used. Although, it has some disadvantages e. g. it is susceptible to changes in the impedance of the skin electrode interface due to movement or sweat [62, 67]. Also placing the electrodes on muscles with high pennation angles affects the EMG signals; the EMG signal read in one region of the muscle may not represent the behavior of all the muscle given that the muscles fibers are shorter than the muscle’s length [68].

The EMG readings from an amputee can be improved with targeted muscle reinnervation, that consists in transferring nerves from the residual limb to an area of the body where EMG can be read more effectively. This region can be the upper arm, chest, or back [22].

The work realized by Dalley et al. [59] consisted of a finite state machine approach, using two surface EMG signals. They define six different hand grasps that span 85% of the ones required in daily basis (hook, cylindrical, spherical, lateral, tip, and
tripod) and two more configurations, pointing and flat hand. The tests were done using non-amputee subjects to control a virtual hand. They present the time that it took the test subjects to go from one position to other, and used the two EMG signals as proportional and logic input at the same time to change between states. In other research, [60] uses three EMG signals also from a non-amputated individual to decode the force at the finger tip. With the data they constructed a transfer function to model the prosthetic. The control strategy tracks the force magnitude and uses it to estimate the force in the model. The three EMG signals were fused since it gave them better results reporting a Pearson’s correlation coefficient of $r = 0.86$.

However, in [61] they consider that given the loss of surface where myoelectric readings can be done by the amputation, and the crosstalk common in this method, it not practical to do multiple-input proportional control of multiple outputs. They compare the efficiency of pattern recognition techniques, needed to actuate multiple outputs, and they comment that the accuracy of this method on line, can be different than the accuracy when tested off-line for four EMG channels. Finally they report off-line accuracy, for individual movements as follow: linear discriminant analysis (LDA) 95.7 %, multi layer perceptron (MLP) 92.8 %, self organizing maps (SOM) 94.5 %, and regulatory feedback networks (RFN) 86.9 %. However, [62] concludes that pattern recognition won’t be sufficient to have a reliable control, but it has to be part of a robust control structure.

In [63], Kondo et al. use EMG signals but they focus on the transitions to try to decode the human intention quickly. The reading was done in windows, and at the end of each window a K nearest neighbors (KNN) algorithm classified the signal as being in a transition or in a stable state using the average of the signals. The whole work is a combination of kinematics decoding and continuous torque estimation. Also they introduce constrictions to reduce misclassification since, once the hand is in one
state, it can only change to the most proximal in shape adding stability to the process. The accuracy reported for the classification during the transitions is 25.8% for simple KNN and 32.5% using KNN with constrictions.

The results obtained by Antuvan et al. [21] are very promising, since they used four EMG inputs and mapped them to two outputs embedding the human in the control loop. The mapping matrices were arbitrarily selected to show that is not necessary to decode the movements from the signals since it is possible for the humans to learn new control schemes. They show that the time to accomplish a given task using this system reduces exponentially with respect the number of trials. They did it for four different mapping functions with five persons. The EMG signals used were from the triceps brachii, biceps brachii, extensor carpi ulnaris, and flexor carpi radialis; and the magnitude of the signals was transformed to a 2 dimensional space.

Timemy et al. [64], use multichannel EMG to decode finger movements testing the system on 6 amputees. They tested the system with several numbers of classes. The biggest classification error reported was 30 % using 12 clases. They report a classification accuracy of 98 % when using 5 classes and 12 EMG sensors. Kinnaird et al. [66], use EMG to control a robotic ankle exoskeleton using proportional control from the medial gastrocnemius. They report a 12 % reduction in the EMG activity compared to literature of normal gait. In a related work, Cavallaro et al. [65] use surface EMG to control an exoskeleton for an arm. Although their processing is more complicated using genetic algorithms to tune a model based on Hill’s muscular model.

In [69] they build a new sensor to use in real-time orthoses control. It combines the EMG signal and near-infrared light detection. Since the tissues are practically transparent for infrared light between 700 and 900 nm it can be used to detect flexion/extension of the muscles since the intensity of the reflected near-infrared light changes when the muscle moves because blood chromophores absorb this light. Then,
Table 2.2: State of the Art in Powered Hand Prostheses.

<table>
<thead>
<tr>
<th>Hand</th>
<th>Input signal</th>
<th>DOF</th>
<th>Manufacturer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensor hand SPEED</td>
<td>EMG</td>
<td>1</td>
<td>Otto Bock</td>
</tr>
<tr>
<td>i-LIMB</td>
<td>EMG</td>
<td>1</td>
<td>Touch Bionics Inc.</td>
</tr>
<tr>
<td>Motion control</td>
<td>EMG</td>
<td>1</td>
<td>Motion control Inc.</td>
</tr>
<tr>
<td>ETD/Hand</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEKA arm</td>
<td>EMG</td>
<td>18</td>
<td>DEKA R&amp;D Corp.</td>
</tr>
<tr>
<td>MANUS-HAND</td>
<td>EMG</td>
<td>3</td>
<td>Pons et al.</td>
</tr>
<tr>
<td>Vanderbilt Multigrasp</td>
<td>EMG</td>
<td>9</td>
<td>Goldfarb et al. [70–72]</td>
</tr>
<tr>
<td>Hand</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The EMG and the near-infrared signals are fused to be used as input to a Guilin-Hills classifier. Although, they do not report percentage of accuracy, they show that the fused signal allowed them to differentiate middle and ring finger movements. Robotic hands that use EMG control are shown in Table 2.2.

2.2.4 Electroencephalography

The research presented in [71, 73–79] use electroencephalography (EEG) to control robotic devices or to evaluate if it is feasible to use it. In the case of [75, 77] they used it to decode hand kinematics; [76] proposes to use it to restore gait in humans, and [73] to control a robotic arm. In [77], Agashe et al. construct a methodology to decode the hand’s kinematics using superficial EEG signals as input. The inputs were low-pass filtered at 1 Hz before processing. They recorded the position of the
hand and the EEG signals. The signals used were acquired when the study subjects were grasping different objects. The principal achievement in this paper is that they sample EEG during grasping tasks since they involve several degrees of freedom and they reconstructed the trajectories of the finger’s joints during those movements with a Pearson’s correlation coefficient $r = 0.736 \pm 0.01$ between the reconstructed trajectory and the real one for the angles of thumb carpometacarpal, and finger and thumb metacarpophalangeal joints. In this work they used five non amputated individuals. They selected the EEG channels with a genetic algorithm to get the most useful ones without having too many dimensions. Also, lags were used as input since it has been shown that this can improve the decoding of movements. No real device was actuated, and they used a simultaneous multi-output system. In a similar work [75] studies the feasibility of using EEG to decode hand kinematics during reach motions. The captured hand motion and EEG signals were band pass filtered from 0.5 Hz to 100 Hz and notch filtered at 60 Hz. They encounter that EEG signals contain information about the motions of the hand; however, they used a linear decoder, and the reconstructed signal and the original position have a correlation values between 0.35 and 0.59. They conclude that the method can be used for kinematics decoding but more research is needed to improve the method.

In [76], Presacco et al. use EEG to generate joint trajectories to restore gait in humans. They used 12 EEG sensors, each signal was band pass filtered from 0.1 to 2 Hz. Wiener filters were used for decoding. They show a comparison of the real trajectory and the reconstructed trajectory using the Wiener filters. They show that it is possible to generate a similar trajectory with this approach with average $r$ values between the actual and predicted values of 0.7. This number is not high enough to be considered highly reliable; however more research is needed using better or different decoding methods.
Lastly, Shedded et al. [73], use EEG to control a robotic arm. They defined three motions, open arm, close arm, and close hand. Using 4 EEG sensors achieved a recognition rate range from 77.8 % to 91.9 %. The data was processed in three different ways using, fast Fourier transform, wavelet transform, and principal components analysis. The classification was done with a multi layer perceptron trained with regular back-propagation. The best results is achieved using wavelet transformation (91.9 %). According to the authors the approach is promising, however it requires a training step that requires to be analyzed off-line and at least 1000 iterations to get a good classification network.

2.2.5 Sonography

Sonography (SMG) is another technique that even though, is not new, just started to be used for the control of prostheses [80–83]. Also known as ultrasonography and ecography; SMG has been used for a long time now as a medical diagnostic tool and so far there are no side effects of its use on the human body [81]. Sawiski et al. [84] is use SMG to understand the underlying muscle movement at the ankle. In [81], one SMG transducer is used in the forearm of healthy individuals, and used regression analysis to correlate the movement of the muscles captured by the transducer to the actual kinematics of the hand; they used least squares to get a regression matrix $M$ in $P = MV$, where $V$ has the features from the SMG images, and $P$ has the positions of the fingers. They considered the flexion extension of the five fingers and the abduction of the thumb. [81] confirmed that the features from the images and the positions of the fingers are highly correlated. [83] use an off-line classification of features extracted from the SMG images using a support vector machine (SVM), getting accuracies greater than 90 percent for the movement of each finger. Also, in a similar work that also used SMG but using a multichannel acquisition system, [58]
compare EMG and SMG in similar setups and the reported tracking error using SMG was 12.8% and 24.1% for EMG. [82] also realized tracking tests using SMG concluding that the technique has potential to be used in clinical trials but the mounting of the transducer is an issue to be solved. [81] mentions that compensation of the transducer motion is needed, therefore this method is sensible to probe position. Another disadvantage is that SMG is based on image processing; the computational power required is very large compared to EMG, myokinematic, and force myography; and this is important in wearable robotics. The main advantage is that this system can be used for simultaneous multi-output systems.

2.2.6 Mechanomyography

Mechanomyography (MMG) is the detection and measurement of muscular vibrations [67]. These vibrations are produced by the muscles during volitional contraction; the exerted force depends on the firing frequency and the motor units used. Higher frequencies and higher amount of motor units produce higher forces [85]. It can be done using microphones, accelerometers, or piezoelectric detectors. In [85], Tarata uses accelerometers to demonstrate that MMG shows the magnitude of muscle activation similarly to EMG, although in this work the comparison is used to measure muscular fatigue, that is defined as reduction of the force capacity. It is shown the link between the MMG and EMG signals with a correlation of 1.55 % for the biceps, so it can be inferred that MMG can be used in a similar way as EMG to control prostheses. The biggest difference between these two kinds of bio-potentials is that MMG contains less frequencies since the tissue acts as a filter [85]. The practical disadvantages of this method are the multiple source of artifacts, since it measured vibrations, the vibrations from the environment, the movement of the human, and noise all affect the signal.
2.2.7 Myokinematic

Another bio-signal available to control prostheses is the myokinematic (MK), that is the estimation of the muscular activity measuring the displacement of the skin produced by the sum of the transversal area changes during muscle contraction. Kenney et al. [86] studies the use of MK as a control signal. They design a sensor to measure the meridional change using Hall Effect. They test the sensor on amputees and using the MK signal they were instructed to contract the muscles on the residual limb to control a cursor on the computer, and track a predefined trajectory; the average error reported is 10%. In [87], the MK signal is compared with EMG as an alternative to control one variable to move a cursor in one direction on the computer. They did it using the MK signal from the biceps brachii with a Hall Effect displacement sensor designed by [86] to read MK signals. However, they do not report concrete quantitative results in that article.

Also using a single channel Hall effect MK system, [88] control proportionally a prosthetic prehensor and compares the angular position of the device and the fingers in a non amputated test subject. They show that the signal can be used in the laboratory, the range of motion in the selected location was 3.8 mm. The Pearson’s correlation for the tests between the myokinematic signal and the actual angle of the prosthetic was 0.96, and the peak error was 15%. The test was done on the forearm of a healthy subject. During the test, a splint was used to constrain the movement of the wrist and thumb. The disadvantage of this approach is that the motions of the socket and sensor introduce undesired measurements, until now MK can be used only under controlled conditions.

In the work done by [67], they utilize a similar approach to decode the movement of facial muscles using thin film piezoelectric sensors, but they measure the displacement
parallel to the skin surface instead of perpendicularly. Therefore, they are measuring the displacement in the parallel direction to the activation of the muscle. For the control, they use a threshold to eliminate false activation and the signals are then used as inputs in a neural network to classify the signals into three different movements. The classification has a 99% precision. The main advantage is that the thickness of the sensor is 40\( \mu m \), more research is needed to evaluate if this method can be used for proportional control.

2.2.8 Force myography

Force myography is defined as the sensing of the radial force on the surface caused by the increment of the muscle’s cross sectional area during activity, and it has been used efficiently to control multi-finger prosthetics [89–91]. Craelius et al. [90, 91] use FMG to control a prosthetic hand with 3 fingers using 3 FMG sensors and defining 3 different states. They approach the problem with a finite state machine showing good qualitative results. In [92] they used FMG in conjunction with EMG as features to decode intended hand configurations. 8 FSRs were collocated around the forearm and processed the outputs from the sensors fusing them. Then, they adaptively detected spikes in the signal. The detection of spikes was the activation signal for the prosthetic control to go from the current position to another; basically, they processed the combination of the signal to use it in a multilevel control. The tests were done with the hand of the subjects always in the neutral position. Likewise, [93, 94] used 8 FRS’s in the referenced works; [93] used it to make amputees perform two tasks, pick and place on a pegboard. This tasks were done virtually. [94] realized tests on the quadriceps of healthy individuals and compared the signal with EMG, finding that the FMG signal lasts longer and provides information also when the muscle is passive.
[95, 96] used 32 FSRs to measure the muscle activity on the forearm; [96] did it on healthy volunteers, and [95] on amputees. In [96], the test were done with a restricted arm posture, and they used a SVM classifier to identify 17 possible fingers postures, getting accuracies greater than 90%. [95] tested their approach with two amputees relating the pressure distribution read by the sensors with the movement of the fingers in a classification task. Other work that also uses FMG is [89]. In their case, 14 FSRs were used around the forearm, the sensors outputs were, rectified, and low pass filtered at 4 Hz. They tested the tracking of the signals of the force applied when gripping a cylindrical object. The reported correlations are 0.86 for a fast tracking task and 0.90 for a slow tracking task. Also they report that their testing was done in a neutral position.

[97] uses 54 FSR’s on the forearm to read the FMG signal, and used a KNN classifier with K=1 and Euclidean distance to differentiate between 6 possible hand postures. Even using a high number of sensors, for 3 of the 6 postures they got a 100% accuracy but, for one of the postures they just got 45 % accuracy.

Lastly, the immediate related work to the one presented here is [98] where the approach of using FSRs in a Wheatstone bridge. In our work [98] the approach was to use 4 FSRs in one full Wheatstone bridge configuration and they were able to control the aperture of a virtual hand continuously. The configuration of the system only requires the healthy person to open and close completely the hand once to get a maximum and minimum value to use as a scale for the operation. This approach showed a flaw because the hand pronation changed the maximum and minimum values and this approach was not able to track this change. The benefits of this scheme are the minimal sensitivity to sensor placement, and reduction of dimensionality having 4 sensors and just one signal.
Table 2.3: Summary of Control Signals for Prostheses, Orthoses, and Exoskeletons.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Other names</th>
<th>Principle of operation</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electro-myography (EMG)</td>
<td>NA</td>
<td>Measures the muscular electric activity, can be superficial or internal.</td>
<td>Widely used and studied. Impedance change due to sweat.</td>
</tr>
<tr>
<td>Electro-encephalography (EEG)</td>
<td>NA</td>
<td>Measures the electrical activity on the scalp.</td>
<td>Does not require activity on the residual muscle, affected by eyes movements and environmental noise.</td>
</tr>
<tr>
<td>Sonography (SMG)</td>
<td>Ecography, ultrasonogra-</td>
<td>Applies ultrasound waves (&gt; 20kHz) to the body and measures the reflected energy and converts it to an image.</td>
<td>It gives a precise image of the muscles. Requires image processing. Large transducer. Sensible to small location changes.</td>
</tr>
<tr>
<td>Mechano-myography (MMG)</td>
<td>Acoustic myogram, phonomyo-</td>
<td>Measures the vibrations caused by the muscles using accelerometers or microphones.</td>
<td>Low frequencies (25 Hz), highly affected by environmental noise.</td>
</tr>
<tr>
<td>Myokinemetric (MK)</td>
<td>Muscular bulge</td>
<td>Measures the dimensional change on the skin surface due to muscular contraction.</td>
<td>Highly sensitivity to probe location changes. Large transducer.</td>
</tr>
<tr>
<td>--------------------</td>
<td>----------------</td>
<td>--------------------------------------------------------------------------------</td>
<td>------------------------------------------------------------------</td>
</tr>
<tr>
<td>Force myography (FMG)</td>
<td>Pressure mapping, force mapping</td>
<td>Measures the change of force/pressure on the skin surface produced by changes in muscular activity.</td>
<td>Low passed signal of muscular activity. No sensitivity to small location changes. Small sensors.</td>
</tr>
</tbody>
</table>
Chapter 3

PHASE BASED OSCILLATOR: CONTROLLING OSCILLATIONS WITH A
PHASE BASED FORCING FUNCTION

3.1 Oscillators and Limit Cycles

Oscillation is a phenomena that occurs in many physical systems. Formally, the representation of a system that oscillates, given that it has a nontrivial periodic solution is [99]:

\[ x(t + T) = x(t), \forall t \geq 0 \] (3.1)

For some \( T > 0 \).

A limit cycle is a non trivial solution of the system that is represented as a closed trajectory in the phase portrait and other trajectories converge to this cycle [99]. This means, nonlinear oscillators can be stable, and the amplitude of the oscillation is independent of the initial conditions[99]. Limit cycles are used to control walking robots, because they can produce trajectories for the joints of the robots in a stable manner and in a repeating fashion [100–102].

Consider a second order dynamic system and an arbitrary limit cycle shown in Figure 3.1 representing a particular solution of this system. Now, consider an external stimulus to the system represented as a vector \( F \) on the limit cycle. The stimulus can be a perturbation, or an external continuous signal affecting the magnitude and frequency of the system. For systems that do not have a limit cycle, this external perturbation can create it; the function of the phase oscillator controller is to create an oscillatory system and to have oscillations with a desired amplitude and frequency.
3.2 Phase Based Oscillator Controller

A forcing function adds energy to the system. In the phase oscillator controller, the forcing function $F$ is periodic, formed from the phase angle of the dynamics of the system to control oscillations. In other words, the forcing function forms an oscillator with a controlled amplitude and frequency, see Figure 3.2. Let

$$
\phi = \arctan 2(\dot{x}, \omega x) \quad (3.2)
$$

Define a forcing function based on the phase angle as

$$
f(x, \dot{x}) = c \sin(\phi) + d \cos(\phi) \quad (3.3)
$$
therefore

\[
\sin(\phi) = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \omega^2 x^2}}
\]  
\hspace{1cm} (3.4)

\[
\cos(\phi) = \frac{\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}}
\]  
\hspace{1cm} (3.5)

\[
\therefore f(\dot{x}, x) = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}}
\]  
\hspace{1cm} (3.6)

This controller is based on the assumption that the system can be closely represented as a general second order dynamic system of the form:

\[
\frac{d^2 x}{dt^2} + 2\zeta \omega_n \frac{dx}{dt} + \omega_n^2 x = 0
\]  
\hspace{1cm} (3.7)

where \(\omega_n\) is the natural frequency of the system, \(\zeta\) is the damping ratio and \(x\) is the phase variable of interest.

This system can have three different behaviors: purely oscillatory \((\zeta = 0)\), damped \((0 < \zeta < 1)\), or over-damped \((\zeta > 1)\) [42]. In real life, systems do not present purely oscillatory behavior, but sometimes this is desired, e.g. for the control of wearable robots [42]. The phase oscillator controller can generate these oscillations injecting energy to the system at the proper time and can do it in a stable manner by transforming the phase angle of the system to the periodic signal that will drive the system to the desired state. The nonlinear controller is robust, time invariant, and forces a limit cycle onto the system. Contrary to linear oscillators, that become unstable with small perturbations, non linear oscillators are robust to these changes.

The sine of the phase, eq.3.4, will adjust the damping of the system; it can diminish it, lower it or increase it. Negative values of \(c\) will add damping to the system; positive values of \(c\) will lower the damping, producing oscillations. The cosine of the phase, eq.3.5, will adjust the stiffness. Negative values of \(d\) increase the stiffness of the system; positive values of \(d\) will decrease the stiffness in the system. Changing the stiffness in the system effectively alters the natural frequency of the whole system.
controlling the frequency of the oscillations. The combination of the sine and cosine of the phase will control the amplitude and frequency of the oscillations of the system altering both the damping and the stiffness. The effects of sine and cosine, when used together are coupled. In the next sections the sine and cosine are used, together and separately, and the altered behavior is described.

Using equations 3.6, and 3.7 the oscillator is defined:

\[ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  

(3.8)

The system \( \dot{x} = f(x) \)

\[ \dot{x} = -2\zeta\omega_n\dot{x} - \omega_n^2 x + \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  

(3.9)

3.3 Existence of a Limit Cycle

The phase oscillator is nonlinear; therefore the existence of a limit cycle can be evaluated using the Poincaré-Bendixson criterion. Using the continuously differentiable function

\[ V(x) = \frac{\omega_n^2 x^2}{2} + \frac{\dot{x}^2}{2} \]  

(3.10)

Its gradient is

\[ \nabla V(x) = \omega_n^2 x \dot{x} + \ddot{x} \dot{x} \]  

(3.11)

So, \( f(x) \cdot \nabla V(x) \) is

\[ f(x) \cdot \nabla V(x) = (\omega_n^2 - \omega_n^2) x \dot{x} - 2\zeta\omega_n\ddot{x} + \frac{c\ddot{x} + d\omega x \ddot{x}}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  

(3.12)

Using the Poincaré-Bendixson criterion it is possible to determine a region \( M = \{ k_1 \leq V(x) \leq k_2 \} \), where \( 0 < k_1 \leq k_2 \), \( M \) is closed and bounded, in which it is guaranteed the existence of a closed orbit. The vector field points inwards for \( f(x) \cdot \nabla V(x) < 0 \),
outwards for $f(x) \cdot \nabla V(x) > 0$, and tangent for $f(x) \cdot \nabla V(x) = 0$. Consequently for this system we have:

for $f(x) \cdot \nabla V(x) < 0$

$$\frac{c\dot{x} + d\omega x}{\sqrt{x^2 + \omega^2 x^2}} < 2\zeta\omega_n \dot{x} + (\omega_n^2 - \omega^2)x \quad (3.13)$$

for $f(x) \cdot \nabla V(x) > 0$

$$\frac{c\dot{x} + d\omega x}{\sqrt{x^2 + \omega^2 x^2}} > 2\zeta\omega_n \dot{x} + (\omega_n^2 - \omega^2)x \quad (3.14)$$

for $f(x) \cdot \nabla V(x) = 0$

$$\frac{c\dot{x} + d\omega x}{\sqrt{x^2 + \omega^2 x^2}} = 2\zeta\omega_n \dot{x} + (\omega_n^2 - \omega^2)x \quad (3.15)$$

Therefore we get the closed orbit

$$\omega^2 x^2 + \dot{x}^2 = \left(\frac{c\dot{x} + d\omega x}{2\zeta\omega_n \dot{x} + (\omega_n^2 - \omega^2)x}\right)^2 \quad (3.16)$$

This means that $M = \{k_1 = V(x) = k_2\}$, and $V(x) = \frac{1}{2} \left(\frac{c\dot{x} + d\omega x}{2\zeta\omega_n \dot{x} + (\omega_n^2 - \omega^2)x}\right)^2$. The orbit is well defined for $c = \alpha 2\zeta\omega_n, d = \frac{\alpha}{\omega} (\omega_n^2 - \omega^2)$, where $\alpha$ is a proportionality variable that defines the radius of the orbit. The next section show that $\alpha$ is the product of amplitude and frequency.

$$\omega^2 x^2 + \dot{x} = \alpha^2 \quad (3.17)$$

Evaluating the simpler case, defining $f(x, \dot{x}) = c \sin(\phi)$, and $V(x) = \frac{\omega_n^2 x^2}{2} + \frac{\dot{x}^2}{2}$ we get:

$$\ddot{x} = -2\zeta\omega_n \dot{x} - \omega_n^2 x + \frac{c\dot{x}}{\sqrt{x^2 + \omega^2 x^2}} \quad (3.18)$$

$$f(x) \cdot \nabla V(x) = -2\zeta\omega_n \dot{x}^2 + \frac{c\dot{x}^2}{\sqrt{x^2 + \omega^2 x^2}} \quad (3.19)$$
Figure 3.3: Resulting Orbit for Eq. 3.16, with a Forcing Function Based on \( \sin \phi + \cos \phi \). The Constants Used to Generate the Solutions were \( \omega_n = 5, \zeta = 0.5, c = 35, d = -24 \).

Therefore, the closed orbit

\[
\omega^2 x^2 + \dot{x}^2 = \left( \frac{c}{2\zeta \omega_n} \right)^2.
\]

In this case \( M = \{ k_1 = V(x) = k_2 \} \), and \( V(x) = \frac{1}{2} \left( \frac{c}{2\zeta \omega_n} \right)^2 \).

The use of a forcing function based on \( \sin \phi + \cos \phi \) and a forcing function based on \( \sin \phi \) produce a well defined closed orbit. See Fig. 3.3, and Fig. 3.4.
Figure 3.4: Resulting Orbit for Eq. 3.20 with a Forcing Function Based on \( \sin \phi \). The Constants Used to Generate the Solutions were \( \omega_n = 1, \zeta = 0.5, c = 5 \).

3.4 Stability of the Limit Cycle

Although the Poincaré-Bendixson criterion guarantees the existence of the limit cycle and therefore the stability. The system can also be analyzed using the Ghaffari criterion [103], which shows the necessary and sufficient conditions for the stability of limit cycles in non-linear systems. These conditions are summarized in Table 3.1.

Using \( V = \frac{\omega_n^2 x^2}{2} + \frac{\dot{x}^2}{2} \), and this criterion and the representation of the limit cycle, where \( N \) is a positive constant \( N > 0 \).

\[
\omega^2 x^2 + \dot{x}^2 = \left( \frac{c\dot{x} + d\omega x}{2\zeta\omega_n \dot{x}^2 N} \right)^2
\]  
(3.21)
Table 3.1: Necessary and sufficient conditions for stability of limit cycles in nonlinear systems according to [103].

<table>
<thead>
<tr>
<th></th>
<th>$dV(x)/dt$</th>
<th>$d^2V(x)/dt^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outside limit cycle</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>On limit cycle</td>
<td>= 0</td>
<td>= 0</td>
</tr>
<tr>
<td>Inside limit cycle</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>

Plugging into eq. 3.12

$$f(x) \cdot \nabla V(x) = -2\zeta \omega_n \dot{x}^2 + \left( \frac{c\dot{x} + d\omega x}{c\dot{x} + d\omega x} \right) \left( \frac{2\zeta \omega_n \dot{x}^2}{N} \right)$$

(3.22)

Therefore, on the limit cycle $f(x) \cdot \nabla V(x) = 0$ for $N = 1$; outside the limit cycle $f(x) \cdot \nabla V(x) < 0$ for $N > 1$; inside the limit cycle $f(x) \cdot \nabla V(x) > 0$ for $0 < N < 1$.

Evaluating the second derivative:

$$\ddot{V} = -4\zeta \omega_n \ddot{x} + \frac{-c\dot{x}^3 \dot{x} - c\omega^2 \dot{x}^3 x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + \frac{2c\dot{x} \ddot{x}}{\sqrt{\dot{x}^2 + \omega^2 x^2}}$$

\( + \frac{-d\omega x \dot{x}^2 \ddot{x} - d\omega^3 \dot{x}^2 x^2}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + \frac{d\omega x \ddot{x} + d\omega \dot{x}^2}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \)

(3.23)

Rearranging.

$$\ddot{V} = \left( -4\zeta \omega_n \ddot{x} + \frac{2c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \right) \ddot{x}$$

\( + \frac{d\omega \dot{x}^2}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + \frac{-c\omega^2 \dot{x}^3 x - d\omega^3 \dot{x}^2 x^2}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \)

(3.24)

Substituting $\ddot{x}$.

$$\ddot{V} = \left( -4\zeta \omega_n \ddot{x} + \frac{2c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + \frac{-c\dot{x}^3 - d\omega x \dot{x}^2}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \right) \left( -2\zeta \omega_n \dot{x} - \omega_n^2 x + \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \right)$$

\( + \frac{d\omega \dot{x}^2}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + \frac{-c\omega^2 \dot{x}^3 x - d\omega^3 \dot{x}^2 x^2}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \)

(3.25)
Plugging in the representation of the limit cycle.

\[
\ddot{V} = \left( -4\zeta \omega_n \dot{x} + \frac{(2\dot{x} + d\omega x)(2\zeta \omega_n \dot{x})}{(\dot{x} + d\omega x)^2} + \frac{8\zeta^3 \omega_n^5 \dot{x}^5}{(\dot{x} + d\omega x)^2} \right) (-\omega_n^2 x)
\]

\[
\ddot{V} = \frac{2\zeta d\omega_n \dot{x}^3}{(\dot{x} + d\omega x)^2} + \frac{8\zeta^3 \omega_n^5 \dot{x}^5 x}{(\dot{x} + d\omega x)^2}
\]

\[
\ddot{V} = \frac{4\zeta \omega_n^3 x \dot{x}^2 + 4d\zeta \omega_n^3 \omega x^2 \dot{x} - 4\zeta \omega_n^3 x^2 \dot{x}^2 - 2d\zeta \omega_n^3 \omega x^2 \dot{x} + 2d\zeta \omega_n \omega \dot{x}^3}{\sqrt{\dot{x}^2 + \omega_n^2 x^2}}
\]

\[
\ddot{V} = \frac{2d\zeta \omega_n \omega \dot{x}(\omega_n^2 x^2 + \dot{x}^2)}{\sqrt{\dot{x}^2 + \omega_n^2 x^2}} + \frac{8\zeta^3 \omega_n^5 \dot{x}^5 (\omega_n^2 - \omega^2)}{(\dot{x} + d\omega x)^2}
\]

The system meets the Ghaffari criterion requirements for stability only when \(\omega = \omega_n\), and \(d = 0\). This means the Ghaffari criterion guarantees the stability of the limit cycle only for the use of the sine of the phase as forcing function. However in section 3.3, it has been shown that the limit cycle exists and is stable.

### 3.5 Solution

Finding the solution for a general periodic function using the Fourier expansion of periodic functions. Consider:

\[
x = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)
\]

\[
\dot{x} = -\sum_{m=1}^{\infty} a_m m \omega \sin(m\omega t) + \sum_{m=1}^{\infty} b_m m \omega \cos(m\omega t)
\]

\[
\ddot{x} = -\sum_{m=1}^{\infty} a_m m^2 \omega^2 \cos(m\omega t) - \sum_{m=1}^{\infty} b_m m^2 \omega^2 \sin(m\omega t)
\]

Consider that the general representation of second order dynamic system excited with a periodic function:

\[
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega_n^2 x^2}} + f
\]
where $f$ is a non periodic component of the excitation.

Plugging equation 3.29 into 3.30:

$$
- \sum_{m=1}^{\infty} a_m m^2 \omega^2 \cos(m \omega t) - \sum_{m=1}^{\infty} b_m m^2 \omega^2 \sin(m \omega t)
+ 2 \zeta \omega_n \left( - \sum_{m=1}^{\infty} a_m m \omega \sin(m \omega t) + \sum_{m=1}^{\infty} b_m m \omega \cos(m \omega t) \right)
+ \omega_n^2 \left( \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m \omega t) + \sum_{m=1}^{\infty} b_m \sin(m \omega t) \right)
$$

$$
= \left[ c \left( - \sum_{m=1}^{\infty} a_m m \omega \sin(m \omega t) + \sum_{m=1}^{\infty} b_m m \omega \cos(m \omega t) \right)
+ d \omega \left( \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m \omega t) + \sum_{m=1}^{\infty} b_m \sin(m \omega t) \right) \right]^2 + f
$$

To simplify notation, let:

$$
R = \sqrt{ \left( - \sum_{m=1}^{\infty} a_m m \omega \sin(m \omega t) + \sum_{m=1}^{\infty} b_m m \omega \cos(m \omega t) \right)^2}
+ \omega_n^2 \left( \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m \omega t) + \sum_{m=1}^{\infty} b_m \sin(m \omega t) \right)^2 ]
$$

(3.31)

(3.32)
Separating the independent terms, cosine, and sine of equation 3.31.

\[ \omega_n^2 \left( \frac{a_0}{2} \right) = \left[ \frac{d\omega \left( \frac{a_0}{2} \right)}{R} \right] + f \]

\[ -\sum_{m=1}^{\infty} a_m m^2 \omega^2 + 2 \zeta \omega_n \sum_{m=1}^{\infty} b_m m \omega + \omega_n^2 \sum_{m=1}^{\infty} a_m \cos(m \omega t) \]

\[ = \left[ \frac{(c \sum_{m=1}^{\infty} b_m m \omega + d\omega \sum_{m=1}^{\infty} a_m) \cos(m \omega t)}{R} \right] \]

\[ -\sum_{m=1}^{\infty} b_m m^2 \omega^2 - 2 \zeta \omega_n \sum_{m=1}^{\infty} a_m m \omega + \omega_n^2 \sum_{m=1}^{\infty} b_m \sin(m \omega t) \]

\[ = \left[ \frac{(-c \sum_{m=1}^{\infty} a_m m \omega + d\omega \sum_{m=1}^{\infty} b_m) \sin(m \omega t)}{R} \right] \]

Rearranging the equations.

\[ f = \frac{\omega_n^2 a_0}{2} - \left[ \frac{d\omega a_0}{2R} \right] \]

\[ \sum_{m=1}^{\infty} \left( -a_m m^2 \omega^2 + 2 \zeta \omega_n b_m m \omega + \omega_n^2 a_m \right) = \left[ \frac{c \sum_{m=1}^{\infty} b_m m \omega + d\omega \sum_{m=1}^{\infty} a_m}{R} \right] \]

\[ \sum_{m=1}^{\infty} \left( -b_m m^2 \omega^2 - 2 \zeta \omega_n a_m m \omega + \omega_n^2 b_m \right) = \left[ \frac{-c \sum_{m=1}^{\infty} a_m m \omega + d\omega \sum_{m=1}^{\infty} b_m}{R} \right] \]

Solving for c and setting the equations equal:

\[ \frac{R \sum_{m=1}^{\infty} (-a_m m^2 \omega^2 + 2 \zeta \omega_n b_m m \omega + \omega_n^2 a_m) - d\omega \sum_{m=1}^{\infty} a_m}{\sum_{m=1}^{\infty} b_m m \omega} = \]

\[ -\frac{R \sum_{m=1}^{\infty} (-b_m m^2 \omega^2 - 2 \zeta \omega_n a_m m \omega + \omega_n^2 b_m) + d\omega \sum_{m=1}^{\infty} b_m}{\sum_{m=1}^{\infty} a_m m \omega} \]

Rearranging:

\[ \frac{R}{\omega} \left[ \frac{\sum_{m=1}^{\infty} (-a_m m^2 \omega^2 + 2 \zeta \omega_n b_m m \omega + \omega_n^2 a_m)}{\sum_{m=1}^{\infty} b_m m} + \frac{\sum_{m=1}^{\infty} (-b_m m^2 \omega^2 - 2 \zeta \omega_n a_m m \omega + \omega_n^2 b_m)}{\sum_{m=1}^{\infty} a_m m} \right] = d \left[ \frac{\sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m}{\sum_{m=1}^{\infty} b_m m} \right] \]

Finding a common denominator on both sides of the equation:

\[ \frac{R}{\omega} \left[ \frac{\sum_{m=1}^{\infty} a_m m \sum_{m=1}^{\infty} (-a_m m^2 \omega^2 + 2 \zeta \omega_n b_m m \omega + \omega_n^2 a_m))}{\sum_{m=1}^{\infty} a_m m \sum_{m=1}^{\infty} b_m m} + \frac{\sum_{m=1}^{\infty} b_m m \sum_{m=1}^{\infty} (b_m m^2 \omega^2 - 2 \zeta \omega_n a_m m \omega + \omega_n^2 b_m))}{\sum_{m=1}^{\infty} a_m m \sum_{m=1}^{\infty} b_m m} \right] = d \left[ \frac{\sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m}{\sum_{m=1}^{\infty} b_m m} \right] \]

(3.37)
Solving for \( d \):

\[
d = \frac{R}{\omega} \left[ \sum_{m=1}^{\infty} a_m m \sum_{m=1}^{\infty} (-a_m m^2 \omega^2 + 2\zeta \omega_n b_m m \omega + \omega_n^2 a_m) \right. \\
- \left. \sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right] \\
+ \sum_{m=1}^{\infty} b_m m \sum_{m=1}^{\infty} \left( -b_m m^2 \omega^2 - 2\zeta \omega_n a_m m \omega + \omega_n^2 b_m \right) \\
\sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right]
\]

(3.38)

Rearranging the equation getting the constant terms out of the summations:

\[
d = \frac{R}{\omega} \omega_n \left[ \sum_{m=1}^{\infty} a_m m \sum_{m=1}^{\infty} a_m m^2 \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right. \\
- \left. \sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right] \\
\sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right]
\]

(3.39)

Plugging into equation 3.34 and solving for \( c \):

\[
c = \frac{R}{\omega} \left[ \sum_{m=1}^{\infty} (-a_m m^2 \omega^2 + 2\zeta \omega_n b_m m \omega + \omega_n^2 a_m) \right. \\
- \left. \sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right] \left[ \sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right]
\]

(3.40)

Plugging equation 3.39 into 3.34:

\[
f = \frac{\omega_n^2 a_0}{2} - \frac{a_0}{2} \left[ \sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m \right]
\]

(3.41)

The system where the integral over one period \( T \) is zero, meaning that there is no offset in the signal, \( a_0 = 0 \), \( \therefore f = 0 \). Also, \( a_m = 0 \forall m \) if the function is odd \( (f(x) = -f(-x)) \). \( b_m = 0 \forall m \) if the function is even \( (f(x) = f(-x)) \). This method allows the calculation of \( c \), and \( d \) using a fast Fourier transform of the human movement.
In this section the behavior of \( x \) is considered to be purely sinusoidal. Let

\[
\begin{align*}
  x(t) &= A \sin(\omega t) \\
  \dot{x}(t) &= A\omega \cos(\omega t) \\
  \ddot{x}(t) &= -A\omega^2 \sin(\omega t)
\end{align*}
\] (3.42)

Substituting 3.42 into 3.8

\[
- A\omega^2 \sin(\omega t) + 2\zeta A\omega_n \cos(\omega t) + A\omega_n^2 \sin(\omega t) = \frac{cA\omega \cos(\omega t) + dA\omega \sin(\omega t)}{\sqrt{A^2\omega^2 \cos^2(\omega t) + A^2\omega^2 \sin^2(\omega t)}}
\] (3.43)

Where the denominator of the right hand side of the equation equals \( A\omega \), therefore it becomes:

\[
- A\omega^2 \sin(\omega t) + 2\zeta A\omega_n \cos(\omega t) + A\omega_n^2 \sin(\omega t) = c \cos(\omega t) + d \sin(\omega t)
\] (3.44)

Solving:

\[
\begin{align*}
  A(\omega_n^2 - \omega^2) &= d \\
  2\zeta A\omega_n &= c
\end{align*}
\] (3.45)

This result is similar to eq. 3.40 for a single sinusoidal component. From this equation, we can find the amplitude and frequency in terms of the model’s parameters \( \omega_n \), \( \zeta \), and \( c \), and \( d \).

\[
\begin{align*}
  A &= \frac{d\zeta + \sqrt{d^2\zeta^2 + c^2}}{2\omega_n^2\zeta} \\
  \omega &= \frac{-d\zeta\omega_n + \omega_n \sqrt{d^2\zeta^2 + c^2}}{c}
\end{align*}
\] (3.46)

From the limit cycle, substituting ea. 3.42 into 3.16

\[
A^2\omega^2 \sin^2(\omega t) + A^2\omega^2 \cos^2(\omega t) = \left( \frac{cA\omega \cos(\omega t) + dA\omega \sin(\omega t)}{2\zeta\omega_n A\cos(\omega t) + (\omega_n^2 - \omega^2)A\sin(\omega t)} \right)^2
\] (3.47)
\[ A\omega = \frac{cA\omega \cos(\omega t) + dA\omega \sin(\omega t)}{2\zeta \omega_n \omega A \cos(\omega t) + (\omega_n^2 - \omega^2) A \sin(\omega t)} \tag{3.48} \]

\[ 2\zeta \omega_n \omega A \cos(\omega t) + (\omega_n^2 - \omega^2) A \sin(\omega t) = c \cos(\omega t) + d \sin(\omega t) \tag{3.49} \]

\[ A(\omega_n^2 - \omega^2) = d \tag{3.50} \]

\[ 2\zeta A\omega_n = c \]

This is the same result as in eq. 3.45.

3.6.1 Stability of the Limit Cycle Using Polar Coordinates

\[ \omega x = r \cos \phi \tag{3.51} \]

\[ \dot{x} = r \sin \phi \]

\[ r \cos \phi = A\omega \sin(\omega t) \tag{3.52} \]

\[ r \sin \phi = A\omega \cos(\omega t) \]

Figure 3.5: Graphic Representation of the Definition of the Phase Angle in Polar Coordinates.
\[ r = A\omega, \phi = \pi/2 - \omega t, \dot{\phi} = -\omega, \ddot{x} = \dot{r} \sin \phi + r \dot{\phi} \cos \phi. \] Substituting into eq 3.8, we get:

\[ \dot{r} \sin \phi + r \dot{\phi} \cos \phi + 2\zeta \omega_n r \sin \phi + \frac{\omega_n^2 r}{\omega} \cos \phi = c \sin \phi + d \cos \phi \quad (3.53) \]

From eq. 3.53, separating the terms multiplying \( \cos \phi \), and \( \sin \phi \), we get these two differential equations.

\[ \dot{r} + 2\zeta \omega_n r = c \]
\[ r \dot{\phi} + \frac{\omega_n^2 r}{\omega} = d \quad (3.54) \]

Solving the first equation, \( r \) behaves like a first order differential equation.

\[ r = e^{-2\zeta \omega_n t} + \frac{c}{2\zeta \omega_n} \quad (3.55) \]

Therefore the steady stable response is

\[ r_{ss} = \frac{c}{2\zeta \omega_n} \]
\[ A\omega = \frac{c}{2\zeta \omega_n} \quad (3.56) \]

The steady state constant is the same as before \( c = 2\zeta A\omega \omega_n \), meaning that the limit cycle is stable and reaches a constant radius. From the second equation:

\[ \dot{\phi} = \frac{2d\zeta \omega_n}{c} - \frac{\omega_n^2}{\omega} \]
\[ \dot{\omega} = \frac{2d\zeta \omega_n}{c} - \frac{\omega_n^2}{\omega} \quad (3.57) \]
\[ \omega^2 + \frac{2d\zeta \omega_n}{c} \omega - \omega_n^2 = 0 \quad (3.58) \]

As calculated before,

\[ \omega = \frac{-d\zeta \omega_n + \omega_n \sqrt{d^2 \zeta^2 + c^2}}{c} \quad (3.60) \]
3.6.2 Comparison with the Time Based Oscillator

A time based oscillator is external and independent to the system; a phase based oscillator applies an external stimulus to modify the dynamic characteristics of the system to be oscillatory. The time based model applies an open-loop forcing function while the phase oscillator is based on the current state of the system \((x, \dot{x})\). Both approaches rely on the accuracy of the model. A time based oscillator will have a phase shift with respect to the system; it affects the amplitude of the oscillations as shown in this section. First, consider this oscillator with a phase shift \(\gamma\).

\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = B \sin(\omega t + \gamma) \tag{3.61}
\]

As in the previous section, \(\omega x = r \cos \phi = A\omega \sin(\omega t)\), and \(\dot{x} = r \sin \phi = A\omega \cos(\omega t)\), therefore \(\dot{x} = \dot{r} \sin \phi + r \dot{\phi} \cos \phi, r = A\omega, \phi = \pi/2 - \omega t, \dot{\phi} = -\omega\)

\[
\dot{r} \sin \phi + r \dot{\phi} \cos \phi + 2\zeta\omega_n r \sin \phi + \frac{\omega_n^2 r}{\omega} \cos \phi = B \sin(\omega t + \gamma) \tag{3.62}
\]

Using the trigonometric identities \(\cos u = \sin(\pi/2 - u)\), \(\cos(u - v) = \cos u \cos v + \sin u \sin v\), eq. 3.62 becomes:

\[
\dot{r} \sin \phi + r \dot{\phi} \cos \phi + 2\zeta\omega_n r \sin \phi + \frac{\omega_n^2 r}{\omega} \cos \phi = B \cos \gamma \cos \phi + B \sin \gamma \sin \phi \tag{3.63}
\]

\[
\dot{r} + 2\zeta\omega_n r = B \sin \gamma \tag{3.64}
\]

\[
r \dot{\phi} + \frac{\omega_n^2 r}{\omega} = B \cos \gamma
\]

Solving the differential equation.

\[
r = e^{-2\zeta\omega_n t} + \frac{B \sin \gamma}{2\zeta\omega_n} \tag{3.65}
\]

The steady state value of the radius is:

\[
r_{ss} = \frac{B \sin \gamma}{2\zeta\omega_n} = A\omega \tag{3.66}
\]
\[ B = \frac{2A\zeta\omega_n\omega}{\sin \gamma} \] (3.67)

From the second differential equation,
\[ \dot{\phi} = -\frac{\omega_n^2}{\omega} + 2\zeta\omega_n \cot \gamma = -\omega \] (3.68)

\[ \frac{\omega_n^2 - \omega^2}{2\zeta\omega_n\omega} = \cot \gamma = \frac{d}{c} \] (3.69)

So, the magnitude of \( \gamma \) is:
\[ \gamma = \tan^{-1} \left( \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right) \] (3.70)

\( B \) depends on the system, the desired frequency \( \omega \), and the phase shift \( \gamma \). For the same amplitude, \( B \) has to be greater for a faster \( \omega \), and the amplitude \( A \) depends directly on \( B \). Also, \( \gamma \) relies on the desired frequency \( \omega \).

Using a time based forcing function, \( f(t) \) equals:
\[ f(t) = \frac{2A\zeta\omega_n\omega}{\sin \left( \tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right)} \sin \left( \omega t + \tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right) \] (3.71)

It is possible to compare directly with the phase oscillator using the trigonometric identity.

\[ c\sin \phi + d\cos \phi = \sqrt{c^2 + d^2} \cos \left( \phi - \tan^{-1} \frac{c}{d} \right) \] (3.72)

\[ c\sin \phi + d\cos \phi = \sqrt{c^2 + d^2} \cos \left( \frac{\pi}{2} - \omega t - \tan^{-1} \frac{c}{d} \right) \] (3.73)

\[ c\sin \phi + d\cos \phi = \sqrt{c^2 + d^2} \sin \left( \omega t + \tan^{-1} \frac{c}{d} \right) \] (3.74)

### 3.6.3 Simulations of the Phase Oscillator and the Time Based Oscillator

Using eq. 3.45 and 3.71, it is possible to generate oscillations of equal magnitude and frequency. For the purely sinusoidal case, the use of the phase based oscillator does not represent a great advantage since the output is the same. Although, it has a better performance to handle external perturbations. In table 3.2 there are summarized three simulated cases. The phase based oscillator adapts to changes in phase.
Table 3.2: Summary of the Three Cases Simulated and Analyzed Using the Time Based Oscillator and the Phase Based Oscillator, and the Values of the Coefficients for Each Case.

<table>
<thead>
<tr>
<th>Case</th>
<th>Phase based oscillator</th>
<th>Time based oscillator</th>
</tr>
</thead>
<tbody>
<tr>
<td>A=1, ω = 5</td>
<td>c=25, d= 0</td>
<td>B = 25, γ = π/2</td>
</tr>
<tr>
<td>A=1, ω = 7</td>
<td>c=35, d=-24</td>
<td>B=-42.43, γ = -0.9697</td>
</tr>
<tr>
<td>A=1, ω = 3</td>
<td>c=15, d= 16</td>
<td>B = 21.93, γ = 0.6839</td>
</tr>
</tbody>
</table>

shift, while the time based oscillator forces the system to go back to the original phase requiring more time/energy.

Both methods depend on the accuracy of the model to produce desired amplitude and frequency. There is better performance against perturbations using the phase based oscillator, see Fig. 3.7. Also, the phase based oscillator can handle high noise in the velocity value, see Fig. 3.8.

Figure 3.6: Output of the System for the 3 Cases Using the Phase Based Oscillator and the Time Based One. The Output of the Nonlinear Phase Based Oscillator is Shown in Blue. The Output of the Time Based Oscillator is Shown in Red. The Constants Used to Generate the Solutions were \( \omega_n = 5, \zeta = 0.5 \).
3.7 Analysis of the Effects of the Forcing Function

Now, the effects of the forcing functions for this case are analyzed. The functions are: \( c \sin \phi, d \cos \phi, c \sin \phi + d \cos \phi. \)
3.7.1 Sine of Phase Angle as a Forcing Function

Using \( f(x, \dot{x}) = c \sin \phi \) and adding it to the second order differential equation model:

\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{c\dot{x}}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \tag{3.75}
\]

The amplitude of the limit cycle is found analytically solving the equation.

\[
2\zeta\omega_n \dot{x} = \frac{c\dot{x}}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \tag{3.76}
\]

Using the eigenfunction 3.42,

\[
2\zeta\omega_n A \cos(\omega t) = \frac{c\omega A \cos(\omega t)}{\sqrt{\omega^2 A^2 \cos^2(\omega t) + \omega^2 A^2 \sin^2(\omega t)}} \tag{3.77}
\]

\[
2\zeta A \omega_n = c \tag{3.78}
\]

\[
A = \frac{c}{2\zeta \omega_n} \tag{3.79}
\]

The amplitude of the oscillations depends directly on the magnitude of \( c \). If \( c = 0 \), there are no oscillations.

In the same way, the frequency can be found solving the equation 3.75 using the eigenfunction 3.42, for \( \sin(\omega t) = 1, \cos(\omega t) = 0 \).

\[
-A^2 \sin(\omega t) + 2\zeta \omega_n A \omega \cos(\omega t) + \omega_n^2 A \sin(\omega t) = \frac{c\omega A \cos(\omega t)}{\sqrt{\omega^2 A^2 \cos^2(\omega t) + \omega^2 A^2 \sin^2(\omega t)}} \tag{3.80}
\]

\[
-A^2 + A^2 = 0 \tag{3.81}
\]

\[
\therefore \omega = \omega_n \tag{3.82}
\]

Therefore, it is not possible to control the frequency of the oscillations, it depends only on the system’s natural frequency.

In the phase plot, using \( \sin \phi \), the amplitude of the limit cycle can be changed. The \( \sin \phi \) is in synchrony with the velocity of the system \( \dot{x} \); consequently it directly affects
the damping of the system. Figure 3.9 shows the output of a the system for three
different values of $c$. For $c > 0$ the amplitude of the oscillations varies proportionally
(there is a limit cycle), and for $c < 0$ the oscillations get damped to zero (no limit
cycle but stable), and the system reaches zero faster for more negative values of $c$.

![Figure 3.9: Results of Simulation Using the Sine of the Phase Angle as a Forcing
Function. The First Plot Shows the Values of $c$ vs Time. The Second Plot Shows the
Output of the Dynamic System. The Oscillation is Damped for the Negative Value of
$c$ and the Amplitude is Different for Both Positive Values of $c$. The Last Plot has the
Value of the Sine of the Phase Angle. The Constants Used to Generate the Solutions
Were $\omega_n = 1$, $\zeta = 0.5$.]

Defining the states as

$$
y_1 = x
$$

$$
y_2 = \dot{y}_1 = \dot{x}
$$

$$
\dot{y}_2 = \ddot{x}
$$

The state space representation of the oscillator is

$$
y_1 = y_2
$$

$$
y_2 = -\omega_n^2 y_1 - 2\zeta \omega_n y_2 + c \frac{y_2}{\sqrt{\omega_n^2 y_1^2 + y_2^2}}
$$

Using this state space representation, it is possible to generate a phase portrait of the
system shown in figure 3.10. It can be seen that a limit cycle exists. Both trajectories converge to the limit cycle.

**Figure 3.10:** Phase Portrait of the Oscillator Using $\sin \phi$ as a Forcing Function. In Blue are Two Solutions, one With Initial Conditions $y_1 = 3$, $y_2 = 0$ and the Second Case $y_1 = -6$, $y_2 = 0$. A Stable Limit Cycle is Shown in Bold Blue. The Start of the Trajectories is Marked With a Square and the end With a Diamond. The Constants Used to Generate the Solutions Were: $\omega_n = 1$, $\zeta = .5$, $c = 5$. 

53
3.7.2 Cosine of Phase Angle as a Forcing Function

Using \( f(x, \dot{x}) = d\cos\phi \) and adding it to the second order differential equation model:

\[
\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{d\omega x}{\sqrt{\dot{x}^2 + \omega_n^2 x^2}} \quad (3.85)
\]

Let

\[
x = A\sin(\omega t) \\
\dot{x} = A\omega\cos(\omega t) + \dot{A}\sin(\omega t) \quad (3.86) \\
\ddot{x} = -A\omega^2\sin(\omega t) + 2\dot{A}\omega\cos(\omega t) + \ddot{A}\sin(\omega t)
\]

\[
- A\omega^2 \sin(\omega t) + 2\dot{A}\omega \cos(\omega t) + \ddot{A}\sin(\omega t) \\
+ 2\zeta\omega_n [A\omega \cos(\omega t) + \dot{A}\sin(\omega t)] + \omega_n^2 A\sin(\omega t)
\]

\[
= \frac{d\omega A\sin(\omega t)}{\sqrt{(A\omega \cos(\omega t) + \dot{A}\sin(\omega t))^2 + \omega^2 A^2 \sin^2(\omega t)}} \quad (3.87)
\]

For \( \sin(\omega t) = 0, \cos(\omega t) = 1 \).

\[
\dot{A} + \zeta\omega_n A = 0 \quad (3.88) \\
A = e^{-\zeta\omega_n} \quad (3.89)
\]

Consequently the amplitude in steady state is zero, so using the cosine of the phase does not produce an oscillator, nor a limit cycle.

For the frequency, solving the equation 3.85, using the eigenfunction 3.42. For \( \sin(\omega t) = 1, \cos(\omega t) = 0 \).

\[
-A\omega^2 + \omega_n^2 A = d \quad (3.90) \\
\omega = \sqrt{\frac{\omega_n^2 A - d}{A}} \quad (3.91)
\]

The frequency of the system at an instant when \( A \neq 0 \), depends on the value of \( d \).

Therefore, using \( \cos\phi \) as a forcing function can only affect the frequency of the system during transitions but can not generate a limit cycle; \( \cos\phi \) is in synchrony.
with the position of the system $x$. Figure 3.11 shows the output of the system for three different values of $d$. Also, observe that the steady state position value depends on $d$; set $x = x_e$, $\dot{x} = \ddot{x} = 0$ in equation 3.85.

$$\omega_n^2 x_e = d \omega$$

(3.92)

$$\therefore x_e = \frac{d \omega}{\omega_n^2}$$

(3.93)

Figure 3.11 shows the output of the system for three different values of $d$. The oscillations decay but there is a change in the frequency of the response for different values of $d$. Observe that for negative values of $d$ the frequency increases.

![Figure 3.11: Results of Simulation Using the Cosine of the Phase Angle as a Forcing Function. The First Plot Shows the Values of $d$ vs Time. The Second Plot Shows the Output of the Dynamic System (Position). The Oscillation Decays and the Frequency Changes With $d$. The Last Plot Has the Value of the Cosine of the Phase Angle. The Constants Used to Generate the Solutions Were $\omega = \omega_n = 1$, $\zeta = 0.5$.](image)

Defining the states as in equation 3.83, the state space representation of the oscillator is

$$\dot{y}_1 = y_2$$

$$y_2 = -\omega_n^2 y_1 - 2\zeta \omega_n y_2 + \frac{\omega d y_1}{\sqrt{\omega_n^2 y_1^2 + y_2^2}}$$

(3.94)
Using this state space representation, it is possible to generate a phase portrait of the system shown in figure 3.12. There is not a limit cycle but the system is stable, and the final position depends on the initial condition. Consequently, \( x = \sin A(\omega t) \) is not a valid solution of the system for all time. It is possible to analyze the transition and its effects on the frequency of the system.

\[
x = \sin A(\omega t)
\]

**Figure 3.12:** Phase Portrait of the Oscillator Using \( \cos \phi \) as Forcing Function. In Blue are Shown Solutions for Initial Conditions \( y_1 = 3 \) and \( y_1 = -6 \); \( y_2 = 0 \). The System with the Cosine Function Does Not Generate a Limit Cycle. The Start of the Trajectories is Marked with a Square and the end with a Diamond. The Constants Used to Generate the Solutions Were \( d = -5, \zeta = 0.5, \omega = \omega_n = 1. \)
3.7.3 Sine + Cosine of Phase Angle as Forcing Function

Using \( f(x, \dot{x}) = c \sin \phi + d \cos \phi \) and adding it to the second order differential equation model:

\[
\ddot{x} + \zeta \omega_n \dot{x} + \omega_n^2 x = \frac{c \dot{x} + d \omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \tag{3.95}
\]

The amplitude of the limit cycle and the frequency are defined by this equation. Both depend on the values of \( c \) and \( d \).

\[
A = \frac{d \zeta + \sqrt{d^2 \zeta^2 + c^2}}{2 \omega_n^2 \zeta} \tag{3.96}
\]

\[
\omega = \frac{-d \zeta \omega_n + \omega_n \sqrt{d^2 \zeta^2 + c^2}}{c} \tag{3.96}
\]

To have oscillations (\( A > 0 \)), these conditions have to be satisfied.

\[
d\zeta + \sqrt{d^2 \zeta^2 + c^2} > 0
\]

\[
-d\zeta + \sqrt{d^2 \zeta^2 + c^2} > 0
\]

\[
\frac{\omega_n}{c} > 0
\]

\[
\omega_n > 0
\]

\[
\therefore c > 0 \tag{3.97}
\]

Figure 3.13 shows the output of the system for two values of \( c \), and three of \( d \). The frequency of the output changes with the value of \( d \), and also the amplitude of the oscillations. When \( c \) changes to a negative value, the oscillations decay to zero. When \( c < 0 \) the value of \( d \) affects the frequency of the system.

Defining the states as in equation 3.83, the state space representation of the oscillator is

\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\omega_n^2 y_1 - 2 \zeta \omega_n y_2 + \frac{d \omega y_1 + cy_2}{\sqrt{\omega^2 y_1^2 + y_2^2}} \tag{3.98}
\end{align*}
\]
Figure 3.13: Results of Simulation Using the Cosine and Sine of the Phase Angle as a Forcing Function. The First Plot Shows the Output of the Dynamic System (Position). The Second Plot Shows the Value of $c$ vs Time. The Third Plot Shows $\sin \phi$ vs Time. The Fourth Plot Shows the Value of $d$ vs Time; and the Last Plot Shows $\cos \phi$ vs Time. The Oscillation Decays for the Negative Value of $c$. The Constants Used to Generate the Solutions Were $\omega_n = 1$, $\zeta = 0.5$.

Using this state space representation, it is possible to generate a phase portrait of the system shown in figure 3.14. It can be seen that a limit cycle exists where other starting trajectories converge.
Figure 3.14: Phase Portrait of the Oscillator Using \( \sin \phi + \cos \phi \) as a Forcing Function. In Blue are Two Solutions, One With Initial Conditions \( y_1 = 3, y_2 = 0 \) and the Second Case \( y_1 = -6, y_2 = 0 \). A Stable Limit Cycle is Formed in Bold Blue. The Start of the Trajectories is Marked with a Square and the End with a Diamond. The Constants Used to Generate the Solutions Were \( c = 5, d = -2, \zeta = 0.5, \omega_n = 1 \).

3.8 Use as Controller

The phase based oscillator can operate as non linear controller using the error as a variable of interest and using \( c < 0 \), therefore the value of the error will converge to zero.

\[
F = \frac{c \dot{e} + d \omega e}{\sqrt{\dot{e}^2 + \omega^2 e^2}}
\]  

(3.99)

where \( e \) is the error defined as \( e = x - x_d \), where \( x_d \) is the desired value.

For \( d = 0 \), the error will converge to zero at the natural frequency of the system,
and faster for \( d < 0 \). A practical consideration about this approach, when the variable of interest is zero and stable, then the system is undetermined because \( x = \dot{x} = 0 \). In this case it is necessary to set a range to activate the controller only when the error is outside of it; or add a small term inside the square root. The forcing function becomes:

\[
F = \frac{c \dot{e} + d \omega e}{\sqrt{\dot{e}^2 + \omega^2 e^2 + n}}
\]  

(3.100)

where \( 0 < n << 1 \).

### 3.8.1 Comparison with Linear Controller

The system was simulated as controlled and compared to the performance of a PID controller with the structure:

\[
F = K_p e + K_i \int e + K_d \dot{e}
\]

(3.101)

The PID was tuned using the MATLAB tuning toolbox, the gains used were \( K_p = 317 \), \( K_i = 505 \), \( K_d = 34.34 \). For the phase based oscillator controller the constants used were \( c = -1000 \), \( d = -8000 \). Both systems were made to follow a step reference at 2 seconds, and 3 perturbations were introduced. The results of the simulations are shown in Fig. 3.15 and Fig. 3.16. Both controllers showed similar performance but the phase oscillator controller was slightly faster.
Figure 3.15: System response with the two controllers. The first plot shows the response using the phase based oscillator controller; the second plot shows the comparison of the two systems; and the third plot shows the response using the PID controller.

Figure 3.16: Close Up of the System Response with the Two Controllers. The First Plot Shows the Response Using the Phase Based Oscillator Controller; the Second Plot Shows the Comparison of the Two Systems; and the Third Plot Shows the Response Using the PID Controller.
3.9 Adding Offset to the Oscillations

For oscillatory output not centered around zero, also need \( c > 0, \ d < 0, \) and \( x_d = \text{constant}. \) This changes the control law to:

\[
\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2(x - x_d) = \frac{c\dot{x} + d\omega(x - x_d)}{\sqrt{x^2 + \omega^2(x - x_d)^2}} \tag{3.102}
\]

Figure 3.17 shows the simulation of a second order dynamic system with mass \( m = 0.1Kg, \ b = 0.5, \) and \( k = 50. \) \( \omega_n = \sqrt{k/m}, \ \zeta = 0.111. \) Using an offset \( x_d = 1 \)

![Simulation Output of the Phase Based Oscillator Adding Offset](image)

**Figure 3.17:** Simulation Output of the Phase Based Oscillator Adding Offset. The System Used for the Simulation Was \( \omega_n = \sqrt{k/m}, \ \zeta = 0.111, \) Using an Offset \( x_d = 1. \)
3.10 Summary

We developed a phase oscillator that was able to create limit cycles of our choosing, adjust the amplitude and/or the frequency. The Poincaré-Bendixson criterion and an analysis in polar coordinates showed that the system creates a globally stable limit cycle.

We compared a standard time based forcing function to the phase oscillator. The time based system is not based on the state variables and a phase shift occurs.

The phase oscillator is based on sine and cosine functions. The $c$ times the sine function determines if the system oscillates. If $c > 0$, the system oscillates and if $c < 0$, the system does not oscillate. If the cosine function is not used, the system oscillates at the natural frequency. If $d$ times the cosine of the phase is added to the sine function, it affects the frequency of the oscillations. $d < 0$ produces an oscillation frequency $\omega > \omega_n$. The sine of the phase affects the damping of the system, and the cosine of the phase affects the stiffness in the system. The values of $c$, and $d$ can be chosen based on a Fourier transform, eq.3.40, or can be chosen based on a sinusoidal response, eq. 3.45.
4.1 Simulation Examples

A rotational spring, mass, damper system was simulated to generate oscillations in three cases; changing the amplitude, changing the frequency and, changing amplitude and frequency. The parameters of the simulated system were: inertia $J = 10 \text{ Kg}m^2$, spring stiffens constant $k = 2.7 \text{ Nm/rad}$, the damper constant $B = 4 \text{ Nms}^{-1}/\text{rad}$. Therefore, $\omega_n = \sqrt{\frac{k}{J}} = 0.5196 \text{ rad/s}$, $\zeta = \frac{c}{2J\omega_n} = 0.3849$.

Figure 4.1 shows the response for when the amplitude of the oscillations is changed. The plot on top shows the response in the phase plane. In red is shown the initial limit cycle ($A = 1.5 \text{ rad}, \omega = 2 \text{ rad/s}$) and in blue the transition and final limit cycle ($A = 2 \text{ rad}, \omega = 2 \text{ rad/s}$). In the plot of the response in the position vs time plane it can be seen how the frequency of the oscillations does not change.

Figure 4.2 shows the system response for 2 changes of frequency. The initial limit cycle ($A = 1.5 \text{ rad}, \omega = 0.7 \text{ rad/s}$) is shown in red, then the system is changed to have a limit cycle of $A = 1.5 \text{ rad}, \omega = 1.2 \text{ rad/s}$, shown in blue, and in cyan is shown the final limit cycle of ($A = 1.5 \text{ rad}, \omega = 1.7 \text{ rad/s}$). The amplitude of the oscillations is maintained constant but the frequency is increased therefore the minor axis of the ellipse in the phase plane increases given that the velocity of the output will reach a higher maximum. Figure 4.1 shows the response for changes in amplitude and frequency and shows smooth transitions even when both the amplitude and frequency are affected.
**Figure 4.1:** Response of the Phase Based Oscillator During a Change of Amplitude Maintaining the Frequency Constant at $\omega_n = 2 \text{ rad/s}$. The Plot on Top Show the Response in the Phase Plane, in Red is Shown the First 40 s of the Simulation Where the Values of $c$, and $d$ are Maintained Constant to Obtain $A = 1.5 \text{ rad}, \omega = 2 \text{ rad/s}$. In Blue is Shown the Response After the Values of $c$ and $d$ are Modified to Obtain Final Conditions $A = 2 \text{ rad}, \omega = 2 \text{ rad/s}$. The Plot on the Bottom Shows the System Response in the Position vs Time Plane.
Figure 4.2: Response of the Phase Based Oscillator when the Frequency is Changed and the Amplitude is Maintained Constant at $A = 1.5$ rad. The Plot on Top Show the Response in the Phase Plane, in Red is Shown the First 50 s of the Simulation Where the Values of $c$ and $d$ are Maintained Constant to Obtain $A = 1.5$ rad, $\omega = 0.7$ rad/s. In Blue is Shown the Response After the Values of $c$ and $d$ are Modified to Generate $A = 1.5$ rad, $\omega = 1.2$ rad/s. In Cyan is Shown the Response After the Values of $c$ and $d$ are Modified to Obtain Final Conditions $A = 1.5$ rad, $\omega = 1.7$ rad/s. The Plot on the Bottom Shows the System Response in the Position vs Time Plane.
Figure 4.3: Response of the Phase Based Oscillator for Three Different Conditions. The Plot on Top Show the Response in the Phase Plane, in Red is Shown the First 50 s of the Simulation Where the Values of $c$, and $d$ are Maintained Constant to Obtain $A = 1.5\text{rad}$, $\omega = 0.8\text{rad/s}$. In Blue is Shown the Response After the Values of $c$ and $d$ are Modified to Generate $A = 1\text{rad}$, $\omega = 1.5\text{rad/s}$. In Cyan is Shown the Response After the Values of $c$ and $d$ are Modified to Obtain Final Conditions $A = 1.5\text{rad}$, $\omega = 2\text{rad/s}$. The Plot on the Bottom Shows the System Response in the Position vs Time Plane.
4.2 Oscillating the Angular Velocity of a DC Motor

The phase based oscillator was implemented to oscillate the angular velocity of a DC Motor. It was implemented in this way because the relationship of the angular velocity to the voltage of a DC motor can be closely represented as a second order dynamic system. The model is:

\[ \ddot{\omega}_M + \left( \frac{JR + bL}{JL} \right) \dot{\omega}_M + \left( \frac{bR + K^2}{JL} \right) \omega_M = 0 \] (4.1)

where \( \omega_M \) is the angular velocity of the motor, \( J \) is the moment of inertia \([Kg \cdot m^2]\), \( b \) is the constant motor viscous friction \([N \cdot m \cdot s]\), \( K \) is the motor torque constant\([N \cdot m/A]\) or \([V \cdot /rad]\), \( R \) is the electric resistance of the motor \([\Omega]\), and \( L \) is the electric inductance of the motor \([H]\).

The system was programmed in Simulink and a PC-104 was used to control the motor. The PC-104 has an I/O card of Sensoray Co. Inc., and was connected to a motor driver model AZBH40A8 of Advanced Motion Controls Inc. The DC motor used for the test was a 24 VDC motor model 407548 of Maxon Precision Motors.

**System Identification**

The first step was to produce oscillations to identify the parameters of the system using as forcing function:

\[ f(\omega_M, \dot{\omega}_M) = \frac{c\omega_M + d\dot{\omega}_M}{\sqrt{\omega_M^2 + \omega^2 \dot{\omega}_M^2}} \] (4.2)

Where \( \omega_M \) is the variable of interest, the angular velocity of the motor, and \( \omega \) is the frequency of the oscillations. The constants used where \( c = 1, d = 0, \) and \( \omega = 1 \).

From the results of the first test shown in figure 4.4 it was found that the natural frequency of the system is close to \( \omega \approx 69 \text{rad/s} \).
It can be seen in figure 4.4 that the output is not perfectly sinusoidal, it is skewed to the right, this is because $\omega = 1$ was used for the first test. The next step was to perform the same test but now using the experimentally determined value of $\omega$, therefore $c = 1$, $d = 0$, and $\omega = 69$ rad/s.

From the results of the second test shown in figure 4.5 it was found a better approximation of the natural frequency of the system $\omega_n \approx 78$ rad/s. This also produces a sinusoidal response that is less skewed.

The next step is to identify the value of the damping coefficient $\zeta$. A step response test was done to identify the overshoot of the system and estimate $\zeta$ using this value. For a second order system the relationship of the percentage of overshoot and the damping is:

$$O_S = 100e^{-\frac{\zeta \omega}{\sqrt{1-\zeta^2}}} \quad (4.3)$$

**Figure 4.4:** System Output $\omega_M$ vs Time with Constant Values $c = 1$, $d = 0$, and $\omega = 1$. The Output Has a Frequency $\omega \approx 69$ rad/s.
where $O_S$ is the percentage of overshoot. Figure 4.6 shows the system response to a unitary step. From this plot the overshoot was identified to be $O_S = 1\%$, therefore $\zeta \approx 0.8262$. 

Figure 4.6: Motor Response for a Unitary Step Input $\omega_M$ vs Time. The Overshoot is Approximately 1\%. 

Figure 4.5: Motor Output $\omega_M$ vs Time with Constant Values $c = 1$, $d = 0$, and $\omega = 69$. The Output Has a Frequency $\omega \approx 78$ rad/s.
In previous sections it was found that

\begin{align*}
c &= 2\zeta A\omega_n \\
d &= A(\omega_n^2 - \omega^2)
\end{align*}  

(4.4)

To identify the gain of the system consider this representation:

\begin{equation}
\ddot{\omega}_M + 2\zeta\omega_n\dot{\omega}_M + \omega_n^2\omega_M = G\frac{c'\dot{\omega}_M + d'\omega_M}{\sqrt{\dot{\omega}_M^2 + \omega^2}}  
\end{equation}  

(4.5)

where $G$ is the gain of the actuation system (motor, driver, and data acquisition), and $c = Gc'$, $d = Gd'$. To identify the gain $G$, the system was run with $c = 1$, $d = 0$; the response is shown in fig 4.7. Using the equation 4.4, $G$ was identified to be $G = 7.10678 \times 10^{-7}$. The values used to calculate $G$ were $\zeta = 0.826$, $A = 140 \text{ rad/s}$, $\omega = \omega_n = 78 \text{ rad/s}$.

**Validation of system parameters**

To validate the experimental values for the parameters of the system, a test was run to obtain the error between the desired and actual oscillation parameters. The desired values for frequency and amplitude for this test were $A_d = 150 \text{ rad/s}$, $\omega_d = 100 \text{ rad/s}$. Using equation 4.4 values for $c$ and $d$ are found.

\begin{align*}
c &= c'G = 2\zeta A\omega_n G \\
c &= 2(0.8260)(150)(100)(78)(7.10678 \times 10^{-7}) \\
c &= 1.37363
\end{align*}  

(4.6)

\begin{align*}
d &= d'G = A(\omega_n^2 - \omega^2)G \\
d &= 150(78^2 - 100^2)(7.10678 \times 10^{-7}) \\
d &= -0.4174
\end{align*}  

(4.7)
The results of the test using the values from equations 4.6 and 4.7 are shown in figures 4.7, 4.8, 4.9, and 4.10. The system output has a frequency of $\omega = 97 \text{ rad/s}$, and amplitude of $A = 153 \text{ rad/s}$. The error in the frequency values is 3%; the error in the amplitude values is 2%.

![DC Motor rotational velocity](image)

**Figure 4.7:** Motor Angular Velocity $\omega_M$ During a Test with $c = 1.373$, and $d = -0.417$. The Frequency of the Oscillations is $\omega = 97 \text{ rad/s}$. The Amplitude of the Oscillations is $A = 153 \text{ rad/s}$. 

72
Figure 4.8: Motor Angular Acceleration $\dot{\omega}_M$ During a Test with $c = 1.373$, and $d = -0.417$. The Frequency of the Oscillations is $\omega = 97$ rad/s. The Amplitude of the Oscillations is $A = 1.6 \times 10^4 \text{ rad/s}^2$.

Figure 4.9: System Output in the Phase Plane During a Test with $c = 1.373$, and $d = -0.417$. The Frequency of the Oscillations is $\omega = 97$ rad/s. The Amplitude of the Oscillations is $A = 153$ rad/s. The Trajectory Starts at $(0,0)$. 
Figure 4.10: On Top, Magnitude of the Forcing Function Output vs Time During the Test with $c = 1.373$, and $d = -0.417$. On bottom, Magnitude of $c$ and $d$ vs Time.
Changing the frequency

Using the identified parameters, values of $c$ and $d$ were calculated for desired initial and final conditions changing the frequency and maintaining the amplitude constant. The desired initial conditions were $A_{d1} = 150$ rad/s and $\omega_{d1} = 100$ rad/s; and the final desired conditions were $A_{d2} = 150$ rad/s and $\omega_{d2} = 80$ rad/s. Calculating $c$ and $d$ as previously shown we got $c_1 = 1.37$, $c_2 = 1.11$, $d_1 = -0.41$, and $d_2 = -0.03$.

After running the system the actual values were $A_1 = 153$ rad/s, $\omega_1 = 103$ rad/s, $A_2 = 148$, and $\omega_2 = 81.6$. The error in amplitude was 2 %. The error in frequency was 3 %. The output of the system can be seen in Figure 4.11, it shows a smooth and fast transition. Figure 4.12 shows the motor acceleration vs time. Figure 4.13 shows the output of the system in the phase plane where it can be seen there are two defined limit cycles. Ideally, the two limit cycles would have the same maximums on the horizontal axis and only change on the vertical axis.

Figure 4.14 shows the output of the phase based forcing function and the magnitude of $c$ and $d$ vs time.
**Figure 4.11:** Motor Angular Velocity $\omega_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.37$, $d_1 = -0.41$. The Final Values are $c_2 = 1.11$, $d_2 = -0.03$. The Initial Amplitude is $A_1 = 153$ rad/s, $A_2 = 148$ rad/s. The Initial Frequency is $\omega_1 = 103$ rad/s, and the Final Value is $\omega_2 = 81.6$ rad/s.

**Figure 4.12:** Motor Angular Acceleration $\dot{\omega}_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.37$, $d_1 = -0.41$. The Final Values are $c_2 = 1.11$, $d_2 = -0.03$. The Initial Amplitude is $A_1 = 1.6 \times 10^4$ rad/s, $A_2 = 1.25 \times 10^4$ rad/s. The Initial Frequency is $\omega_1 = 103$ rad/s, and the Final Value is $\omega_n = 81.6$ rad/s.
Figure 4.13: System Output in the Phase Plane During the Frequency Change Test. The Initial Values are $c_1 = 1.37$, $d_1 = -0.41$. The Final Values are $c_2 = 1.11$, $d_2 = -0.03$. The Initial Amplitude is $A_1 = 153$ rad/s, the Final Amplitude is $A_2 = 148$ rad/s. The Initial Frequency is $\omega_1 = 103$ rad/s, and the Final Value is $\omega_2 = 81.6$ rad/s.
Figure 4.14: On Top, Magnitude of the Forcing Function Output vs Time During the Test. On Bottom, Magnitude of $c$ and $d$ vs time. With $c_1 = 1.37$, $d_1 = -0.41$, $c_2 = 1.11$, and $d_2 = -0.03$
Changing the amplitude

The goal in this test is to maintain the frequency constant and change the amplitude of the oscillations. As in the previous section, using the identified parameters, values of $c$ and $d$ were calculated for the desired initial and final conditions changing the amplitude and maintaining the frequency constant. The desired initial conditions were $A_{d1} = 150$ rad/s and $\omega_{d1} = 80$ rad/s; and the final desired conditions were $A_{d2} = 100$ rad/s and $\omega_{d2} = 80$ rad/s. Calculating $c$ and $d$ as previously shown:

$c_1 = 1.09$, $c_2 = 0.732$, $d_1 = -0.3$, and $d_2 = -0.022$.

After running the system the actual values were $A_1 = 146$ rad/s, $\omega_1 = 80.55$ rad/s, $A_2 = 96.5$, and $\omega_2 = 79.53$. The error in amplitude was 3%. The error in frequency was 2%. The output of the system can be seen in fig. 4.15, it shows a smooth and fast transition. Figure 4.16 shows the motor acceleration vs time. Figure 4.17 shows the output of the system in the phase plane where it can be seen there are two defined limit cycles with clear different amplitudes. Figure 4.18 shows the output of the phase based forcing function and the magnitude of $c$ and $d$ vs time.
Figure 4.15: Motor Angular Velocity $\omega_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.09$, $d_1 = -0.3$. The Final Values are $c_2 = 0.732$, $d_2 = -0.022$. The Initial Amplitude is $A_1 = 146 \text{ rad/s}$, $A_2 = 96.5 \text{ rad/s}$. The Initial Frequency is $\omega_1 = 80.55 \text{ rad/s}$, and the Final Value is $\omega_n = 79.53 \text{ rad/s}$.

Figure 4.16: Motor Angular Acceleration $\dot{\omega}_M$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.09$, $d_1 = -0.3$. The Final Values are $c_2 = 0.732$, $d_2 = -0.022$. The Initial Amplitude is $A_1 = 1.3 \times 10^4 \text{ rad/s}$, $A_2 = 0.8 \times 10^4 \text{ rad/s}$. The Initial Frequency is $\omega_1 = 80.55 \text{ rad/s}$, and the Final Value is $\omega_n = 79.53 \text{ rad/s}$.
Figure 4.17: System Output in the Phase Plane During the Amplitude Change Test. The Initial Values are $c_1 = 1.09$, $d_1 = -0.3$. The Final Values are $c_2 = 0.732$, $d_2 = -0.022$. The Initial Amplitude is $A_1 = 146$ rad/s, the Final Amplitude is $A_2 = 96.5$ rad/s. The Initial Frequency is $\omega_1 = 80.55$ rad/s, and the Final Value is $\omega_2 = 79.53$ rad/s.
Figure 4.18: On Top, Magnitude of the Forcing Function Output vs Time During the Test. On Bottom, Magnitude of $c$ and $d$ vs Time. With $c_1 = 1.09$, $d_1 = -0.3$, $c_2 = 0.732$, and $d_2 = -0.022$
4.3 Oscillating a Pendulum

Pendulum

The theoretical findings were also used to oscillate a pendulum that was constructed as seen in Fig. 4.19, with an aluminum beam with length 42 cm, and a steel mass at the end of 100 g. From the equations of motion of a simple pendulum, the natural frequency of the system is \( \omega_n \approx \sqrt{g/l} = 4.83 \text{rad/s} \).

Control implementation

As shown in the previous section, the signal was processed using a PC-104 with a Sensoray IO card, and programmed using Simulink/ Real Time Workshop from MathWorks. The motor used was a 12 VDC model EYQF-33300-640 of Barber-Colman Co.

For the implementation, the system is modeled as:

\[
\ddot{\theta} + 2\zeta \omega_n \dot{\theta} + \omega^2 \theta = G \frac{c' \dot{\theta} + d' \omega \theta}{\sqrt{\dot{\theta}^2 + \omega^2 \theta^2}}
\]  

(4.8)

where \( G \) is the gain of the actuation system (motor, driver, and data acquisition), and \( c = Gc' \), \( d = Gd' \).
System Identification

To identify the system parameters a step test response was done to obtain the damping ratio and the natural frequency of the system. Figure 4.20 shows the results of the test. \( \omega_n \approx 4.882 \text{ rad/s}, \; \zeta = 0.06782. \)

![Figure 4.20: Step Test Response of the Pendulum \( \theta \) vs Time. The Overshoot is Approximately 80%](image)

The next step was to identify the value of the system gain \( G \) in equation 4.8. The system was run using \( c = 2, \; d = 0 \). The position of the pendulum is shown in figure 4.21; the velocity in figure 4.22. From the test, using the equation 4.4 \( G \) was found to be \( G = 0.465671 \).

Then three tests were realized, one to validate the identified parameters, one changing the amplitude and one changing the frequency. The values of desired frequency and amplitude where selected having in mind that the implementation of the system is limited by the bandwidth of the actuators.
Validation of system parameters

To validate the experimental values for the parameters of the system, a test was run to obtain the error between the desired and actual oscillation parameters. The desired
values for frequency and amplitude for this test were $A_d = 1 \text{ rad}, \omega_d = 5 \text{ rad/s}$. Using equation 4.4 values for $c$ and $d$ are found to be $c = 1.5411, d = -0.552$. The results of the test are shown in figures 4.23, 4.24, 4.25, and 4.26. The system output has a frequency of $\omega = 4.92 \text{ rad/s}$, and amplitude of $A = 1.018 \text{ rad}$. The error in the frequency values is 1.6%; the error in the amplitude values is 1.8%.

Figure 4.23: Pendulum Position $\theta$ During a Test with $c = 1.5411$, and $d = -0.552$. The Frequency of the Oscillations is $\omega = 4.92 \text{ rad/s}$. The Amplitude of the Oscillations is $A = 1.018 \text{ rad}$. 
Figure 4.24: Pendulum Velocity $\dot{\theta}$ During a Test with $c = 1.5411$, and $d = -0.552$.

Figure 4.25: System Output in the Phase Plane During a Test with $c = 1.5411$, and $d = -0.552$. The Frequency of the Oscillations is $\omega = 4.92$ rad/s. The Amplitude of the Oscillations is $A = 1.018$ rad. The trajectory Starts at (0, 0).
Figure 4.26: On Top, Magnitude of the Forcing Function Output vs Time During the Test with $c = 1.5411$, and $d = -0.552$. On Bottom, Magnitude of $c$ and $d$ vs Time.
Change of frequency

This test consisted in changing the values of $c$ and $d$ according to eq. 4.4 to change the frequency of the oscillations maintaining the amplitude at the same magnitude, for a frequency different from the natural frequency of the system $\omega \neq \omega_n$.

Using the identified parameters, values of $c$ and $d$ were calculated for desired initial and final conditions changing the frequency and maintaining the amplitude constant. The desired initial conditions were $A_{d1} = 1$ rad and $\omega_{d1} = 6$ rad/s; and the final desired conditions were $A_{d2} = 1$ rad and $\omega_{d2} = 5$ rad/s. Calculating $c$ and $d$ as previously shown: $c_1 = 1.849$, $c_2 = 1.5411$, $d_1 = -5.6744$, and $d_2 = -0.5521$.

After running the system the actual values were $A_1 = 1.07$ rad, $\omega_1 = 6.04$ rad/s, $A_2 = 1.06$, and $\omega_2 = 4.79$. The error in amplitude was 7%. The error in frequency was 4.2%. The output of the system can be seen in fig. 4.27, it shows a smooth and fast transition. Figure 4.28 shows the pendulum angular velocity $\theta$ vs time. Figure 4.29 shows the output of the system in the phase plane where it can be seen there are two defined limit cycles, Ideally, the two limit cycles would have the same maximums on the horizontal axis and only change on the vertical axis.

Figure 4.30 shows the output of the phase based forcing function and the magnitude of $c$ and $d$ vs time.
Figure 4.27: Pendulum Angular Position $\theta$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.849$, $d_1 = -5.6744$. The Final Values are $c_2 = 1.5411$, $d_2 = -0.5521$. The Initial Amplitude is $A_1 = 1.07$ rad, $A_2 = 1.06$ rad. The Initial Frequency is $\omega_1 = 6.04$ rad/s, and the Final Value is $\omega_2 = 4.79$ rad/s.

Figure 4.28: Pendulum Angular Velocity $\dot{\theta}$ vs Time for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.849$, $d_1 = -5.6744$. The Final Values are $c_2 = 1.5411$, $d_2 = -0.5521$. The Initial Amplitude is $A_1 = 1.07$ rad, $A_2 = 1.06$ rad. The Initial Frequency is $\omega_1 = 6.04$ rad/s, and the Final Value is $\omega_2 = 4.79$ rad/s.
Figure 4.29: System Output in the Phase Plane During the Frequency Change Test. The Initial Values are $c_1 = 1.849$, $d_1 = -5.6744$. The Final Values are $c_2 = 1.5411$, $d_2 = -0.5521$. The Initial Amplitude is $A_1 = 1.07$ rad, $A_2 = 1.06$ rad. The Initial Frequency is $\omega_1 = 6.04$ rad/s, and the Final Value is $\omega_2 = 4.79$ rad/s.
Figure 4.30: On Top, Magnitude of the Forcing Function Output vs Time During the Test. On Bottom, Magnitude of $c$ and $d$ vs Time. With $c_1 = 1.849$, $d_1 = -5.6744$, $c_2 = 1.5411$, and $d_2 = -0.5521$
Change of amplitude

This test consisted in changing the values of $c$ and $d$ according to eq. 4.4 to change the amplitude of the oscillations, maintaining the frequency at the same magnitude, for a frequency different from the natural frequency of the system $\omega \neq \omega_n$. As in the previous section, using the identified parameters, values of $c$ and $d$ were calculated for the desired initial and final conditions changing the amplitude and maintaining the frequency constant. The desired initial conditions were $A_{1d} = 1.2 \text{ rad}$ and $\omega_{1d} = 5 \text{ rad/s}$; and the final desired conditions were $A_{2d} = 0.8 \text{ rad}$ and $\omega_{2d} = 5 \text{ rad/s}$.

Calculating $c$ and $d$: $c_1 = 1.84$, $c_2 = 1.23$, $d_1 = -0.6625$, and $d_2 = -0.4416$.

After running the system the actual values were $A_1 = 1.31 \text{ rad}$, $\omega_1 = 4.8 \text{ rad/s}$, $A_2 = 0.69$, and $\omega_2 = 4.9$. The error in amplitude was 13.7 %. The error in frequency was 4 %. The output of the system can be seen in fig. 4.31, it shows a smooth transition. It is clear the exponential decay in the transitions as seen in equation 3.55. Figure 4.32 shows the pendulum angular velocity $\dot{\theta}$ vs time. Figure 4.33 shows the output of the system in the phase plane where it can be seen there are two defined limit cycles with clear different amplitudes. Figure 4.18 shows the output of the phase based forcing function and the magnitude of $c$ and $d$ vs time.
Figure 4.31: Pendulum Angular Position $\theta$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.84$, $d_1 = -0.6625$. The Final Values are $c_2 = 1.23$, $d_2 = -0.4416$. The Initial Amplitude is $A_1 = 1.31$ rad, $A_2 = 0.69$ rad. The Initial Frequency is $\omega_1 = 4.8$ rad/s, and the Final Value is $\omega_2 = 4.9$ rad/s.

Figure 4.32: Pendulum Angular Velocity $\dot{\theta}$ for a Change of Values of $c$ and $d$. The Initial Values are $c_1 = 1.84$, $d_1 = -0.6625$. The Final Values are $c_2 = 1.23$, $d_2 = -0.4416$. The Initial Amplitude is $A_1 = 1.31$ rad, $A_2 = 0.69$ rad. The Initial Frequency is $\omega_1 = 4.8$ rad/s, and the Final Value is $\omega_2 = 4.9$ rad/s.
Figure 4.33: System Output in the Phase Plane. The Initial Values are $c_1 = 1.84$, $d_1 = -0.6625$. The Final Values are $c_2 = 1.23$, $d_2 = -0.4416$. The Initial Amplitude is $A_1 = 1.31$ rad, $A_2 = 0.69$ rad. The Initial Frequency is $\omega_1 = 4.8$ rad/s, and the Final Value is $\omega_2 = 4.9$ rad/s.
Figure 4.34: On Top, Magnitude of the Forcing Function Output vs Time During the Test. On Bottom, Magnitude of $c$ and $d$ vs time. With $c_1 = 1.84$, $d_1 = -0.6625$, $c_2 = 1.23$, and $d_2 = -0.4416$
4.4 Transitory Response During Perturbation Recovery

This section analyzes the recovery behavior for both, the time based oscillator and the phase based oscillator, after a perturbation occurs. The main difference is the recovery trajectory in the phase plane. In the case of the phase based oscillator, the output of the system stays on the same side of the limit cycle. For example, after a perturbation, the system output is outside of the limit cycle, the response decreases until it reaches the limit cycle again. When the initial conditions are inside the limit cycle, the response slowly increases until it reaches the limit cycle. The response does not cross over the limit cycle.

For the time based oscillator, the trajectory of the output in the phase plane overshoots the amplitude of the limit cycle. The output crosses over the limit cycle. The response of both systems is presented in figures 4.36, 4.37, 4.38, and 4.39 for simulated results, and the response of the experimental system is presented in figures 4.40, 4.41, 4.42, and 4.43.

Recovery after perturbation using the phase based oscillator

When using the phase based oscillator, the oscillations grow or shrink exponentially until the limit cycle is reached. This behavior is non linear and described by eq.3.12. The direction of the field in the phase plane is given:

\[ f(x) \cdot \nabla V(x) = (\omega^2 - \omega_n^2)x\dot{x} - 2\zeta \omega_n \dot{x}^2 + \frac{c\dot{x}^2 + d\omega x\dot{x}}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  \hspace{1cm} (4.9)

The response can be approximated using eq. 3.55.

\[ A\omega = e^{-2\zeta \omega_n t} + \frac{c}{2\zeta \omega_n} \]  \hspace{1cm} (4.10)

The exponential decay has the form \( N(t) = N_0 e^{-Lt} \). \( L \) is the decay rate; and the time constant is \( \tau = 1/L \). Therefore the time constant for the amplitude of the oscillations
using the phased based oscillator is $\tau = \frac{1}{2\zeta \omega_n}$. Using only the exponential term of the equation it is possible to approximate the time as:

$$\alpha = \omega (A_f - A_0) e^{-2\zeta \omega_n t} \quad (4.11)$$

where $0 < \alpha << 1$, $A_f$ is the final amplitude of oscillations, $A_0$ is the initial oscillation amplitude, or the amplitude after the perturbation.

Equation 4.10 gives a good approximation of the response, however the simulation of the nonlinear equation 3.12 gives a better result. The frequency of the system during the transition does not change because the the values of $c$ and $d$ stay constant.

The frequency is given by equation 3.46:

$$\omega = -d \zeta \omega_n + \omega_n \sqrt{d^2 \zeta^2 + c^2} \quad (4.12)$$

**Recovery after perturbation using the time based oscillator**

For the time based oscillator, the recovery time is governed by the transition matrix $A$ of the system and the phase difference between the input and the position of the pendulum immediately after the perturbation. Given the system response in the time domain:

$$x(t) = e^{At}x_0 + \int_{t_0}^{t} e^{t-\tau}BU(\tau)d\tau \quad (4.13)$$

where $e^{At}x_0$ is the zero input response of the system and $\int_{t_0}^{t} e^{t-\tau}BU(\tau)d\tau$ is the zero state response. The zero input response is the transition of the system. Let the pendulum model be

$$\ddot{\theta} + 2\zeta \omega n \dot{\theta} + \omega_n^2 \theta = \beta \sin(\omega_i t) \quad (4.14)$$

where $\beta \sin(\omega_i t)$ is the external time based forcing function. The time constant of the system can be identified using the inverse of the Laplace representation of the transfer
function of the system for the zero input response (not considering the external input $\beta \sin(\omega_i t)$).

$$F(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ \hspace{1cm} (4.15)

Transforming to the time domain.

$$\theta(t) = \frac{e^{-\zeta\omega_n t}}{\omega_n \sqrt{1 - \zeta^2}} \sin \left( \omega_n \sqrt{1 - \zeta^2} t \right)$$ \hspace{1cm} (4.16)

Consequently, the time constant of the response is $\tau = \frac{1}{\zeta \omega_n}$, which is slower than the time constant of the phase based system, $\tau = \frac{1}{2 \zeta \omega_n}$. The recovery time for the time based oscillator depends on the respective time constant and the phase shift of the input with respect the pendulum motion. This phase shift determines the initial conditions for the zero input response.

$$x_0 = \begin{bmatrix} \theta(t_0) - \sin^{-1}(\omega_i t_0) \\ \dot{\theta}(t_0) - \cos^{-1}(\omega_i t_0) \end{bmatrix}$$ \hspace{1cm} (4.17)

In matrix form, the zero input response is

$$\begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} = x(t) = e^{At}x_0 = e^{\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} t}x_0$$ \hspace{1cm} (4.18)

Considering that for the pendulum $\zeta < 1$, the eigenvalues of the system are complex. Using the diagonalization property of the transition matrix it is possible to construct the response of the system summing the effect of each eigenvalue (sum of modal components). For the $j$th eigenvalue $\lambda_j = \sigma_j + i \omega_j$, where $i = \sqrt{-1}$, the zero input response is:

$$X(t) = \sum_{j=1}^{n} [r_j^H x_0] e^{\lambda_j t} q_j$$ \hspace{1cm} (4.19)
where \( q_j \) are the right eigenvectors that form \( Q \), and \( r_j \) are the left eigenvectors. The eigenvalues of \( A \) are \( \lambda_{1,2} = -\zeta_\omega n \pm \omega_n \sqrt{1 - \zeta^2} i \). In matrix form

\[
x(t) = e^{At}x_0 = Qe^\Lambda t R^H x_0
\]

(4.20)

where \( Q \) is the matrix of right eigenvectors, \( R^H \) is the conjugate transpose of the left eigenvectors matrix, and \( \Lambda \) is the diagonal matrix that contains the eigenvalues of \( A \). Therefore:

\[
x(t) = Q \begin{bmatrix} e^{(-\zeta_\omega n + \omega_n \sqrt{1 - \zeta^2}) t} & 0 \\ 0 & e^{(-\zeta_\omega n - \omega_n \sqrt{1 - \zeta^2}) t} \end{bmatrix} R^H x_0
\]

(4.21)

\[
Q = \begin{bmatrix} \frac{\zeta + \sqrt{1 - \zeta^2} i}{\omega_n} & \frac{\zeta - \sqrt{1 - \zeta^2} i}{\omega_n} \\ \frac{\omega_n i}{2 \sqrt{1 - \zeta^2}} & \frac{\omega_n i}{2 \sqrt{1 - \zeta^2}} \end{bmatrix}
\]

(4.22)

\[
R^H = Q^{-1} = \begin{bmatrix} \frac{\omega_n i}{2 \sqrt{1 - \zeta^2}} & \frac{\omega_n i}{2 \sqrt{1 - \zeta^2}} \\ -\frac{\omega_n i}{2 \sqrt{1 - \zeta^2}} & -\frac{\omega_n i}{2 \sqrt{1 - \zeta^2}} \end{bmatrix}
\]

(4.23)

Using the Euler’s formula.

\[
e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)
\]

(4.24)

\[
e^{-i\omega t} = \cos(\omega t) - i \sin(\omega t)
\]

Let

\[
M = Q e^{\Lambda t} R^H
\]

(4.25)

we get

\[
M = \begin{bmatrix} e^{-\zeta_\omega n t} \left[ \cos(\omega_n \sqrt{1 - \zeta^2} t) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) \right] & \frac{e^{-\zeta_\omega n t}}{\omega_n \sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) \\ -\frac{\omega_n e^{-\zeta_\omega n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) & -e^{-\zeta_\omega n t} \sin(\omega_n \sqrt{1 - \zeta^2} t) \end{bmatrix}
\]

(4.26)
Therefore the zero input response of the position of the pendulum is:

\[
\theta(t) = e^{-\zeta \omega_n t} \left[ \cos(\omega_n \sqrt{1 - \zeta^2} t) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) \right] x_0_1 \\
+ \frac{e^{-\zeta \omega_n t}}{\omega_n \sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t)x_0_2
\] (4.27)

The time constant is confirmed to be \( \tau = \frac{1}{\zeta \omega_n} \). As expected, the response of the value of \( \theta(t) \) is oscillatory with a different frequency than the input frequency, therefore the frequency during the transition is affected.

The zero state response is

\[
\int_{t_0}^{t} e^{t-\tau} B U(\tau) d\tau = \int_{t_0}^{t} \left[ \frac{e^{-\zeta \omega_n (t-\tau)} \sin(\omega_n \sqrt{1 - \zeta^2} (t-\tau)) \beta \sin(\omega_i \tau) d\tau}{\omega_n \sqrt{1 - \zeta^2}} \right] \\
- \frac{e^{-\zeta \omega_n (t-\tau)} \sin(\omega_n \sqrt{1 - \zeta^2} (t-\tau)) \beta \sin(\omega_i \tau) d\tau}{\sqrt{1 - \zeta^2}}
\] (4.28)

Therefore, the complete response is

\[
\begin{bmatrix}
\theta(t) \\
\dot{\theta}(t)
\end{bmatrix} = M \begin{bmatrix}
\theta(t_0) - \sin^{-1}(\omega_i t_0) \\
\dot{\theta}(t_0) - \cos^{-1}(\omega_i t_0)
\end{bmatrix} + \int_{t_0}^{t} \left[ \frac{e^{-\zeta \omega_n (t-\tau)} \sin(\omega_n \sqrt{1 - \zeta^2} (t-\tau)) \beta \sin(\omega_i \tau)}{\omega_n \sqrt{1 - \zeta^2}} \right] \\
- \frac{e^{-\zeta \omega_n (t-\tau)} \sin(\omega_n \sqrt{1 - \zeta^2} (t-\tau)) \beta \sin(\omega_i \tau)}{\sqrt{1 - \zeta^2}}
\right]
\] (4.29)

**Comparison**

For the phase based oscillator, the recovery time depends on how far the amplitude of the oscillations is from the limit cycle, and it is not affected by the phase shift since the system is in sync all the time. For the time based oscillator the recovery time depends on the phase shift of the system with respect the input signal. Based on the time constants, we can say the recovery time, after a perturbation, is shorter when using the phase based oscillator. See table 4.1, and figure 4.35.

The exponential decay equations for both systems are:

\[
\alpha = [\theta(t_0) - \sin^{-1}(\omega_i t_0)] e^{-\zeta \omega_n t}
\] (4.30)

\[
\alpha = [\omega(A_f - A_0)] e^{-2\zeta \omega_n t}
\]
Table 4.1: Comparison of the Transitory Responses of the Time Based Oscillator and the Phase Based Oscillator.

<table>
<thead>
<tr>
<th></th>
<th>Time constant</th>
<th>Initial condition of exponential decay</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time based oscillator</td>
<td>$\tau = \frac{1}{\zeta \omega_n}$</td>
<td>$\theta(t_0) - \sin^{-1}(\omega_i t_0)$</td>
<td>The frequency changes during the transition, it is a combination of $\omega_i$ from the input, and $\left(\omega_n \sqrt{1 - \zeta^2}\right)$, from the zero input response.</td>
</tr>
<tr>
<td>Phase based oscillator</td>
<td>$\tau = \frac{1}{2 \zeta \omega_n}$</td>
<td>$\omega(A_f - A_0)$</td>
<td>The frequency stays constant during the transition $\omega' = \frac{-d \zeta \omega_n + \omega_n \sqrt{d^2 \zeta^2 + c^2}}{c}$</td>
</tr>
</tbody>
</table>

To compare the recovery times in terms of the proportion of initial conditions let:

$$\alpha = \delta_P e^{-\zeta \omega_n t}$$

$$\alpha = \delta_A e^{-2 \zeta \omega_n t} \quad (4.31)$$

Where $\delta_P$ is the proportion of difference in phase between the initial conditions and the input normalized to $[0, 1]$. And $\delta_A$ is the proportion of difference in amplitude between the initial conditions and the final amplitude also normalized to $[0, 1]$. Figure 4.35 shows a color map of the shortest recovery time for different phase and amplitude initial conditions expressed as proportion, and using $\alpha = 0.01$. To increase the contrast, when the shorter recovery time was achieved with the time base oscillator, the time is shown as negative. It is shown that, in percentage terms, the phase based oscillator recovers in less time in most of the initial conditions. Solving $t$ from both equations in 4.31, and setting them equal we get the curve that separates the two
\[ \delta_P = \frac{\delta_A^2}{\alpha} \] (4.32)

Figure 4.35: Shorter Recovery Time for the Exponential Decay of the Time Based Oscillator and the Phase Based Oscillator. When the Time Was Smaller for the Time Based Oscillator it is Shown as Negative. The Values Used to Generate the Plot Were: \( \zeta = 0.06782, \omega_n = 4.88 \text{ rad/s}, \alpha = 0.01. \)
4.4.1 Simulation of Perturbation Recovery

The system was simulated in four different cases. In each case the response is compared using the phase oscillator controller and the equivalent time based forcing function. The system used for the simulations was: $\omega_n = 1$ rad/s, $\zeta = 0.5$. The results are shown in figures 4.36, 4.37, 4.38, and 4.39. In the phase plots, the phase oscillator controller produces a response that does not cross the limit cycle, meaning that there is no overshoot in the oscillations, the amplitude grows or declines until it reaches the limit cycle; on the contrary, the time based oscillator can produce significant overshoot in the amplitude. The plots of angular position vs time, show that the recovery time is slightly faster using the phase oscillator. In the power plots, the phase oscillator controller requires less power to recover from a perturbation; the over shoot caused by the time based oscillator causes an overshoot in the power requirements. The magnitude of the overshoot depends on the phase of the time based oscillator and the initial conditions as seen in the previous section. Since the phase oscillator controller is intrinsically in synchrony with the system, there is no phase difference between the system and the oscillations produced by the forcing function.
Figure 4.36: System Response for Initial Conditions Not on the Limit Cycle. In this case, the initial conditions are $\theta_0 = -0.1$ rad, $\dot{\theta}_0 = 0$ rad/s. In red, the response using the phase oscillator controller and in blue the response using the equivalent time based forcing function. The plot on top shows the response in the phase plane. The plot in the middle shows the angular position vs time. The plot at the bottom shows power in the system $P = \tau \omega$. 
Figure 4.37: System Response for Initial Conditions Not on the Limit Cycle. In This Case, the Initial Conditions are $\theta_0 = 3 \text{ rad}$, $\dot{\theta}_0 = 0 \text{ rad/s}$. In Red, the Response Using the Phase Oscillator Controller and in Blue the Response Using the Equivalent Time Based Forcing Function. The Plot on Top Shows the Response in the Phase Plane. The Plot in the Middle Shows the Angular Position vs Time. The Plot at the Bottom Shows Power in the System $P = \tau \omega$. 
Figure 4.38: System Response for Initial Conditions Not on the Limit Cycle. In this case, the initial conditions are $\theta_0 = 0$ rad, $\dot{\theta}_0 = 4$ rad/s. In red, the response using the Phase Oscillator Controller and in blue the response using the Equivalent Time Based Forcing Function. The plot on top shows the response in the Phase Plane. The plot in the middle shows the angular position vs time. The plot at the bottom shows power in the system $P = \tau \omega$. 

107
Figure 4.39: System Response for Initial Conditions Not on the Limit Cycle. In this case, the initial conditions are $\theta_0 = 0$ rad, $\dot{\theta}_0 = 2$ rad/s. In red, the response using the phase oscillator controller and in blue the response using the equivalent time-based forcing function. The plot on top shows the response on the phase plane. The plot in the middle shows the angular position vs time. The plot at the bottom shows power in the system $P = \tau \omega$. 
4.4.2 Real data of Perturbation Recovery

The same pendulum shown in section 4.3 was controlled with the phase based forcing function with constant values of $c$ and $d$, and with a sinusoidal time based forcing function. A external force was added to disturb the trajectory of the pendulum to determine the response. The results show how the system returns to the initial amplitude and frequency.

Figures 4.40 and 4.41 show the response to similar perturbations around $t=20$ s, and $t=40$ s. Figure 4.40 has the response of the system with the phase based forcing function and figure 4.41 has the output response using the time based sinusoidal forcing function. The response using the phase based controller is smoother. The recovery with the phase based oscillator is faster. However, the time based oscillator response approaches quicker to the desired steady oscillations, but it has an oscillatory response that takes more time to stabilize, (see equation 4.16).

In a similar way, figures 4.42 and 4.43 show the response to similar perturbations around $t=17$ s. Figure 4.42 has the response of the system with the phase based forcing function and figure 4.43 has the output response using the time based sinusoidal forcing function. Again, the response using the phase based controller is smoother while the response using the time based oscillator allows the motion to go above and below the desired amplitude.
Figure 4.40: Pendulum Angular Position $\theta$ vs Time Using the Phase Based Forcing Function. $c = 1.849$, $d = -5.6744$, $\omega = 6$. A External Force is Introduced at $t = 21$ s, and $t = 40$ s.

Figure 4.41: Pendulum Angular Position $\theta$ vs Time Using the Time Based Forcing Function. A External Force is Introduced at $t = 20$ s, and $t = 40$ s.
Figure 4.42: Pendulum Angular Position $\theta$ vs Time Using the Phase Based Forcing Function. $c = 1.849$, $d = -5.6744$, $\omega = 6$. Trajectory Perturbed from $t = 17s$ to $t = 20s$. 

Figure 4.43: Pendulum Angular Position $\theta$ vs Time Using the Time Based Forcing Function. Trajectory Perturbed from $t = 17s$ to $t = 19s$. 

111
For the results shown in Fig. 4.42. The parameters of the system are: $\omega = 6$ rad/s, $\omega_n = 4.88$ rad/s, $\zeta = 0.06782$. Therefore, $\tau = 1.51$ s. From the plot 4.42, the recovery time is close to $t = 10$ s. Considering that at the end of the perturbation the amplitude was $A_0 = 0.70$ rad, and the final value was $A_f = 1.07$ rad. The recovery time can be approximated as

$$\alpha = (1.07 - 0.75)e^{-0.6619t}$$  \hspace{1cm} (4.33)

The recovery time, using equation 4.33 and $\alpha = 0.005$ is $t = 6.50259$ s. The calculated recovery time using $\alpha = 0.0001$ is $t = 12.4129$ s. Simulating equation 3.12 in MATLAB, the recovery time was found to be $t = 7.703$ s for 97% amplitude; and $t = 14.5$ s for 100%.

For the time based oscillator shown in Fig. 4.43, considering a phase shift of $\pi/8$. The equation becomes

$$\alpha = \frac{\pi}{8}e^{-0.3311t}$$  \hspace{1cm} (4.34)

With this method the recovery time is $t = 9.4083$ s for $\alpha = 0.01745$ rad, that is approximately $1^\circ$. Therefore, if the phase shift is small enough the recovery time will be faster with the time based oscillator, if the shift angle is not small, the recovery time will be larger than with the phase based oscillator.

In conclusion, although, the time constant of the exponential decay is faster for the phase based oscillator, it cannot be said the phase based oscillator will respond faster for all the cases as compared to the time based oscillator. The recovery time using the phase based oscillator depends on how far the initial condition is from the limit cycle in the phase plane. The recovery time using the time based oscillator depends on the phase shift between the input and the initial conditions of the system.

Additionally, the response of the phase based oscillator does not overshoot the limit cycle, the amplitude of the oscillations increases and decreases exponentially,
and the frequency is not affected. These qualities make the system suitable to be used for periodic motion assistance. The time based oscillator overshoots the limit cycle, the amplitude goes back to a steady state value in a exponential oscillatory way, affecting the frequency during the transition.
In a real system, it is difficult to estimate the exact values of system parameters. In actual operation, depending upon changes in mass, stiffness and damping properties system parameters change. Sometimes due to changes in geometry, nonlinearity and operating region the controller that is designed to control the ideal system may or may not work.

In a wearable robotic device, a robot designed for a particular operator, may or may not fit the other operator correctly. Changes in loading, variations in physiological parameters, and usage can lead to changes in mounting configuration. In case of soldiers the variations could be significant and robot working correctly for one soldier may not work for other.

Thus, the question would be - Is the controller robust enough to accommodate the possible stochastic and time dependent variations? Would controller work reliably over intended operating period? The phase oscillator control discussed earlier can stabilize the "deterministic" system and drive it to periodic orbit.

The objective of this chapter is to investigate how robust the controller is if some of the system parameters are stochastic. First the stability conditions and bounds for a linear system with random perturbations is discussed and later this approach is
extended to a system with phase oscillator controller.

5.1 Bounds for Perturbation and Uncertainty

5.1.1 Linear System with Stochastic Perturbation

This method is adopted from [104], and modified to be used with the phase based forcing function. First consider the system.

\[
\ddot{x} + \left(\frac{b + b(t)}{m}\right) \dot{x} + \left(\frac{k + k(t)}{m}\right)x = 0 \tag{5.1}
\]

where \(b(t)\) and \(k(t)\) are the random perturbations in damping \(b\), and stiffness \(k\).

Rewriting equation 5.1 in state space form:

\[
\dot{x} = Ax + H(t)x \tag{5.2}
\]

where

\[
x = \{x, \dot{x}\}^T, A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}, H(t) = \begin{bmatrix} 0 & 1 \\ -k(t)/m & -b(t)/m \end{bmatrix}
\]

\(A\) is the transition matrix of the system, is asymptotically stable (controllable) with eigenvalues that have negative real parts. \(H(t)\) is a matrix of stochastic processes that represent random perturbations to the system and uncertainties. The stochastic processes in \(H(t)\) are measurable, stationary and ergodic; meaning that the samples average is constant over time instead of random variables. This model allows to find bounds for the perturbations that do not affect the general stability of the system.

Based on the theorem by Infante [105], it can be shown that for a symmetric positive semidefinite matrix \(P\) and some \(\epsilon > 0\),

\[
E \left\{ \lambda_{max}[(A + H(t))^T + P(A + H(t))P^{-1}] \right\} \leq -\epsilon \tag{5.3}
\]

where \(E\) is the expected value, and \(\lambda_{max}\) is the maximum eigenvalue of the matrix \(P\).

The details of the proof can be found in [105], and its generalization to time periodic systems in [104]. For clarity, an outline of proof is included here.
Consider a quadratic Lyapunov function $V(x) = x^T P x$. For the system in equation 5.2, define

$$
\lambda(t) = \frac{V(x)}{V(x)} = \frac{x^T[(A + H(t))^T P + P(A + H(t))]x}{x^T P x}
$$

(5.4)

Referencing [106], for the norm $\|x\|_p = (x^T P x)^{1/2}$, it can be shown that

$$
\frac{d}{dt} \log \|x\|_p = \frac{x^T[(A + H(t))^T P + P(A + H(t))]x}{x^T P x}
$$

(5.5)

Integrating 5.5 and dividing it by $t$ we get:

$$
\frac{\log \|x(t)\|_p - \log \|x(0)\|_p}{t} = \frac{1}{t} \int_0^t \frac{x^T[(A + H(s))^T P + P(A + H(s))]x}{x^T P x} ds
$$

(5.6)

if the left hand side of equation 5.5 is negative as $t$ approaches 0, it follows

$$
\lim_{t \to \infty} \|x(t)\|_p = 0
$$

(5.7)

Therefore the system is asymptotically stable, consequently eq. 5.8 is a stability condition for the system in eq. 5.2.

$$
\lim_{t \to \infty} \left[ \frac{1}{t} \int_0^t \frac{x^T[(A + H(s))^T P + P(A + H(s))]x}{x^T P x} ds \right] < 0
$$

(5.8)

From the properties of pencil of quadratic forms it is known that the quotient 5.9 satisfies the condition in eq. 5.10.

$$
\frac{x^T[(A + H(s))^T P + P(A + H(s))]x}{x^T P x}
$$

(5.9)

$$
\lambda_{\text{min}}(s) \leq \lambda^T(s) [(A + H(s))^T P + P(A + H(s))] \lambda(s) \leq \lambda_{\text{max}}(s)
$$

(5.10)

where $\lambda_{\text{min}}(s)$, and $\lambda_{\text{max}}(s)$ are the minimum and maximum characteristic values of the matrix $[(A + H(s))^T P + P(A + H(s))] P^{-1}$. Equation 5.10 can be rewritten as:

$$
\lambda_{\text{min}}\{(A + H(t))^T P + P(A + H(t))\} P^{-1} \leq \lambda(t) \leq
\lambda_{\text{max}}\{(A + H(t))^T P + P(A + H(t))\} P^{-1}
$$

(5.11)
If the elements of the matrix \((A + H(t))\) are stationary, ergodic random processes; the condition in Equation 5.8 holds, and consequently, Equation 5.3 holds. From Equations 5.4 and 5.11:

\[
V[x(t)] = V[x(t_0)]e^{\int_{t_0}^{t_0} \lambda(r)dr} \equiv V[x(t_0)]e^{(t-t_0)\int_{t_0}^{t} \lambda(r)dr} \tag{5.12}
\]

It can be observed that, if \(E\{\lambda(t)\} \leq \epsilon\) for some \(\epsilon > 0\), \(V[x(t)]\) is bounded and that \(V[x(t)] \to 0\) as \(t \to \infty\). This is the condition imposed by the inequality given by Equation 5.3, which proves the results. For the detailed proof, refer to [105].

### 5.1.2 Extension to System with Phase Oscillator Controller

To use this theorem for the phase based oscillator we add the nonlinear phase based forcing function. It can be noted that \(f(x, \dot{x})\) satisfies the sector condition, therefore Equation 5.3 becomes:

\[
E\{\lambda_{\text{max}}[(A + H(t))^T + P(A + H(t))P^{-1}]} \leq -\|f(x, \dot{x})\| - \epsilon \tag{5.13}
\]

where \(\|f(x, \dot{x})\|\) is the norm of the nonlinear phase based forcing function.

\[
\|f(x, \dot{x})\| = \left\| \begin{array}{c} c\dot{x}_2 + d\omega x_1 \\ \sqrt{\dot{x}_2^2 + \omega^2 x_1^2} \end{array} \right\| \tag{5.14}
\]

To find the bounds, using Equation 5.13, consider general representation of a second order dynamic system (cf. equation 5.1):

\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{5.15}
\]

In matrix form:

\[
A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \tag{5.16}
\]
Let the matrix of stochastic processes be:

\[ H = \begin{bmatrix}
0 & 0 \\
-h_1(t) & -h_2(t)
\end{bmatrix} \]  
(5.17)

The quadratic Lyapunov function is

\[ V = x^T P x \]  
(5.18)

where P is symmetric positive semidefinite.

\[ P = P^T \]  
(5.19)

The gradient of the Lyapunov function is:

\[ \dot{V} = \dot{x}^T P x + x^T P \dot{x} \]  
(5.20)

Using the theorem 5.13, let

\[ D = (A + H)^T + P(A + H)P^{-1} \]  
(5.21)

Given that \((A + H)^T = A^T + H^T\), equation 5.21 becomes:

\[ D = A^T + H^T + P(A + H)P^{-1} \]  
(5.22)

Plugging in equations 5.16, and 5.17, into eq. 5.22:

\[ D = \begin{bmatrix}
0 & -\omega_n^2 \\
1 & -2\zeta \omega_n
\end{bmatrix} + \begin{bmatrix}
0 & -h_1(t) \\
0 & -h_2(t)
\end{bmatrix} + P \begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta \omega_n
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
-h_1(t) & -h_2(t)
\end{bmatrix} P^{-1} \]  
(5.23)

From previous sections, P is known to be (see equation 3.10)

\[ P = \begin{bmatrix}
\frac{\omega_n^2}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix} \]  
(5.24)
plugging in eq. 5.24 into eq. 5.23

\[
D = \begin{bmatrix}
0 & -\omega_n^2 - h_1(t) \\
1 & -2\zeta\omega_n - h_2(t)
\end{bmatrix} + \begin{bmatrix}
\frac{\omega_n^2}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
-\omega_n^2 - h_1(t) & -2\zeta\omega_n - h_2(t)
\end{bmatrix} \begin{bmatrix}
\frac{4}{\omega^2} & \frac{1}{2} & 0 \\
0 & \frac{\omega_n^2}{2}
\end{bmatrix}
\]

(5.25)

Multiplying the last terms of eq. 5.25

\[
D = \begin{bmatrix}
0 & -\omega_n^2 - h_1(t) \\
1 & -2\zeta\omega_n - h_2(t)
\end{bmatrix} + \begin{bmatrix}
0 & \omega^2 \\
-\frac{h_1(t) + \omega_n^2}{\omega^2} & -2\zeta\omega_n - h_2(t)
\end{bmatrix}
\]

(5.26)

Adding the terms of eq. 5.26

\[
D = \begin{bmatrix}
0 & \omega^2 - \omega_n^2 - h_1(t) \\
\frac{\omega^2 - \omega_n^2 - h_1(t)}{\omega^2} & -4\zeta\omega_n - 2h_2(t)
\end{bmatrix}
\]

(5.27)

The eigenvalues of D are found setting \( \text{det}(\lambda I - D) = 0 \).

\[
0 = \lambda^2 + (4\zeta\omega_n + 2h_2(t))\lambda - \frac{(\omega_n^2 + h_1(t) - \omega^2)^2}{\omega^2}
\]

(5.28)

Solving equation 5.28

\[
\lambda_{1,2} = -2\zeta\omega_n - h_2(t) \pm \sqrt{\frac{4}{\zeta\omega_n + \frac{h_2(t)}{2}} + \frac{(\omega_n^2 + h_1(t) - \omega^2)^2}{\omega^2}}
\]

(5.29)

From eq. 5.13, it can be said

\[
E\{\lambda_{\max}(D)\} \leq -\|f(x, \dot{x})\| - \epsilon
\]

(5.30)

If \( \epsilon = 0 \)

\[
E\{\lambda_{\max}(D)\} \leq -\sqrt{c^2 + d^2}
\]

(5.31)

Plugging in equation 5.29 into eq. 5.31

\[
E\left\{-2\zeta\omega_n - h_2(t) \pm \sqrt{\frac{4}{\zeta\omega_n + \frac{h_2(t)}{2}} + \frac{(\omega_n^2 + h_1(t) - \omega^2)^2}{\omega^2}}\right\} \leq -\sqrt{c^2 + d^2}
\]

(5.32)
Adding $E\{h_2(t)\} + 2\zeta \omega_n$ to both sides of eq. 5.32

$$E \left\{ \sqrt{\frac{4}{\omega^2} \left( \zeta \omega_n + \frac{h_2(t)}{2} \right)^2 + \frac{(\omega_n^2 + h_1(t) - \omega^2)^2}{\omega^2}} \right\} \leq E\{h_2(t)\} + 2\zeta \omega_n - \sqrt{c^2 + d^2}$$

(5.33)

Ussing Schwartz'z inequality, $(E\{f(t)\})^2 \leq E\{f^2(t)\}$.

$$E \left\{ 4\left( \zeta \omega_n + \frac{h_2(t)}{2} \right)^2 + \frac{(\omega_n^2 + h_1(t) - \omega^2)^2}{\omega^2} \right\} \leq \left( E\{h_2(t)\} + 2\zeta \omega_n - \sqrt{c^2 + d^2} \right)^2$$

(5.34)

Expanding

$$E \left\{ \frac{\omega^2 (4\zeta^2 \omega_n^2 + 4\zeta \omega_n h_2(t) + h_2^2(t)) + h_1^2(t) + 2h_1(t)(\omega_n^2 - \omega^2) + (\omega_n^2 - \omega^2)^2}{\omega^2} \right\} \leq$$

$$E\{h_2^2(t)\} + 4\zeta^2 \omega_n^2 + c^2 + d^2 - 2E\{h_2^2(t)\}\sqrt{c^2 + d^2} - 4\zeta \omega_n \sqrt{c^2 + d^2} + 4\zeta \omega_n E\{h_2(t)\}$$

(5.35)

Rearranging the terms

$$E \left\{ h_1^2(t) + 2h_1(t)(\omega_n^2 - \omega^2) + h_2^2(t)\left[ \omega^2 + 2\omega \sqrt{c^2 + d^2} \right] + \omega^2 h_2(t) \right\} \leq$$

$$- (\omega_n^2 - \omega^2)^2 + \omega^2[c^2 + d^2 - 4\zeta \omega_n \sqrt{c^2 + d^2}]$$

(5.36)

This system is guaranteed to be stable for this expected value of uncertainty.

Figure 5.1 shows the bounds for the system with $\omega_n = 4.88$ rad/s, and $\zeta = 0.06782$, that are the same parameters as the pendulum used in Chapter 4. The values of $c$ and $d$ are obtained using eq. 3.45.

$$d = A(\omega_n^2 - \omega^2)$$

$$c = 2\zeta A \omega \omega_n$$

(5.37)
Figure 5.1: Bound for \( E \{ h_1^2(t) + 2h_1(t)(\omega_n^2 - \omega^2) + h_2^2(t)[\omega^2 + 2\omega^2\sqrt{c^2 + d^2}] + \omega^2h_2(t) \} \) that Guarantee the Stability of the System. \( \omega_n = 4.88 \text{ rad/s}, \zeta = 0.06782 \).
5.2 Handling Perturbations and Uncertainty with Lyapunov Redesign

It can be seen that the phase oscillator controller controls the linear system with random perturbations. Sometimes there are un-modeled nonlinearities in system that could destabilize the system. So, a controller is designed to handle un-modeled nonlinearities that will work in conjunction with the phase oscillator controller.

Lyapunov redesign is used to construct a robust controller using a bounding function [104]; here the method is used to design an additional controller to handle the nonlinearities that were not considered in the model, noise, and perturbations. Lyapunov Redesign guarantees stability. The method consists in a series of transformations that allow to calculate a controller considering bounding conditions for the perturbation and noise. The resulting controller is Lyapunov stable.

Consider the system

\[
\ddot{x} = -2\zeta\omega_n \dot{x} - \omega_n^2 x + \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \quad (5.38)
\]

In matrix form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta\omega_n
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
cx_2 + d\omega x_1
\end{bmatrix} \quad (5.39)
\]

\[
\dot{x} = Ax + f(x, \dot{x}) \quad (5.40)
\]

Adding random noise and a controller

\[
\dot{x} = Ax + f(x, \dot{x}) + B\Delta u + f(x, t) + B\nu^* \quad (5.41)
\]

where \(\Delta u\) is random noise, \(f(x, t)\) is un-modeled perturbations and nonlinearities, \(\nu^*\) is the controller, and \(B\) is the input vector of dimension \([n \times 1]\).

The uncertainty is assumed to satisfy the matching condition; this means the controller can handle the uncertainty because the term appear in the input equation.
$B$ can be factored out of the last three terms. Let:

$$\dot{x} = Ax + f(x, \dot{x}) + B(\Delta u + \nu^* + B^g f(x, t)) \quad (5.42)$$

where $B^g$ is the generalized inverse of $B$, or pseudo-inverse, that satisfies

$$B^g B = I_n \quad (5.43)$$

Equation 5.42 contains the system, the controller, uncertainty and the nonlinear forcing function $f(x, \dot{x})$. The transformed system now is used to robustify the controller with the Lyapunov redesign. Assume the noise and the uncertainty/perturbation satisfies the bounding conditions:

$$\|\Delta u + B^g f(x, t)\| \leq \sigma(x)\|x\| + \rho\|\nu^*\| \quad (5.44)$$

$$\sigma(x) > 0; 0 \leq \rho < 1 \quad (5.45)$$

where $\sigma(x)$ is a bounding function. Therefore equation 5.44 establishes that the norm of non linearities and noise has to be less than the bounding function plus the controller.

Using a quadratic Lyapunov function:

$$V(x) = x^T P x \quad (5.46)$$

$$P = P^T$$

The gradient of the Lyapunov function is

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} \quad (5.47)$$

Plugging in eq. 5.42 into eq. 5.47.

$$\dot{V} = [x^T A^T + f^T(x, \dot{x}) + (\nu^* + \Delta u + B^g f(x, t)) B^T] P x$$

$$+ x^T P [Ax + f(x, \dot{x}) + B(\nu^* + \Delta u + B^g f(x, t))] \quad (5.48)$$
Grouping terms.

$$\dot{V}(x) = x^T [A^T P + PA] x + 2x^T P[f(x, \dot{x})] + 2x^T PB(v^* + \Delta u + B^g f(x, t)) \quad (5.49)$$

Let

$$A^T P + PA = -Q \quad (5.50)$$

Therefore Eq. 5.49 becomes

$$\dot{V}(x) = -x^T Q x - x^T I x + 2x^T P[f(x, \dot{x})] + 2x^T PB(v^* + \Delta u + B^g f(x, t)) \quad (5.51)$$

To simplify, let

$$w^T = 2x^T PB \quad (5.52)$$

Equation 5.51 becomes

$$\dot{V}(x) = -x^T Q x - x^T I x + w^T B^g f(x, \dot{x}) + w^T (v^* + \Delta u + B^g f(x, t)) \quad (5.53)$$

Separating the last term of the equation 5.53, and making it negative semidefinite,

$$w^T (v^* + \Delta u + B^g f(x, t)) \leq 0 \quad (5.54)$$

To apply the bounding conditions the equation 5.54 is rewritten separating the controller term $v^*$

$$w^T v^* + w^T (\Delta u + B^g f(x, t)) \leq 0 \quad (5.55)$$

This can be rewritten as

$$w^T v^* + w^T (\Delta u + B^g f(x, t)) \leq w^T v^* + |w^T (\Delta u + B^g f(x, t))| \quad (5.56)$$

Since $\|AB\| \leq \|A\|\|B\|$, eq. 5.56 implies:

$$w^T v^* + w^T (\Delta u + B^g f(x, t)) \leq w^T v^* + \|w\|\|\Delta u + B^g f(x, t)\| \quad (5.57)$$
Applying the bounding conditions of equation 5.44 we get.

\[ w^T \nu^* + w^T (\Delta u + B^g f(x, t)) < w^T \nu^* + \|w\| (\sigma(x) \|x\| + \rho \|\nu^*\|) \] (5.58)

Reincorporating the term \(-xIx\), the equation 5.58 becomes:

\[ -x^T Ix + w^T \nu^* + w^T (\Delta u + B^g f(x, t)) \]

\[ < -x^T Ix - w^T \nu^* + \|w\| (\sigma(x) \|x\| + \rho \|\nu^*\|) \] (5.59)

Let \( \nu^* = -w \kappa \), where \( \kappa(t) > 0 \).

\[ -x^T Ix + w^T \nu^* + w^T (\Delta u + B^g f(x, t)) \]

\[ < -x^T Ix - w^T w \kappa + \|w\| (\sigma(x) \|x\| + \rho \|w\kappa\|) \] (5.60)

Defining the norm of a vector as \( \|a\|^2 = a^T I a \)

\[ -x^T Ix + w^T \nu^* + w^T (\Delta u + B^g f(x, t)) \]

\[ < -\|x\|^2 - \kappa \|w\|^2 + \|w\| (\sigma(x) \|x\| + \rho \kappa \|w\|) \] (5.61)

Reordering the terms on the right hand side of the equation

\[ -x^T Ix + w^T \nu^* + w^T (\Delta u + B^g f(x, t)) \]

\[ < -\|w\|^2 \kappa (1 - \rho) + \|w\| \|x\| \sigma(x) - \|x\|^2 \] (5.62)

Choose

\[ \kappa = \frac{1}{4(1-\rho)} \sigma^2(x) \] (5.63)

Plugging in eq. 5.63 into eq. 5.62

\[ -x^T Ix + w^T \nu^* + w^T (\Delta u + f(x, t)) \]

\[ < -\frac{1}{4} \|w\|^2 \sigma^2(x) + \|w\| \|x\| \sigma(x) - \|x\|^2 \] (5.64)

Rearranging the right hand side of eq. 5.64.

\[ -x^T Ix + w^T \nu^* + w^T (\Delta u + f(x, t)) \]

\[ < -\left( \frac{1}{2} \|w\| \sigma(x) - \|x\| \right)^2 \] (5.65)
Hence, the term is negative definite and the controller $\kappa$ guarantees the stability of the system for the established bounding conditions. The controller $\nu^*$ is

$$\nu^* = -w\kappa = - \left( \frac{\sigma^2(x)}{4(1 - \rho)} \right) w$$

(5.66)

As before, using the transformation $w^T = 2x^TPB$

$$\nu^* = -2 \left( \frac{\sigma^2(x)}{4(1 - \rho)} \right) B^TPx$$

(5.67)

Therefore the controller $\nu^*$ affects the direction of the vector field, reducing the effect of the noise and perturbations that satisfy the bounding conditions, but also reducing the amplitude of the limit cycle. The system in eq. 5.41 gets the form:

$$\dot{x} = Ax + f(x, \dot{x}) + B\Delta u + f(x, t) - 2B \left( \frac{\sigma^2(x)}{4(1 - \rho)} \right) B^TPx$$

(5.68)

Effectively the controller $\nu^*$ changes the direction of the vector field of the system reducing the amplitude and shape of the limit cycle. This can be used to handle noise and small perturbations. Practically, to use this controller with the phase based oscillator, the norm of the bounding function $\|\sigma(x)\|$ has to be a small proportion of $\sqrt{c^2 + d^2}$. If it is not small, it will reduce considerably the amplitude of the limit cycle.

5.2.1 Simulation Example: Using the Controller Obtained by Lyapunov Redesign on a Pendulum.

A simulation was generated applying equation 5.68 to the original system and using the phase based forcing function. For the system $\omega_n = 1$, $\zeta = 0.5$. The results are shown in figures 5.2 and 5.3. Using $\rho = 0.1$ and;

Input vector

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(5.69)
Modeled noise

\[ \Delta u = [0.1 \sin(50\pi t)]x_1 + [0.1 \sin(50\pi t) + 0.7 \sin(70\pi t)]x_2 \] (5.70)

Nonlinearity

\[ f(x, t) = -[0.03 \sin(\pi t)]x_1^2 \] (5.71)
\[ \sigma(x) = \frac{0.03x_1 - 0.03x_2}{\sqrt{x_1^2 + x_2^2}} \] (5.72)

And, as shown in Chapter 3 (eq. 3.10), the \( P \) matrix for the quadratic Lyapunov function is:

\[
P = \begin{bmatrix}
\frac{\omega^2}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\] (5.73)

In the results it is clear that the use of the added controller reduces the variability of the response in the phase plane. The system response gets closer to the ideal values. Without the controller \( \nu^* \) the position varies with range \( r = 0.2 \text{ rad} \). With controller the range is \( r = 0.1 \text{ rad} \). The controller \( \nu^* \) is contra resting some of the effect of the noise.
Figure 5.2: System Response in the Phase Plane. In Red the System Without Noise and Without Non Linearities. In Blue the System with Added Noise $\Delta u$ and Non Linearity $f(x,t)$. In Green the Response of the System with Noise $\Delta u$ and Non Linearity $f(x,t)$ and the Controller $\nu^*$. 

Figure 5.3: Close View of the System Response in the Phase Plane. In Red the System Without Noise and Without Non Linearities. In Blue the System with Added Noise $\Delta u$ and Non Linearity $f(x,t)$. In Green the Response of the System with Noise $\Delta u$ and Non Linearity $f(x,t)$ and the Controller $\nu^*$. 

128
5.2.2 Implementation: Reducing the Trajectory Variability Around the Limit Cycle.

The controller was added to the pendulum control system described in Chapter 4. For this case, the controller \( \nu^* \) takes the form:

\[
\nu^* = -2 \left( \frac{\sigma^2(x)}{4(1 - \rho)} \right) B^T P x
\]

\[
\nu^* = -2 \left( \frac{0.05x_1 - 0.05x_2}{4(1 - 0.1)\sqrt{x_1^2 + x_2^2}} \right) \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

Therefore the forcing function becomes:

\[
f(x) = \frac{1.849x_2 - 0.6625(6)x_1}{\sqrt{x_2^2 + 36x_1^2}} + \left( \frac{0.05x_1 - 0.05x_2}{3.6\sqrt{x_1^2 + x_2^2}} \right) x_2
\]

For the system

\[
\ddot{\theta} + 2\zeta \omega_n \dot{\theta} + \omega_n^2 \sin(\theta) + \Delta = f(x)
\]

Figure 5.4 shows the response of the pendulum in the phase plane without adding the controller \( \nu^* \), and Fig. 5.5 shows the response with the controller. Both trajectories start at \((0,0)\), and in both cases the system was run for 90 seconds. In the plots it can be seen that the use of the controller produces a better defined limit cycle, the dispersion of the trajectory is lower when using the controller \( \nu^* \).

Figures 5.6 and 5.7 show close up views of the pendulum response in the phase plane where is clear the difference in variability. Without the controller \( \nu^* \) the position varies with range \( r = 0.08 \) rad. With controller the range is \( r = 0.04 \) rad.
Figure 5.4: System Response in the Phase Plane Without Controller $v^*$. The Trajectory Starts at (0, 0). The System was Run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5 \text{ rad/s}$.

Figure 5.5: System Response in the Phase Plane with Controller $v^*$. The Trajectory Starts at (0, 0). The System was Run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5 \text{ rad/s}$.
Figure 5.6: Close View of the System Response in the Phase Plane Without Controller $\nu^*$. The Trajectory Starts at $(0, 0)$. The System was run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5$ rad/s.

Figure 5.7: Close View of the System Response in the Phase Plane with Controller $\nu^*$. The Trajectory Starts at $(0, 0)$. The System was Run for 90 Seconds. The Values for the Phase Oscillator Controller $f(x, \dot{x})$ Were $c = 1.849$, $d = -0.6625$, and $\omega = 5$ rad/s.
5.2.3 Using the Lyapunov Redesign Controller with Nonlinear Pendulum Models.

Consider the nonlinear system:

\[
I \ddot{\theta} + b \dot{\theta} + k \sin(\theta) = \frac{c \dot{\theta} + d \omega \theta}{\sqrt{\dot{\theta}^2 + \omega^2 \theta^2}} + \nu^* \tag{5.77}
\]

where \( I \) is the inertia, \( b \) is the damping constant, and \( k \) is the stiffness. Figure 5.8 shows the limit cycles for the nonlinear model with and without using the controller \( \nu^* \). \( \omega_n = \sqrt{\frac{k}{I}} \), and \( \zeta = \frac{b}{2 \omega_n} \).

Choosing \( \rho = 0.4 \) and

\[
\sigma(\theta) = \frac{2}{\dot{\theta}^2 + 1} \tag{5.78}
\]

The controller is

\[
\nu^* = -2 \left( \frac{\sigma^2(\theta)}{4(1 - \rho)} \right) B^T P \dot{\theta} \\
\nu^* = -0.8 \left( \frac{1}{2(\dot{\theta}^2 + 1)^2} \right) \dot{\theta} \tag{5.79}
\]

The system was simulated adding Gaussian noise \( \Delta u = f(x||\mu = 0, \sigma^2 = 1) \). Figures 5.8, and 5.9 show the response on the phase plane for the nonlinear model with and without using the controller \( \nu^* \). The amplitude of the limit cycle is reduced when adding the controller \( \nu^* \), but it also reduces the variability.
Figure 5.8: Nonlinear Model Response in the Phase Plane. In Green, the System with $\Delta = 0$ and $\nu^* = 0$. In Blue, the System with Added Noise $\Delta u$. In Red, the Response of the System with Noise $\Delta u$ and the Controller $\nu^*$. $c = 1.849$, $d = -0.6625$, $\omega = 5$, $\omega_n = 4.88$, and $\zeta = 0.0678$.

Figure 5.9: Close View of the Nonlinear Model Response in the Phase Plane. In green, the System with $\Delta = 0$ and $\nu^* = 0$. In Blue, the System with Added Noise $\Delta u$. In red, the Response of the System with Noise $\Delta u$ and the Controller $\nu^*$. $c = 1.849$, $d = -0.6625$, $\omega = 5$, $\omega_n = 4.88$, and $\zeta = 0.0678$. 

133
Consider the simplified nonlinear system

\[ I \ddot{\theta} + b \dot{\theta} + k \left( \theta - \frac{\theta^3}{6} \right) = \frac{c \dot{\theta} + d \omega \theta}{\sqrt{\theta^2 + \omega^2 \theta^2}} + \nu^* \]  

(5.80)

The system was simulated adding Gaussian noise \( \Delta u = f(x\|\mu = 0, \sigma^2 = 1) \). Figures 5.9, and 5.11 show the response on the phase plane for the nonlinear simplified model with and without using the controller \( \nu^* \). The amplitude of the limit cycle is reduced when adding the controller \( \nu^* \), but it also reduces the variability.

**Figure 5.10:** Simplified Nonlinear Model Response in the Phase Plane. In green, the System with \( \Delta = 0 \) and \( \nu^* = 0 \). In Blue, the System with Added Noise \( \Delta u \). In Red, the Response of the System with Noise \( \Delta u \) and the Controller \( \nu^* \). \( c = 1.849, d = -0.6625, \omega = 5, \omega_n = 4.88 \), and \( \zeta = 0.0678 \).
Figure 5.11: Close View of the Simplified Nonlinear Model Response in the Phase Plane. In green, the System with $\Delta = 0$ and $\nu^* = 0$. In Blue, the System with Added Noise $\Delta u$. In Red, the Response of the System with Noise $\Delta u$ and the Controller $\nu^*$. $c = 1.849$, $d = -0.6625$, $\omega = 5$, $\omega_n = 4.88$, and $\zeta = 0.0678$. 
5.3 Robust Stability Analysis of the Phase Based Oscillator

5.3.1 Small Gain Theorem

The small gain theorem is a tool to evaluate the stability robustness of feedback systems [107]. This theorem gives conservative results, when the theorem conditions are met it can be said there are sufficient conditions for the system to be stable [99, 107].

Consider the system in Fig 5.12.

\[ ||S_1|| \cdot ||S_2|| < 1 \] (5.81)

for any induced norm. In other form if \( ||S_2|| \leq M \) for some \( M > 0 \), then the system is stable for \( ||S_1|| \leq \frac{1}{M} \). Where \( S_1 \) and \( S_2 \) are stable time invariant systems.

The Small Gain Theorem says that the output of the system is stable if

\[ ||S_1|| \cdot ||S_2|| \leq 1 \]

for any induced norm. In other form if \( ||S_2|| \leq M \) for some \( M > 0 \), then the system is stable for \( ||S_1|| \leq \frac{1}{M} \). Where \( S_1 \) and \( S_2 \) are stable time invariant systems.

For the stability analysis consider that \( S_2 \) is the phase based forcing function and \( S_1 \) represents a system with a parametric uncertainty. The uncertainty, perturbation, or nonlinearities no considered in the model, can be represented as multiplicative, additive, divisive, and feedback signal; the stability of the system is robust if the small gain theorem can be met for all the models.

The phase based oscillator is:

\[ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \] (5.82)
Consider the controller is the nonlinear phase based forcing function

\[ S_2 = f(x, \dot{x}) = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  \hspace{1cm} (5.83)

The infinity norm of a system is the maximum gain of the system over all the frequencies, therefore,

\[ ||S_2||_\infty = \sqrt{c^2 + d^2} \]  \hspace{1cm} (5.84)

The system will be stable even when uncertainty or perturbations are present when

\[ ||S_1||_\infty \leq \frac{1}{\sqrt{c^2 + d^2}} \]  \hspace{1cm} (5.85)

5.3.2 Uncertainty Models

Additive uncertainty/perturbation

\[ \begin{array}{c}
\text{\textbf{\text{Figure 5.13: Block Diagram of the Model of Additive Uncertainty.}}}
\end{array} \]

Consider an additive uncertainty as shown in Fig. 5.13. \( P \) is the plant and \( \Delta \) is a stable perturbation (i.e. it vanishes on time). The transfer function of the system becomes \( S_1 = P + \Delta \). The plant is the general second order differential equation

\[ P = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]  \hspace{1cm} (5.86)

Therefore

\[ ||S_1||_\infty = \left\| \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \Delta \right\|_\infty \leq \frac{1}{\sqrt{c^2 + d^2}} \]  \hspace{1cm} (5.87)

Using the triangle inequality \( ||a|| + ||b|| \geq ||a + b|| \).

\[ \left\| \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right\|_\infty + ||\Delta||_\infty \leq \frac{1}{\sqrt{c^2 + d^2}} \]  \hspace{1cm} (5.88)
\[ \|\Delta\|_\infty \leq \frac{1}{\sqrt{c^2 + d^2}} - \left\| \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right\|_\infty \] (5.89)

The infinity norm can is the maximum gain value of the system. In the case of the second order system, this value is \( \|P\|_\infty = 2\zeta\sqrt{1 - \zeta^2} \). Therefore the max allowed infinity norm of the uncertainty transfer function to have robust stability for an additive perturbation is

\[ \|\Delta\|_\infty \leq \frac{1}{\sqrt{c^2 + d^2}} - 2\zeta\sqrt{1 - \zeta^2} \] (5.90)

If the system satisfies this inequality the stability is guaranteed for the additive case.

**Feedback uncertainty/perturbation**

![Block Diagram of the Model of Feedback Uncertainty](image)

**Figure 5.14:** Block Diagram of the Model of Feedback Uncertainty.

Consider a feedback uncertainty as shown in Fig. 5.13. As before, P is the plant and \( \Delta \) is a stable perturbation. The transfer function of the system becomes \( S_1 = \frac{P}{1 - P\Delta} \).

\[ \|S_1\|_\infty = \left\| \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2 - \Delta} \right\|_\infty \leq \frac{1}{\sqrt{c^2 + d^2}} \] (5.91)

If the uncertainty transfer function is scalar; using the general form of second order systems, define \( \omega_T = \sqrt{\omega_n^2 - \Delta} \), \( 2\zeta\omega_n = 2\zeta_T\omega_T \), therefore \( \zeta_T = \frac{\zeta\omega_n}{\sqrt{\omega_n^2 - \Delta}} \).

\[ 2 \left( \frac{\zeta\omega_n}{\sqrt{\omega_n^2 - \Delta}} \right) \sqrt{1 - \left( \frac{\zeta\omega_n}{\sqrt{\omega_n^2 - \Delta}} \right)^2} \leq \frac{1}{\sqrt{c^2 + d^2}} \] (5.92)
Rearranging the inequality.

\[ \frac{2\zeta \omega_n \sqrt{1 - \zeta^2 \omega_n^2}}{\sqrt{\omega_n^2 - \Delta}} \leq \frac{1}{\sqrt{c^2 + d^2}} \]  

(5.93)

\[ 2\zeta \omega_n \sqrt{1 - \zeta^2 \omega_n^2} \leq \frac{\sqrt{\omega_n^2 - \Delta}}{\sqrt{c^2 + d^2}} \]  

(5.94)

Then, for the case of uncertainty in the feedback, the stability is guaranteed for:

\[ \left(2\zeta \omega_n \sqrt{1 - \zeta^2 \omega_n^2} \sqrt{c^2 + d^2}\right)^2 - \omega_n^2 \geq \Delta \]  

(5.95)

**Multiplicative uncertainty/perturbation**

When the perturbation is modeled as multiplicative the transfer function of the system becomes \( S_1 = P(1 + \Delta) \). The block diagram is shown in Fig. 5.15.

![Block Diagram](image)

**Figure 5.15:** Block Diagram of the Model of Multiplicative Uncertainty.

\[ \|S_1\| = \left\| \frac{1 + \Delta}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right\| < \frac{1}{\sqrt{c^2 + d^2}} \]  

(5.96)

\[ (1 + \Delta)2\zeta \sqrt{1 - \zeta^2} \leq \frac{1}{\sqrt{c^2 + d^2}} \]  

(5.97)

Rearranging the equation.

\[ 1 + \Delta \leq \frac{1}{2\zeta \sqrt{1 - \zeta^2 \sqrt{c^2 + d^2}}} \]  

(5.98)

Then, for the case of multiplicative uncertainty, the stability is guaranteed for:

\[ \Delta \leq \frac{1}{2\zeta \sqrt{1 - \zeta^2 \sqrt{c^2 + d^2}}} - 1 \]  

(5.99)
Divisive uncertainty/perturbation

In this case the transfer function of the system becomes \( S_1 = \frac{P}{1-\Delta} \).

\[
\begin{align*}
S_1 &= \frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(1 - \Delta)} \\
\text{Therefore, the infinity norm of the system has to be:} \\
&= \left\| \frac{1 - \Delta}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \right\|_\infty \leq \frac{1}{\sqrt{c^2 + d^2}} \\
\text{Rearranging the equation:} \\
&= \frac{2\zeta \sqrt{1 - \zeta^2}}{1 - \Delta} \leq \frac{1}{\sqrt{c^2 + d^2}} \\
&= 2\zeta \sqrt{1 - \zeta^2} \sqrt{c^2 + d^2} \leq 1 - \Delta
\end{align*}
\] (5.100)

Then, for the case of divisive uncertainty, the stability is guaranteed for:

\[
2\zeta \sqrt{1 - \zeta^2} \sqrt{c^2 + d^2} - 1 \geq \Delta
\] (5.104)

Since there exists a bound for each uncertainty model, and the small gain theorem is conservative, it can be said that the stability of the phase based oscillator is robust.
Table 5.1: Stability Bounds for the Phase Based Oscillator for Each Uncertainty Model, Using the Small Gain Theorem.

<table>
<thead>
<tr>
<th>Model</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive</td>
<td>$|\Delta|_\infty \leq \frac{1}{\sqrt{c^2+d^2}} - 2\zeta \sqrt{1-\zeta^2}$</td>
</tr>
<tr>
<td>Feedback</td>
<td>$\Delta \leq \left(2\zeta \omega_n \sqrt{1-\zeta^2} \omega_n^2 \sqrt{c^2+d^2}\right)^2 - \omega_n^2$</td>
</tr>
<tr>
<td>Multiplicative</td>
<td>$\Delta \leq \frac{1}{2\zeta \sqrt{1-\zeta^2} \sqrt{c^2+d^2}} - 1$</td>
</tr>
<tr>
<td>Divisive</td>
<td>$\Delta \leq 2\zeta \sqrt{1-\zeta^2} \sqrt{c^2+d^2} - 1$</td>
</tr>
</tbody>
</table>

5.3.3 Stability Analysis of a Nonlinear Pendulum with the Phase Based Forcing Function Using the Small Gain Theorem.

Consider the model formed by a nonlinear representation of a pendulum motion, the phase based forcing function, and an external torque function of time.

$$ I\ddot{\theta} + b \dot{\theta} + k \sin \theta = \frac{c\dot{\theta} + d\omega \dot{\theta}}{\sqrt{\dot{\theta}^2 + \omega^2 \theta^2}} + \tau(t) $$ (5.105)

where $I$ is the inertia, $b$ is the damping constant, $k$ is the stiffness, and $\tau(t)$ is an external torque of the form $\tau(t) = a_0 + a_1 \sin(\omega_1 t + \eta_1) + a_2 \sin(\omega_2 t + \eta_2) + \ldots + a_n \sin(\omega_n t + \eta_n)$. Separating the system and rearranging terms we get:

$$ S_1 = \ddot{\theta} + \frac{b}{I} \dot{\theta} + \frac{k}{I} \sin \theta $$

$$ S_2 = \frac{c\dot{\theta} + d\omega \dot{\theta}}{\sqrt{\dot{\theta}^2 + \omega^2 \theta^2}} + \tau(t) $$ (5.106)

Introducing uncertainty $\Delta$ as an additive term to the phase based forcing function.

$$ S_2 = \frac{c\dot{\theta} + d\omega \dot{\theta}}{\sqrt{\dot{\theta}^2 + \omega^2 \theta^2}} + \tau(t) + \Delta $$ (5.107)

$$ \|S_2\|_\infty = \left\| \frac{c\dot{\theta} + d\omega \dot{\theta}}{\sqrt{\dot{\theta}^2 + \omega^2 \theta^2}} + \tau(t) + \Delta \right\|_\infty $$ (5.108)
Using the triangle inequality $\|a\| + \|b\| \geq \|a + b\|$. 

$$||S_2||_\infty \geq \left\| \frac{c\theta + d\omega \theta}{\sqrt{\theta^2 + \omega^2 \theta^2}} \right\|_\infty + \|\tau(t)\|_\infty + \|\Delta\|_\infty$$  

(5.109)

$$||S_2||_\infty \geq \sqrt{c^2 + d^2} + \|\tau(t)\|_\infty + \|\Delta\|_\infty$$  

(5.110)

According to the small gain theorem, to guarantee stability of the system the infinity norm of $S_1$ has to be:

$$||S_2||_\infty \leq \frac{1}{||S_1||_\infty}$$  

(5.111)

Therefore

$$\sqrt{c^2 + d^2} + \|\tau(t)\|_\infty + \|\Delta\|_\infty \leq \frac{1}{||S_1||_\infty}$$  

(5.112)

$$\|\Delta\|_\infty \leq \frac{1}{||S_1||_\infty} - \sqrt{c^2 + d^2} - \|\tau(t)\|_\infty$$  

(5.113)

$||u||_\infty = sup|u(t)|$, because $S_1$ is nonlinear it is not possible to use the Laplace transform to obtain a transfer function to then get its norm [108]. Solving numerically in Matlab, Figure 5.17 shows the gain response vs amplitude and frequency. The response is close to the the linear system, and the gain is $<< 1$. The norm of the linear approximation of the system is $2\zeta \sqrt{1 - \zeta^2}$. The norm $||\tau||_\infty$ is the maximum amplitude of the signal. Therefore, according to the small gain theorem, the stability bound 5.113 can exist if $||\tau(t)||$ is small enough.
Figure 5.17: DC Gain of the Nonlinear Model of the Pendulum vs Amplitude and Frequency of the Input. The Parameters Used for the Simulation were: $\zeta = 0.5$, $\omega_n = 4.66$ rad/s.
ASSISTING OSCILLATORY MOTION USING THE PHASE BASED FORCING FUNCTION

6.1 Periodic Motion

This chapter explores the use of the nonlinear phase based oscillator to model periodic motion and to assist this motion. Examples will include the assistance of hip and knee motion. Consider the general second order dynamic system:

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \]  
(6.1)

To move the system, consider an external excitation \( E(t) \). The system becomes:

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = E(t) \]  
(6.2)

If the system has a periodic trajectory with no offset, it can be represented with the phase based oscillator.

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  
(6.3)

therefore

\[ E(t) = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  
(6.4)

If the trajectory of the system is periodic but not purely sinusoidal, with an offset, it can be approximated as:

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{c\dot{x} + d\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + f \]  
(6.5)

where \( f \) is a constant component of the excitation \( E(t) \). From Chapter 3, it is known that, if the trajectory of the system has the form \( x = A\sin(\omega t) \), the values of \( c \), and
When the trajectory is periodic but not purely sinusoidal, it can be represented with the Fourier series expansion, 

\[ x = a_0/2 + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t), \]

from equations 3.39, and 3.40, the values of \( c, d, \) and \( f \) are:

\[
d = R \frac{\omega^2}{\omega} \left[ \omega_n^2 - \frac{\omega^2 [\sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m^2 + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m^2]}{\sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m} \right] (6.7)
\]

\[
c = R \frac{\omega}{\omega} \left[ \sum_{m=1}^{\infty} a_m \omega^2 (a_0^2 + \sum_{m=1}^{\infty} b_m m^2) \right] - \sum_{m=1}^{\infty} a_m \left[ \omega_n^2 - \frac{\omega^2 [\sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m^2 + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m^2]}{\sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m} \right] (6.8)
\]

\[
f = \frac{\omega^2}{2} a_0 - \frac{\omega^2}{2} \left[ \sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} a_m m^2 + \sum_{m=1}^{\infty} b_m \sum_{m=1}^{\infty} b_m m^2 \right] (6.9)
\]

where:

\[
R = \sqrt{\left( -\sum_{m=1}^{\infty} a_m m \omega \sin(m\omega t) + \sum_{m=1}^{\infty} b_m m \omega \cos(m\omega t) \right)^2 + \left( \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t) \right)^2} (6.10)
\]

When there is no constant component, then \( f = 0. \)

Oscillatory motion of a rigid body, including the motion of human limbs, can be approximated with equation 6.5 and equations 6.7, 6.8, and 6.9. From the definitions shown in Chapter 3, the phase based oscillator is:

\[
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = c \sin \phi + d \cos \phi (6.11)
\]
where the values of $c$ and $d$ change over time depending on the trajectory. For simplicity we use $c$ and $d$ instead of $c(t)$ and $d(t)$, but these terms change over time when the trajectory is not purely sinusoidal. $c$ and $d$ change proportionally to the amplitude and the shape of the trajectory. It is proposed to assist periodic motion using wearable robots reducing the magnitude of torque exerted by the human by applying an assistive force governed by $c$ and $d$.

### 6.2 Assisting Periodic Motion

To assist periodic motion of a human limb, let this motion be represented as a nonlinear phase based oscillator.

\[
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{c \dot{x} + d \omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + f = E(t) \quad (6.12)
\]

where $E(t)$ is the torque/inertia exerted by the human to control the trajectory of the limb. It has been shown that the human modulates the torque applied to the joints when some kind of assistance is provided to have the same output total torque as when no assistance is provided [32]. The magnitude of the assisting torque can not be too large, because it is assumed the human has control over the trajectory of the limb. If the assistance is too large, the exoskeleton will be in charge of generating the trajectory and this is a different problem. Also, in this work it is proposed to assist the system only using $\sin \phi$ and $\cos \phi$, therefore the constant component $f$ is not considered for assistance.

Consider the model:

\[
E(t) = c \sin \phi + d \cos \phi + f \quad (6.13)
\]

\[
E(t) = (c_h + c_a) \sin \phi + (d_h + d_a) \cos \phi + f = E_{\text{human}}(t) + E_{\text{assist}}(t)
\]

where $c = c_h + c_a$, $d = d_h + d_a$, the torque to produce the motion is divided between
the human and the external assistance torque.

\[ E_{\text{human}}(t) = c_h \sin \phi + d_h \cos \phi + f \]
\[ E_{\text{assist}}(t) = c_a \sin \phi + d_a \cos \phi \]
\[ E(t) = E_{\text{human}}(t) + E_{\text{assist}}(t) = \frac{(c_h + c_a)\dot{x} + (d_h + d_a)\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} + f \] (6.15)

This approach simplifies the control of the wearable robot to assist the motion because \( c_a \sin \phi + d_a \cos \phi \) do not necessarily have to be proportional to \( c \sin \phi + d \cos \phi \). If these are proportional, it will reduce the torque exerted by the human during all time and therefore reduce the power. However, to get \( c_a \sin \phi + d_a \cos \phi \) proportional to \( c \sin \phi + d \cos \phi \) requires the estimation in real time of the values of \( c, d, \) and \( \omega \). Estimating these values is as complicated as the algorithms used in the current literature to assist periodic motion. It requires the knowledge of the harmonic components of \( x \) in real time. Also, using the angle of the phase of the system to assist the motion guarantees, the assisting torque is in synchrony with the system, and the control signal is bounded.

We propose the use of the nonlinear phase based forcing function with arbitrary values for \( c_a \) and \( d_a \) based on the findings from Chapter 3 to assist periodic motion. To control a wearable robot to assist the limbs it is only required to know \( x(t), \dot{x}(t), \) and an approximation of \( \omega(t) \). \( \omega(t) \) is the frequency of the first harmonic. This will reduce the amplitude and maximum of torque required from the human, but not for all time. The advantage is that the power peaks are reduced with a simple control algorithm.

\( \theta, \) and \( \dot{\theta} \) are the variables of interest for a rotational systems. The angular position and angular velocity can be easily calculated. Using these two values provides a forcing function that is only one sampling period behind the system. When using a microcontroller/microprocessor with a small sampling period this delay can be
neglected.

\[ \dot{\theta}(t) = \frac{\theta(t) - \theta(t - T)}{T_s} \quad (6.16) \]

where \( T_s \) is the sampling period.

If the correct value of \( \omega \) is not known, the controller will produce an assistance torque that is in synchrony in frequency, but with different trajectory. \( \omega \) affects the shape of the limit cycle of \( E_{\text{assist}} \). Figure 6.1 shows the phase plot representation of two limit cycles, one produced with \( \omega x, \dot{x} \), and the second one produced with \( \tilde{\omega} x, \dot{x} \), where \( \tilde{\omega} \) is the approximate value of \( \omega \), \( \omega \neq \tilde{\omega} \). It can be seen that even if the value of \( \tilde{\omega} \) is not close to \( \omega \) when the phase angle is \( \pi/2 \) or \( 3\pi/2 \), both limit cycles match.

![Phase plot of two limit cycles](image)

**Figure 6.1:** Two Limit Cycles are Shown, One With \( \omega x \) on the Horizontal Axis and the Second One with \( \tilde{\omega} x \) on the Horizontal Axis. The Systems Match when \( x = 0 \).

### 6.2.1 Phase Angle Error

**Error caused by the frequency value**

In practice, the calculated values of \( \sin \phi \) and \( \cos \phi \) will have an error caused by the difference between the actual frequency of the motion \( \omega \), and the value used in the non linear forcing function. The forcing function is ideally:

\[ E(t)_{\text{assist}} = \frac{c_a \dot{x} + d_a \omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \quad (6.17) \]
In practice, it becomes:

\[ E(t)_{\text{assist}} = \frac{c_a \dot{x} + d_a \ddot{\omega}x}{\sqrt{\dot{x}^2 + \ddot{\omega}^2 x^2}} \]  \hspace{1cm} (6.18)

Where \( \tilde{\omega} \) is the approximate value of \( \omega \). Figure 6.2 shows a graphic representation of the difference between the two angles. The one at \( \omega x, \dot{x} \) is the actual system’s phase angle defined as \( \phi \). The angle with the error is define as \( \tilde{\phi} \). And the error is \( \psi \).

![Diagram showing the difference between actual and approximated phase angles](image)

**Figure 6.2:** Graphic Representation of the Error Caused by the Difference Between \( \omega \) and \( \tilde{\omega} \). The Error is Defined at \( \psi \), the System’s Phase Angle is \( \phi \) and the Approximated Phase Angle is \( \tilde{\phi} \).

The approximated phase angle is

\[ \tilde{\phi} = \phi + \psi \]  \hspace{1cm} (6.19)

As defined before:

\[ \sin \phi = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  \hspace{1cm} (6.20)

\[ \cos \phi = \frac{\omega x}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  \hspace{1cm} (6.21)

From figure 6.2, \( \cos \psi \) and \( \sin \psi \) are:

\[ \sin \psi = \frac{x(\omega - \tilde{\omega}) \sin \phi}{\sqrt{\dot{x}^2 + \omega^2 x^2}} \]  \hspace{1cm} (6.22)

\[ \therefore \sin \psi = \frac{x\dot{x}(\omega - \tilde{\omega})}{\sqrt{\dot{x}^2 + \omega^2 x^2} \sqrt{\dot{x}^2 + \tilde{\omega}^2 x^2}} \]  \hspace{1cm} (6.23)
\[
\cos \psi = \frac{\sqrt{x^2 + \omega^2 x^2} - x(\omega - \tilde{\omega}) \cos \phi}{\sqrt{x^2 + \tilde{\omega}^2 x^2}} \\
\therefore \cos \psi = \frac{\dot{x}^2 + \omega \tilde{\omega} x^2}{\sqrt{\dot{x}^2 + \omega^2 x^2} \sqrt{\dot{x}^2 + \tilde{\omega}^2 x^2}} 
\] (6.24)

The sine and cosine of the approximated angle are:

\[
\sin \tilde{\phi} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \tilde{\omega}^2 x^2}} 
\] (6.26)

\[
\cos \tilde{\phi} = \frac{\tilde{\omega} x}{\sqrt{\dot{x}^2 + \tilde{\omega}^2 x^2}} 
\] (6.27)

Therefore the error in the sine and cosine is:

\[
e_{\sin} = \sin \phi - \sin \tilde{\phi} = \frac{(\sqrt{x^2 + \omega^2 x^2} - \sqrt{\dot{x}^2 + \omega^2 x^2}) \dot{x}}{\sqrt{x^2 + \tilde{\omega}^2 x^2} \sqrt{\dot{x}^2 + \tilde{\omega}^2 x^2}} 
\] (6.28)

\[
e_{\cos} = \cos \phi - \cos \tilde{\phi} = \frac{(\omega \sqrt{x^2 + \omega^2 x^2} - \tilde{\omega} \sqrt{\dot{x}^2 + \omega^2 x^2}) x}{\sqrt{x^2 + \tilde{\omega}^2 x^2} \sqrt{\dot{x}^2 + \tilde{\omega}^2 x^2}} 
\] (6.29)

Consequently \( \sin \phi = \sin \tilde{\phi} = 1 \), for \( x = 0 \). \( \sin \phi = \sin \tilde{\phi} = 0 \), for \( \dot{x} = 0 \). It can be said, \( \sin \phi \approx \sin \tilde{\phi} \), when \( x \approx 0 \), and \( \dot{x} \approx 0 \). And \( \cos \phi = \cos \tilde{\phi} = 1 \) for \( \dot{x} = 0 \). \( \cos \phi = \cos \tilde{\phi} = 0 \) when \( x = 0 \). Figure 6.3 shows the trajectory of \( \sin \tilde{\phi} \) when using \( \tilde{\omega} = \omega \), \( \tilde{\omega} = 0.5 \omega \), and \( \tilde{\omega} = 2 \omega \), shown in blue, red and green respectively. The trajectory of the angle differs but converge at \( \tilde{\phi} = \phi = 0, \pi/2, \pi, 3\pi/4 \). Figure 6.4 shows the results for \( \cos \tilde{\phi} \). The trajectory used for these examples was \( x = 2 \sin(2t) + \cos(4t + 1), \dot{x} = 4 \cos(2t) - \sin(4t + 1) \). Consequently, if the assistance is provided with a discrete actuator like a piston; activating the actuator at any of these points \( \tilde{\phi} = \phi = 0, \pi/2, \pi, 3\pi/4 \) guarantees that the angle error is minimum.

**Error caused by the offset value**

The lack of offset in the position values also introduces error in the calculated values of \( \sin \phi \) and \( \cos \phi \). This is important to consider in practice given that, in some applications, it is not possible to have the value of the offset component of the position

150
Figure 6.3: The Position $x = 2 \sin(2t) + \cos(4t + 1)$ is Shown in Black. The Velocity $\dot{x} = 4 \cos(2t) - \sin(4t + 1)$ is Shown in Dotted Black Line. The Sine of the System was Plotted Using $\sin \tilde{\phi} = \frac{\dot{x}}{\sqrt{x^2 + \omega^2 x^2}}$. The Blue Line Shows the Output for $\tilde{\omega} = \omega$ ($\tilde{\omega} = 2 \text{ rad/s}$). The Red Line Shows the Output for $\tilde{\omega} = 0.5 \omega$ ($\tilde{\omega} = 1 \text{ rad/s}$). The Green Line Shows the Output for $\tilde{\omega} = 2 \omega$ ($\tilde{\omega} = 4 \text{ rad/s}$).

signal. This occurs when the position is calculated integrating the acceleration of the system. What happens is the values of $x$ and $\dot{x}$ obtained by the sensor are in a different coordinate system than the one expected to be used. Therefore there is an error caused in the calculated value of the phase angle.

Consider a system with a trajectory represented as $x = 2 + 2 \sin(2t) + \cos(4t + 1)$. If the sensors can not measure the offset in the signal, then the measured trajectory is $x = 2 \sin(2t) + \cos(4t + 1)$, meaning the term $a_0$ of the Fourier series representation is zero. This has an effect in the phase angle of the system, therefore in the calculated assistance. Figure 6.5 shows position and the limit cycle for both trajectories, in blue with no offset ($a_0 = 0$), and in red with an offset ($a_0 = 2$). The offset changes the
Figure 6.4: The Position $x = 2 \sin(2t) + \cos(4t + 1)$ is Shown in Black. The Velocity $\dot{x} = 4 \cos(2t) - \sin(4t + 1)$ is Shown in Dotted Black Line. The Cosine of the System was Plotted Using $\cos \tilde{\phi} = \frac{\tilde{\omega} \dot{x}}{\sqrt{\tilde{\omega}^2 + \dot{x}^2}}$. The Blue Line Shows the Output for $\tilde{\omega} = \omega$ ($\tilde{\omega} = 2$ rad/s). The Red Line Shows the Output for $\tilde{\omega} = 0.5 \omega$ ($\tilde{\omega} = 1$ rad/s). The Green Line Shows the Output for $\tilde{\omega} = 2 \omega$ ($\tilde{\omega} = 4$ rad/s).

shape of the trajectory of the phase angle, see Figures 6.6, and 6.7. If the sine of the phase angle is used to assist the periodic motion when the position has an offset, then the points when $\sin \phi = 1$ (maximums) do not coincide with the maximum value of the velocity $\dot{x}$ as seen in Figure 6.6. This does not occur with the cosine, as seen in Figure 6.7. However the trajectory changes for both cases. Assisting periodic motion using the sine and cosine of the phase angle calculated with a position signal with no offset will have a trajectory different from the one intended.
Figure 6.5: On Left, Plot of the Position vs Time. In Blue \( x = 2 \sin(2t) + \cos(4t + 1) \) (No Offset). In Red \( x = 2 + 2 \sin(2t) + \cos(4t + 1) \). Right, Plot of the Limit Cycle. In Blue the Limit Cycle with No Offset, and in Red the Limit Cycle With an Offset.

Figure 6.6: Sine of the Phase Angle \( \sin \phi = \frac{\dot{x}}{\sqrt{x^2 + \omega^2x^2}} \). In blue, \( \sin \phi \) is Shown Calculated with No Offset in the Position Signal, \( x = 2 \sin(2t) + \cos(4t + 1) \). In Red, \( \sin \phi \) is Shown Calculated with Offset in the Position Signal \( x = 2 + 2 \sin(2t) + \cos(4t + 1) \). In Solid Black is Shown the Position and in Dotted Black is Shown the Velocity.
Figure 6.7: Cosine of the Phase Angle $\cos \phi = \frac{\omega x}{\sqrt{x'^2 + \omega^2 x^2}}$. In Blue, $\cos \phi$ is Shown Calculated with No Offset in the Position Signal, $x = 2 \sin(2t) + \cos(4t + 1)$. In red, $\sin \phi$ is Shown Calculated with Offset in the Position Signal $x = 2 + 2 \sin(2t) + \cos(4t + 1)$. In Solid Black is Shown the Position and in Dotted Black is Shown the Velocity.
Consider the phase plot in Figure 6.8, it shows two limit cycles, one with an offset and one without it. It shows the systems phase angle $\phi$, the phase angle with error $\tilde{\phi}$, and the angle error $\psi$.

![Figure 6.8: Graphic Representation of the Error Caused by the Difference in Offset in $x$. The Error is Defined at $\psi$, the System’s Phase Angle is $\phi$ and the Phase Angle with the Error is $\tilde{\phi}$.]

From figure 6.8, the sine and cosine of the approximated angle with offset error are:

$$\sin \tilde{\phi} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \tilde{\omega}^2(x-a_0)^2}} \quad (6.30)$$

$$\cos \tilde{\phi} = \frac{\tilde{\omega}x}{\sqrt{\dot{x}^2 + \tilde{\omega}^2(x-a_0)^2}} \quad (6.31)$$

Therefore the error in the sine and cosine is:

$$e_{\sin} = \sin \phi - \sin \tilde{\phi} = \frac{(\sqrt{\dot{x}^2 + \tilde{\omega}^2(x-a_0)^2} - \sqrt{\dot{x}^2 + \omega^2x^2})\dot{x}}{\sqrt{\dot{x}^2 + \omega^2x^2}\sqrt{\dot{x}^2 + \tilde{\omega}^2(x-a_0)^2}} \quad (6.32)$$

$$e_{\cos} = \cos \phi - \cos \tilde{\phi} = \frac{(\omega \sqrt{\dot{x}^2 + \omega^2(x-a_0)^2} - \tilde{\omega} \sqrt{\dot{x}^2 + \tilde{\omega}^2}x + \tilde{\omega}a_0 \sqrt{\dot{x}^2 + \tilde{\omega}^2}x)}{\sqrt{\dot{x}^2 + \omega^2x^2}\sqrt{\dot{x}^2 + \tilde{\omega}^2(x-a_0)^2}} \quad (6.33)$$

As seen in Figure 6.6, the error in the sine value is zero only when the $\dot{x} = 0$, or $\sin \phi = 0$. For the cosine the error is zero for $\dot{x} = 0$, meaning the error is zero for $\cos \phi = \pm 1$ as shown in Figure 6.7.
6.2.2 Assisting Periodic Motion with Constant Values of c and d.

From the results shown in Chapter 3, it is known that $c > 0$ is necessary to have oscillations, $d > 0$ produces oscillations with frequency $\omega < \omega_n$, and $d < 0$ produces oscillations with frequency $\omega > \omega_n$. To assist periodic motion with an unknown trajectory it is proposed to use $c_a > 0$, $d_a > 0$ for $\omega < \omega_n$, and $d_a < 0$ for $\omega > \omega_n$. Equation 3.72 repeated here for simplicity:

$$c \sin \phi + d \cos \phi = \sqrt{c^2 + d^2} \cos \left( \phi - \tan^{-1} \frac{c}{d} \right)$$  \hspace{1cm} (6.34)

It is known that the ratio of $c$ and $d$ produces a phase shift. Figure 6.9 shows the plot of the inverse tangent where it can be seen that the phase shift term $(-\tan^{-1} \frac{c}{d})$ is positive for $d < 0$, and it is negative for $d > 0$, given that $c > 0$. This means the phase shift is positive if the frequency of the oscillations is higher than the natural frequency of the system ($\omega > \omega_n$) and it is negative for the contrary case.

![Inverse Tangent](image)

**Figure 6.9:** Inverse Tangent.

Assisting the system motion using values of $c_a$ and $d_a$ not proportional to the values of $c$ and $d$ of the system produces an assisting torque that has the same frequency of the system’s torque, but is not perfectly in phase. We believe that using constant values of $c_a$ and $d_a$, although is not optimal, it helps to assist the motion of the system. Using values of $c_a \geq 0$, and any value for $d_a$ will help the motion at least
during part of the cycle. The assisting torque will never be completely out of phase if the correct sign is used for the value of $d_a$, therefore it will reduce the torque and power peaks of the system.

In the specific case of assisting the human motion, usually the frequency of the motion will be less than the natural frequency of the system $\omega < \omega_n$ therefore the best values for $c_a$ and $d_a$ will be in the ranges $c_a > 0$, and $d_a > 0$. In the next section it is shown that using a combination of positive values for $c_a$ and $d_a$ produces better results than using only $c_a$ or $d_a$.

Figure 6.10: On Left, Joint Trajectories of the Hip, Knee and Ankle During Normal Gait Cycle in the Sagittal Plane. On right, Sagittal Plane Internal Joint Moments (Nm/Kg) During a Single Gait Cycle of Right Hip (Extensor Moment Positive), Knee (Extensor Moment Positive), and Ankle (Plantarflexor Moment Positive) [6].

157
Assisting the knee

From the data reported in [6] and shown in Figure 6.10 it was obtained a truncated Fourier series representation of the angle and the torque of the knee. These models are Equations 6.35 and 6.38 respectively. From the data reported in [109], for a male with mass \(m = 80.5\) Kg, the calculated distance of the center of mass of the leg and foot from the knee is \(h = 0.2362\) m. The moment of inertia of the leg and foot around the knee is \(I_{knee} = 0.3349\) Kg m\(^2\). The trajectory of the knee angle is:

\[
\theta(t) = 0.266 + 0.2722 \cos(2\pi t + 1.7787) + 0.2181 \cos(4\pi t - 2.29) + 0.0483 \sin(6\pi t - 1.39181)\text{[rad]}
\]

(6.35)

Therefore:

\[
\dot{\theta}(t) = -0.532 \sin(2\pi t + 1.7787) - 0.8726 \sin(4\pi t - 2.29) + 0.2898 \cos(6\pi t - 1.39181)\text{[rad/s]}
\]

(6.36)

\[
\ddot{\theta}(t) = -1.064 \cos(2\pi t + 1.7787) - 3.4906 \cos(4\pi t - 2.29) - 1.7391 \sin(6\pi t - 1.39181)\text{[rad/s\(^2\)]}
\]

(6.37)

And the torque, adjusted using the mass and inertia is:

\[
E(t)_{knee} = 1.7374 + 2.8114 \cos(2\pi t + 2.7796) + 10.1731 \cos(4\pi t - 1.3234) + 7.1367 \cos(6\pi t - 2.3928) + 1.6026 \cos(8\pi t - 2.9492) + 1.6763 \cos(10\pi t - 2.8) + 1.8058 \cos(12\pi t + 1.5605) + 0.7286 \cos(14\pi t - 0.4716) + 0.3373 \cos(16\pi t + 2.2038) + 0.3586 \cos(18\pi t - 1.5356) + 0.7546 \cos(20\pi t + 3.1416)
\]

(6.38)

For this simulation, it is assumed the human controls the total torque and it remains the same as without assistance. Also, it is assumed there is no alteration in the
trajectory of the limb, therefore the human controls the trajectory with no delay, adjusting $E_{human}(t)$ instantly.

Figure 6.11 shows the angle vs time of equation 6.35. The points shown in the trajectory are the approximated gait states shown in Figure 6.12. Figure 6.13 has the limit cycle of the trajectory on the phase plane and also shows the points of the four gait phases shown in Figure 6.12.

![Figure 6.11](image)

**Figure 6.11:** Truncated Fourier Series Representation of the Knee Trajectory in the Sagittal Plane During Normal Gait. Angle vs Time. $\theta(t) = 0.266 + 0.2722 \cos(2\pi t + 1.7787) + 0.2181 \cos(4\pi t - 2.29) + 0.0483 \sin(6\pi t - 1.39181)$.

![Figure 6.12](image)

**Figure 6.12:** Gait Cycle Phases Shown. 1 Initial Contact. 2 Opposite Toe Off. 3 Heel Rise. 4 Toe Off. 5 Tibia Vertical. Diagram Modified From [6]

The system without assistance is:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{c\dot{\theta} + d\omega\theta}{\sqrt{\dot{\theta}^2 + \omega^2\dot{\theta}^2}} + f = E(t) \quad (6.39)$$

159
Figure 6.13: Limit Cycle of the Fourth Order Fourier Series Representation of the Knee Trajectory in the Sagittal Plane During Normal Gait. Degrees vs Degrees/s

Figure 6.14 shows the torque exerted by the human $E_{human}$ and the power $P_{human}$, the trajectory of the knee angle $\theta$, and the angular velocity $\dot{\theta}$ of the knee during normal gait in the sagittal plane when no assistance is provided.

When assistance is provided, the system becomes:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{c\dot{\theta} + d\omega\theta}{\sqrt{\dot{\theta}^2 + \omega^2\theta^2}} + f = E_{human}(t) + E_{assist} \quad (6.40)$$

Assisting the system with the nonlinear phase based forcing function, the system used for the simulations is:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = E_{human} + c_a\sin \phi + d_a\cos \phi \quad (6.41)$$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = E_{human} + \frac{c_a\dot{\theta} + d_a\tilde{\omega}\theta}{\sqrt{\dot{\theta}^2 + \tilde{\omega}^2\theta^2}} \quad (6.42)$$

The goal is to get $E_{human}(t) < E(t)$. Assuming there is available a wearable robot with a maximum torque of 1.5 Nm to assist this motion. Because the moment of
The inertia of the leg and foot around the knee is $I_{knee} = 0.3349$, the assisting excitation $E_{assist}$ has a maximum value of 4.4763 (Torque/Inertia). The system was simulated in three different cases. Case 1: $c_a = d_a = 3.1651$. Case 2: $c_a = 4.4763$, $d_a = 0$. Case 3: $c_a = 0$, $d_a = 4.4763$. It was decided to test these three cases because it gives insight of the effect of using only $\sin \phi$, $\cos \phi$ and the combination of both. The values of $d_a$ were selected to be positive because the walking frequency is less than the natural frequency ($\omega < \omega_n$).

The results of the simulation are shown in Figure 6.15. It can be observed that the maximum positive value of torque $E_{human}$ is reduced in the three cases. The greater reduction is achieved using $c_a = d_a = 3.1651$. However, the torque $E_{human}$ is not reduced all of the time. For the case $c_a = 0$ and $d_a = 4.4763$, the positive peak is reduced but the negative peak increases.

Figure 6.16 shows the effect of using values of $\bar{\omega} \neq \omega$ in the same three cases. In black is shown $E_{human}$ when no assistance is provided. In blue when the system is
**Figure 6.15:** On Top, Plot of the Simulated torque Exerted by the Human $E_{\text{human}}$ at the Knee During a Gait Cycle in the Sagittal Plane. On Bottom, Plot of the Power Required From the Human to Flex the Knee $P_{\text{human}}$. On Both, the Black Line Shows the Values with No Assistance Provided, in Blue When $c_a = d_a = 3.1651$, in Red $c_a = 4.4763$, $d_a = 0$, and in Green $c_a = 0$, $d_a = 4.4763$.

assisted using $\tilde{\omega} = \omega$, in red the results using $\tilde{\omega} = 0.5\omega$, and in green $\tilde{\omega} = 2\omega$. On top, the results of case 1, $c_a = d_a = 3.1651$, in the middle of the plot, the results for case 2: $c_a = 4.4763$, $d_a = 0$, and on the bottom the results for case 3: $c_a = 0$, $d_a = 4.4763$. 

162
It is seen the maximum value of $E_{human}$ is reduced in all cases, even when the value of $\tilde{\omega}$ has a 100% error. Although, $c_a$, $d_a$ and $\tilde{\omega}$ are not optimal, the system is stable and in synchrony with the motion.
Figure 6.16: Results of the Simulation Assistance of the Knee Motion Using the Forcing Function \( E_{\text{assist}} = \frac{c_a \dot{x} + d_a \ddot{x}}{\sqrt{\dot{x}^2 + \ddot{x}^2}} \). On Top, Torque and Power Using \( c_a = d_a = 3.1651 \), in the Middle \( c_a = 4.4763, d_a = 0 \), on Bottom \( c_a = 0, d_a = 4.4763 \). Each Case was Simulated Using \( \dot{\omega} = \omega, \ddot{\omega} = 0.5\omega, \) and \( \ddot{\omega} = 2\omega \), Shown in Blue, Red and Green.
Assisting the hip

Form the data in [6] shown in Figure 6.10, a truncated Fourier series representation was obtained for the position and torque of the hip in the sagittal plane during normal gait. The resulting trajectories are:

$$\theta(t) = 0.12268 + 0.3675 \sin(2\pi t + 2.03) + 0.04 \sin(4\pi t + 4.76) + 0.02 \sin(6\pi t + 0.24) [rad]$$  \hspace{1cm} (6.43)

$$E(t)_{\text{hip}} = 2.194 + 21.67 \cos(2\pi t + 0.2703) + 2.174 \cos(4\pi t + 1.2092) + 4.368 \cos(6\pi t + 0.8343) [1/s^2]$$  \hspace{1cm} (6.44)

Using a general representation for a second order system, $a\ddot{\theta} + b\dot{\theta} + c\sin(\theta) = E_{\text{hip}}$, the parameters were calculated using least squares estimation that approximates the relationship between Equations 6.43 and 6.44. The model is:

$$1.4048\ddot{\theta} - 3.6208\dot{\theta} + 191.32\sin(\theta) = E(t) [Nm]$$  \hspace{1cm} (6.45)

Dividing by 1.4048, Equation 6.45 becomes:

$$\ddot{\theta} - 2.5774\dot{\theta} + 136.19\sin(\theta) = E(t)$$  \hspace{1cm} (6.46)

From this approximation the natural frequency of the system is $\omega_n = 11.67 \text{ rad/s}$. And the damping ratio $\zeta = 2.5774/(2\omega_n) = 0.11$. The inertia $I_{\text{hip}} = 1.4048 \text{ Kgm}^2$. The offset of the model was adjusted as well. The model used for simulations is:

$$\ddot{\theta} - 2.5774\dot{\theta} + 136.19\sin(\theta) - 9.25 = E(t)$$  \hspace{1cm} (6.47)

Figure 6.17 shows the trajectory of the hip angle over time using equation 6.43. The points marked in the trajectory are the approximated gait states shown in Figure 6.18. The limit cycle of the hip is shown in Figure 6.19. This plot also shows the location of the approximated gait states of Figure 6.18.
Figure 6.17: Truncated Fourier Series Representation of the Hip Trajectory in the Sagittal Plane During Normal Gait. Angle vs Time. \( \theta(t) = 0.12268 + 0.3675 \sin(2\pi t + 2.03) + 0.04 \sin(4\pi t + 4.76) + 0.02 \sin(6\pi t + 0.24) \).

Figure 6.18: Gait Cycle Phases Shown for the Dark Leg. 1 Initial Contact. 2 Heel Rise. 3 Toe Off. 4 Feet Adjacent. Diagram Modified from [6]

The plot in Figure 6.20 shows the torque \( E_{human} \) (Nm/Kg), the power \( P_{human} \) (W/Kg), the trajectory of the hip angle \( \theta \) (rad), and the angular velocity \( \dot{\theta} \) (rad/s) of the hip during normal gait in the sagittal plane when no assistance is provided.

Using a wearable robot with a maximum torque of 11.5 Nm to assist the hip motion, \( E_{assist} \) has a maximum value of 8.186 rad/s\(^2\) (Torque/Inertia) \( (I_{hip} = 1.40485) \). The system was simulated in three cases. Case 1: \( c_a = d_a = 8.1305 \) Nm. Case 2: \( c_a = 2.9764 \) Nm, \( d_a = 11.1081 \) Nm. Case 3: \( c_a = 2.9764 \) Nm, \( d_a = 11.1081 \) Nm. Figure 6.21 shows the results of the simulation. The results show that the maximum value of \( E_{human} \) is reduced.
The results of the simulation are shown in Figure 6.21. It can be observed that the maximum value of torque $E_{human}$ is reduced in all cases. Although, in the third case, the reduction of the peak is smaller. The largest torque reduction is achieved using $c_a = 2.9764$ Nm and $d_a = 11.1081$ Nm, meaning the cosine of the phase angle has a greater effect on the torque because of the difference between the natural frequency of the system and the frequency of the motion. For the three cases the peak positive power is reduced, however cases 1 and 3 produce a larger power peak in the negative direction at $t = 0.14s$, and $t = 0.6$ s. This also indicates the cosine component in the torque is more important as compared to the sine element.

Figure 6.22 shows the effect of using values of $\tilde{\omega} \neq \omega$ in the same three cases. In black $E_{human}$ is shown when no assistance is provided. In blue when the system is assisted using $\tilde{\omega} = \omega$, in red $\tilde{\omega} = 0.5\omega$, and in green $\tilde{\omega} = 2\omega$. On top, the
Figure 6.20: Representation Over Time of the Hip Torque (Nm/Kg), Shown in Solid Blue; the Power (W/Kg), Shown in Dotted Blue; the Angle $\theta$ (rad) in Solid Black and the Angular Velocity $\dot{\theta}$ (rad/s) in Dotted Black. Torque and Power Scale on the Left. Angle and Angular Velocity Scale on the Right.

results of case 1, $c_a = d_a = 8.1305$ Nm, in the middle of the plot, the results for case 2: $c_a = 2.9764$ Nm, $d_a = 11.1081$ Nm. On the bottom the results for case 3: $c_a = 11.1081$ Nm, $d_a = 2.9764$ Nm. The maximum value of $E_{human}$ is reduced in the three cases, including when the value of $\bar{\omega}$ has a 100% error.

Figure 6.23 shows the response to a linear frequency sweep with initial frequency $\omega_0 = 3.76$ rad/s (0.6 Hz), and final frequency $\omega_f = 18.84$ rad/s (3 Hz). The system was assisted using $c_a = 2.9764$ Nm, $d_a = 11.1081$ Nm, shown in blue; and $c_a = 11.1081$ Nm, $d_a = 2.9764$ Nm shown in red. It shows the both cases reduce the maximum torque at low frequencies. However the first case is more effective.
Figure 6.21: On Top, Plot of the Simulated Torque Exerted by the Human $E_{human}$ on the Hip During a Gait Cycle in the Sagittal Plane. On Bottom, Plot of the Power Required From the Human to Flex the Hip $P_{human}$. On Both, the Black Line Shows the Values with No Assistance Provided, in Blue When $c_a = d_a = 8.1305$ Nm, in Red $c_a = 2.9764$ Nm, $d_a = 11.1081$ Nm, and in Green $c_a = 11.1081$ Nm, $d_a = 2.9764$ Nm.
Figure 6.22: Results of the Simulation Assistance of the Hip Motion Using the Forcing Function $E_{\text{assist}} = c_{a}\dot{x} + d_{a}\ddot{x}$. On Top, Torque and Power Using $c_{a} = d_{a} = 8.1305 \text{Nm}$, in the Middle $c_{a} = 2.9764 \text{Nm}$, $d_{a} = 11.1081 \text{Nm}$, on Bottom $c_{a} = 11.1081 \text{Nm}$, $d_{a} = 2.9764 \text{Nm}$. Each Case was Simulated Using $\ddot{\omega} = \omega$, $\ddot{\omega} = 0.5\omega$, and $\ddot{\omega} = 2\omega$, Shown in Blue, Red and Green.
Figure 6.23: Linear Frequency Sweep Simulation. In Black is Shown the Torque $E_{human}$ When No Assistance is Provided, in Blue for $c_a = 2.9764 \text{ Nm}$, $d_a = 11.1081 \text{ Nm}$; and in Red for $c_a = 11.1081 \text{ Nm}$, $d_a = 2.9764 \text{ Nm}$. The Initial Frequency is $\omega_0 = 3.76 \text{ rad/s}$, and the Final Frequency is $\omega_f = 18.84 \text{ rad/s}$.
6.2.3 Considering the Neural Delay.

In reality, the human cannot have instantaneous control over the trajectory. It has been documented that there is a neural delay, or neural latency that is the time it takes the brain to control a muscle. This time is in the range of 100-500 ms [110]. To model how the delay changes the torque $E_{human}$, the human control was modeled with a PID tuned to have a delay of 0.103 s. The model is shown in the block diagram in Figure 6.24. The gains of the controller are: $K_p = 250$, $K_i = 861$, and $K_d = 15.2$. This model assumes the PID can represent the trajectory control performed by the human and the neural delay of the muscle activation signals. Introducing a controller to simulate this delay can help to have a better representation and better understanding of the effects of the phase based forcing function. The human will try to keep the trajectory of the limb unaltered modifying the torque, and keeping the total torque also unaltered [32].

Figure 6.24: Block Diagram of the Proposed Model to Represent How the Phase Based Oscillator Can Be Used to Assist the Human.

Figure 6.25 shows torque from the hip simulation results using the model in figure 6.24. The parameters used for the simulation were $c_a = 3.0825$, $d_a = 3.0825$, and $\tilde{\omega} = \omega = 2\pi$ rad/s. In black it is shown the required torque from the human when no assistance is provided and in blue when assistance is provided. The results show that
the maximum torque required from the human when assistance is provided is lower. The power when assisted is less than when no assistance is provided for almost the entire gait cycle. Figure 6.26 shows the hip power $P_{\text{human}}$ from the same simulation. The delay caused by the PID does not affect significantly the trajectory of the torque as seen when comparing the results in Figures 6.21, 6.25, and 6.26. Although, the trajectory of the power is significantly different when the time delay is considered, the magnitude of human power is reduced all the time.

**Figure 6.25:** Simulated Hip Torque $E_{\text{human}}$ During Normal Gait in the Sagittal Plane. In Black it is Shown the Torque When No Assistance is Provided. In Blue is Shown the Torque When Assistance is Provided.

**Figure 6.26:** Simulated Hip Power $P_{\text{human}}$ During Normal Gait. In Black the Torque When No Assistance is Used, and in Blue with Assistance. ($\tau \times \dot{\theta}$)
6.2.4 Assisting Periodic Motion Adapting the Values of c and d.

Consider the trajectory of the hip from Equation 6.43. Using the trigonometric identity \( \sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \), \( \theta(t) \) can be represented as:

\[
\theta(t) = 0.12268 + 0.32942 \cos(2\pi t) - 0.039955 \cos(4\pi t) + 0.004754 \cos(6\pi t) - 0.162889 \sin(2\pi t) + 0.001904 \sin(4\pi t) + 0.019427 \sin(6\pi t) \text{[rad]}
\]

(6.48)

From the Fourier expansion of periodic functions in Equation 3.29:

\[
x = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)
\]

(6.49)

Consequently; \( a_0 = 0.24536 \), \( a_1 = 0.3294 \), \( a_2 = -0.039955 \), \( a_3 = 0.004754 \), \( b_1 = -0.16289 \), \( b_2 = 0.001904 \), \( b_3 = 0.019427 \), and \( \omega = 2\pi \). \( c \) and \( d \) are found using equations 3.39, and 3.40.

\[
c = 11.81
\]

\[
d = 112.27
\]

(6.50)

Figure 6.27 shows the values of \( c \) and \( d \) over time during one cycle. The direction of the manipulation on the phase plane stays constant, it only changes the amplitude. Consider that the values of \( c \) and \( d \), from equations 3.39, and 3.40, only produce an approximation of the system because it is nonlinear.
The system was simulated using the adapting values for $c_a = 0.2c$, and $d_a = 0.2d$ from Equation 6.50; and $\tilde{\omega} = \omega = 2\pi$, and compared with the system using constant values of $c_a$ and $d_a$ selected to be $c_a = 1.1983$ Nm, $d_a = 11.479$ Nm that is in the same angular direction of 6.50. The simulated systems are:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{0.2c\dot{\theta} + 0.2d\tilde{\omega}\theta}{\sqrt{\dot{\theta}^2 + \tilde{\omega}^2\theta^2}}$$ (6.51)

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 8.1862\frac{0.1042\dot{\theta} + 0.9946\tilde{\omega}\theta}{\sqrt{\dot{\theta}^2 + \tilde{\omega}^2\theta^2}}$$ (6.52)

The results are shown in Figure 6.28. In general, the output is better when using the adapting values of $c_a$ and $d_a$. The curves match from $t = 0.2$ s to $t = 6$ s. The performance adapting $c_a$ and $d_a$ is better from $t = 0$ s to $t = 0.2$ s and from $t = 0.6$ to $t = 1$ s. This difference in the torque values is more noticeable in the power plot between $t = 0.6$ s and $t = 0.8$ s.

In summary, from the simulations, the best result is accomplished using a combination of positive values of $c_a$ and $d_a$. This means the assisting torque will have a trajectory of the form $\cos(\phi + \alpha)$, where $\phi$ is the phase angle and $\alpha$ is a negative phase shift. Using constant, arbitrary values of $c_a$ and $d_a$ will not produce optimal results but will assist the system reducing both the torque and power. The advantage is that
Figure 6.28: On Top, Plot of the Simulated Torque Exerted by the Human $E_{human}$ on the Hip During a Gait Cycle in the Sagittal Plane. On Bottom, Plot of the Power Required From the Human to Flex the Hip $P_{human}$. The Simulation was Done With $c_a = 0.2c$, $d_a = 0.2d$ from Equation 6.50, and for the Constant Case $c_a = 1.45188$, $d = 4.10974$

the controller is very simple and produces a signal that is bounded and always with the correct frequency. It does not require the calculation in real time of the frequency components of the signal, therefore it is easily implemented in a microcontroller. This is implemented and tested in the next section using a wearable robot.
6.3 Implementation to Assist Hip Motion

6.3.1 Hip Wearable Robot

The use of the phase based forcing function to assist the hip motion was tested using the robot developed by the students in the Human Machine Integration laboratory. The robot is shown in Figure 6.29 and has a capacity of 11.5 Nm on each side. Table 6.1 lists the technical specifications of the hip robot. The pseudo-code of the robot is shown in 1 where the values of $c$ and $d$ changed from test to test.

Figure 6.29: HESA Hip Robot Developed at the Human-Machine Integration Laboratory.
Table 6.1: Technical Specifications of the Wearable Hip Robot.

<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motor Torque</td>
<td>11.2</td>
<td>Nm</td>
</tr>
<tr>
<td>Motor max current</td>
<td>5</td>
<td>A</td>
</tr>
<tr>
<td>Motor nominal power</td>
<td>180</td>
<td>W</td>
</tr>
<tr>
<td>Motor nominal voltage</td>
<td>24</td>
<td>V</td>
</tr>
<tr>
<td>Robot weight</td>
<td>6.35</td>
<td>Kg</td>
</tr>
</tbody>
</table>

Data: acceleration $\ddot{\theta}$, velocity $\dot{\theta}$, position $\theta$, frequency $\omega$, $c$, $d$

Result: Main motor controller loop

Set initial values: $\omega \leftarrow 2\pi$, $c \leftarrow 0.7937$, $d \leftarrow 0.2062$;

while $stop=0$ do

read velocity $\dot{\theta}$;

$\theta \leftarrow$ integrate velocity $\dot{\theta}$;

$\sin \phi \leftarrow \dot{\theta}/\sqrt{\dot{\theta}^2 + \omega^2\theta^2}$;

$\cos \phi \leftarrow [\omega\theta]/\sqrt{\dot{\theta}^2 + \omega^2\theta^2}$;

$M \leftarrow c\sin \phi + d\cos \phi$;

$M \leftarrow M \ast 100$;

Output port $\leftarrow M$;

end

Algorithm 1: Robot Pseudo-Code

6.3.2 Tests

The system was tested in two sessions, one done on October 10th, and the second one on October 13th. Each session included 9 tests. Each test consisted in using the
hip robot during normal walking for three minutes on a instrumented treadmill using a different ratio of $c_a$ and $d_a$. The speed was maintained constant at 1.34 m/s. The subject was a healthy male, 23 years old, 165 cm of height, and 68 Kg mass. Reaction forces and $VO_2$ were gathered and reported. Figure 6.30 shows the setting. The tests during the first testing session were done in decremental order, going from a angle shift of -10 deg to -80. During the second session, the order was randomized to see if fatigue was affecting the results.

![Test Subject Walking on Instrumented Treadmill Wearing the HESA Robot During One of the Tests.](image)

**Figure 6.30:** Test Subject Walking on Instrumented Treadmill Wearing the HESA Robot During One of the Tests.

The robot was tested using 8 different phase shifts or ratios of $c_a$ and $d_a$ as shown in table 6.2, and an additional test was done with the subject wearing the robot with no power to use it as a reference. The value used in the robot code for the frequency was $\omega = 2\pi$ rad/s (1 Hz). The actual waking frequency obtained from the ground reaction forces data was $\omega = 5.78$ rad/s (0.9199 Hz).
Session 1

Figure 6.31 shows the values of oxygen consumption \((VO_2)\) for the first session in ml/(min Kg) during 500 samples during steady state for all the tests. The \(VO_2\) values were gathered at a rate of 4 Hz. Table 6.3 shows the average and maximum values for each one of the tests and these same results are shown in Figure 6.32. It can be seen that the minimum oxygen consumption value is reached for the 10 degrees shift \((c_a = 0.984, d_a = -0.17)\), and this was the first test of the session.

Table 6.4 has the average ground reaction force in each axis during each test. Figure 6.33 shows the plot of the average ground reaction forces along the z axis for each leg during each test. The average value was preferred over the maximum values because the signal had some outliers. The torque at the hip is directly related to the magnitude of the reaction force along the z axis, and in this case the results shows the same trend as the results for oxygen consumption.

\textbf{Table 6.2:} Summary of Values of \(c_a\) and \(d_a\) and the Respective Shift Angle Equivalence that Were Tested.

<table>
<thead>
<tr>
<th>Angle shift (deg)</th>
<th>-10</th>
<th>-20</th>
<th>-30</th>
<th>-40</th>
<th>-50</th>
<th>-60</th>
<th>-70</th>
<th>-80</th>
<th>Power OFF</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_a)</td>
<td>0.984</td>
<td>0.939</td>
<td>0.866</td>
<td>0.766</td>
<td>0.642</td>
<td>0.499</td>
<td>0.342</td>
<td>0.173</td>
<td>0</td>
</tr>
<tr>
<td>(d_a)</td>
<td>-0.17</td>
<td>-0.34</td>
<td>-0.50</td>
<td>-0.64</td>
<td>-0.77</td>
<td>-0.87</td>
<td>-0.94</td>
<td>-0.98</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 6.31: Oxygen Consumption (VO₂) in ml/(min Kg) Gathered After Steady State was Achieved. The Sampling Frequency was 4 Hz. SESSION 1.
Table 6.3: Average and Maximum Oxygen Consumption \((VO_2)\) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 1.

<table>
<thead>
<tr>
<th>Angle shift (deg)</th>
<th>Average (VO_2) (ml/(min Kg))</th>
<th>Maximum (VO_2) (ml/(min Kg))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>13.43067128</td>
<td>13.88748736</td>
</tr>
<tr>
<td>-20</td>
<td>13.74993877</td>
<td>14.33333673</td>
</tr>
<tr>
<td>-30</td>
<td>13.64844157</td>
<td>14.07309432</td>
</tr>
<tr>
<td>-40</td>
<td>13.73638054</td>
<td>14.16204266</td>
</tr>
<tr>
<td>-50</td>
<td>13.69372666</td>
<td>14.24872292</td>
</tr>
<tr>
<td>-60</td>
<td>13.83737569</td>
<td>14.16204266</td>
</tr>
<tr>
<td>-70</td>
<td>13.91962372</td>
<td>14.41606786</td>
</tr>
<tr>
<td>-80</td>
<td>13.8569266</td>
<td>14.16204266</td>
</tr>
<tr>
<td>Power OFF</td>
<td>14.0796348</td>
<td>14.49708381</td>
</tr>
</tbody>
</table>
Figure 6.32: Average (Blue) and Maximum (Red) Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 1.
Table 6.4: Average Ground Reaction Force Along Each Axis for Both Legs During Each Test SESSION 1.

<table>
<thead>
<tr>
<th>Angle shift</th>
<th>Left leg</th>
<th>Right leg</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>-10</td>
<td>18.6591</td>
<td>-0.6247</td>
</tr>
<tr>
<td>-20</td>
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<td>-30</td>
<td>19.0014</td>
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<td>-1.6493</td>
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<td>-60</td>
<td>19.7855</td>
<td>-1.0145</td>
</tr>
<tr>
<td>-70</td>
<td>18.7613</td>
<td>-1.0236</td>
</tr>
<tr>
<td>-80</td>
<td>19.4759</td>
<td>-1.5963</td>
</tr>
</tbody>
</table>
Figure 6.33: Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right Leg. SESSION 1.
Session 2

During the second testing session, the order of the test was randomized. Figures 6.34 and 6.35 show the values of oxygen consumption ($VO_2$) for the second session in ml/(min Kg). The results in Figure 6.34 are in the same order than the tests were done. Figure 6.35 shows the results in descending order. The $VO_2$ values were gathered at a rate of 4 Hz. Table 6.5 shows the average and maximum values for each one of the tests. In this case the minimum oxygen consumption value is reached when the robot was off; this was the first test of the session.

Table 6.6 has the average ground reaction force in each axis during each test. Figures 6.36 and 6.37 show the average ground reaction forces along the z axis for each leg during each test. In Figure 6.36 the results are in the order that the tests were done. Figure 6.37 shows the results in descending order as in the results for session 1.

Additionally, a test was done using a positive angle shift, meaning using $c_a = 0.707$ and $d_a = 0.707$. This test was not completed because the subject reported this settings made him work against the robot to walk, meaning that instead of assisting the motion, the robot was acting as an exercise machine.

The results from both testing sessions show that the order of the test is important, therefore it can be assumed fatigue is affecting the results. Since the use of positive values of $d_a$ were not effective, it can be said that the system has a natural frequency that is lower than the walking frequency. This does not coincide with the estimated parameters for the second order system of the previous section in which the estimated value for the natural frequency is $\omega_n = 11.67$ rad/s. In general the test were inconclusive. The test subject reported feeling the force of the assistance and being more comfortable when the angle shift was -70 deg.
Table 6.5: Average and Maximum Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 2.

<table>
<thead>
<tr>
<th>Angle shift (deg)</th>
<th>Average $VO_2$ (ml/(min Kg))</th>
<th>Maximum $VO_2$ (ml/(min Kg))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>13.64844157</td>
<td>14.07309432</td>
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<td>-80</td>
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<td>14.16204266</td>
</tr>
<tr>
<td>Power OFF</td>
<td>13.43067128</td>
<td>13.88748736</td>
</tr>
</tbody>
</table>
Figure 6.34: Average (Red) and Maximum (Green) Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test. SESSION 2.
Figure 6.35: Average (Blue) and Maximum (Red) Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test. SESSION 2.
Table 6.6: Average Ground Reaction Force Along Each Axis for Both Legs During Each Test SESSION 2.

<table>
<thead>
<tr>
<th>Angle shift</th>
<th>Left leg</th>
<th></th>
<th></th>
<th>Right leg</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
<td>Z</td>
<td>X</td>
<td>Y</td>
<td>Z</td>
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<td>-0.9690</td>
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</tr>
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<td>-0.091</td>
<td>148.4458</td>
<td>-9.1182</td>
<td>-1.0036</td>
<td>149.2310</td>
</tr>
<tr>
<td>Power OFF</td>
<td>9.2084</td>
<td>0.0287</td>
<td>145.7392</td>
<td>-9.2745</td>
<td>-0.7405</td>
<td>146.4735</td>
</tr>
</tbody>
</table>
Figure 6.36: Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Red, Left Leg. In Green, Right Leg. SESSION 2.
Figure 6.37: Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right Leg. SESSION 2.
Session 3

The robot was modified for a third testing session. The motors were changed for smaller and lighter ones. Also, the program of the robot was modified to not limit the output. The order of the test was randomized. The subject was instructed to rest for 10 minutes after 3 tests. The goal was to reduce the effect of fatigue in the results. Figures 6.38 and 6.39 show the values of oxygen consumption ($VO_2$) for the third session in ml/(min Kg). The results in Figure 6.38 are in the same order than the tests were done. Figure 6.39 shows the results in descending order. The $VO_2$ values were gathered at a rate of 4 Hz. Table 6.7 shows the average and maximum values for each one of the tests. In this case the minimum oxygen consumption value is reached when the robot was off; this was the first test of the session. Figures 6.40 and 6.41 are shown as supplementary material. The heart rate is proportional to the oxygen consumption and shows the same trend.

The average ground reaction force in each axis during each test, is shown in table 6.9. Figure 6.42 shows the plot of the average ground reaction forces along the z axis for each leg during each test. The torque at the hip is directly related to the magnitude of the reaction force along the z axis, and in this case the results shows the same trend as the results for oxygen consumption.

The results from this session show, in smaller proportion, the effect of the fatigue but it is still present, in part because the subject was instructed to rest, and also because the robot was re-programed to not limit the output. There is a significant reduction when the angle shift is -20 degrees. This could be because the subject rested before the test and or because this shift is close to be the optimal relationship between c and d for the nonlinear phase based forcing function. More tests are needed to confirm this.
Table 6.7: Average and Maximum Oxygen Consumption \((VO_2)\) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test SESSION 3.

<table>
<thead>
<tr>
<th>Angle shift (deg)</th>
<th>Average (VO_2) (ml/(min Kg))</th>
<th>Maximum (VO_2) (ml/(min Kg))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>9.9733</td>
<td>10.6033</td>
</tr>
<tr>
<td>-20</td>
<td>9.2593</td>
<td>9.8502</td>
</tr>
<tr>
<td>-30</td>
<td>9.4076</td>
<td>9.8502</td>
</tr>
<tr>
<td>-40</td>
<td>9.4699</td>
<td>10.0720</td>
</tr>
<tr>
<td>-50</td>
<td>9.5726</td>
<td>10.2881</td>
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<tr>
<td>-60</td>
<td>9.5636</td>
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<tr>
<td>-70</td>
<td>9.2677</td>
<td>9.8502</td>
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<tr>
<td>-80</td>
<td>9.5438</td>
<td>10.0720</td>
</tr>
<tr>
<td>Power OFF</td>
<td>9.1279</td>
<td>9.6217</td>
</tr>
</tbody>
</table>

Table 6.8: Average and Maximum Heart Rate in Beats per Minute Gathered After Steady State was Achieved for Each Test SESSION 3.

<table>
<thead>
<tr>
<th>Angle shift (deg)</th>
<th>Average Beats per Minute</th>
<th>Maximum Beats per Minute</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>94.1308</td>
<td>100</td>
</tr>
<tr>
<td>-20</td>
<td>87.9767</td>
<td>93</td>
</tr>
<tr>
<td>-30</td>
<td>89.1835</td>
<td>93</td>
</tr>
<tr>
<td>-40</td>
<td>89.6962</td>
<td>95</td>
</tr>
<tr>
<td>-50</td>
<td>90.5991</td>
<td>97</td>
</tr>
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<td>-60</td>
<td>90.54852</td>
<td>97</td>
</tr>
<tr>
<td>-70</td>
<td>88.0</td>
<td>93</td>
</tr>
<tr>
<td>-80</td>
<td>90.3312</td>
<td>95</td>
</tr>
<tr>
<td>Power OFF</td>
<td>86.9071</td>
<td>91</td>
</tr>
</tbody>
</table>
Figure 6.38: Average (Blue) and Maximum (Orange) Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test. SESSION 3.
Figure 6.39: Average (Blue) and Maximum (Orange) Oxygen Consumption ($VO_2$) in ml/(min Kg) Gathered After Steady State was Achieved for Each Test. SESSION 3.
Figure 6.40: Average (Blue) and Maximum (Orange) Heart Rate in Beats per Minute Gathered After Steady State was Achieved for Each Test. SESSION 3.
Figure 6.41: Average (Blue) and Maximum (Orange) Heart Rate in Beats per Minute Gathered After Steady State was Achieved for Each Test. SESSION 3.
Table 6.9: Average Ground Reaction Force Along Each Axis for Both Legs During Each Test SESSION 3.

<table>
<thead>
<tr>
<th>Angle shift</th>
<th>Left leg</th>
<th>Right leg</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>-10</td>
<td>9.1264</td>
<td>-0.8734</td>
</tr>
<tr>
<td>-20</td>
<td>9.1054</td>
<td>-0.4476</td>
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<tr>
<td>-30</td>
<td>8.9187</td>
<td>-0.2199</td>
</tr>
<tr>
<td>-40</td>
<td>9.0457</td>
<td>-0.1035</td>
</tr>
<tr>
<td>-50</td>
<td>8.5229</td>
<td>-0.0036</td>
</tr>
<tr>
<td>-60</td>
<td>9.6069</td>
<td>-1.3569</td>
</tr>
<tr>
<td>-70</td>
<td>9.3449</td>
<td>-0.1379</td>
</tr>
<tr>
<td>-80</td>
<td>9.0716</td>
<td>-1.4712</td>
</tr>
<tr>
<td>Power OFF</td>
<td>8.1840</td>
<td>0.1587</td>
</tr>
</tbody>
</table>
Figure 6.42: Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right Leg. SESSION 3.
Figure 6.43: Average Ground Reaction Force Along the z Axis for Both Legs During Each Test. In Blue, Left Leg. In Red, Right Leg. SESSION 3.
Chapter 7

CONCLUSIONS

7.1 Dissertation Contributions

The contributions from this dissertation are as follows. The existence of a limit cycle for the proposed phase based oscillator was shown. The stability of the system was analyzed in the state space using the Poincaré-Bendixson criterion and was analyzed using polar coordinates and simple differential equations. It was shown the direct relationship between the phase based oscillator and a time based oscillator. It was shown that the phase based oscillator has a closed solution for any type of periodic function using a Fourier expansion representation. The effects of the different versions of the forcing function for the phase based oscillator were analyzed and simulated.

The theoretical findings were validated using a pendulum and a DC motor changing the amplitude and frequency of the oscillations. Also, the robustness of the phase oscillator was evaluated finding stability bounds assuming stochastic perturbations, and using the small gain theorem. A controller was added using the Lyapunov redesign method to handle small disturbances. Lastly, the interaction human-phase based assistance was modeled and simulated for periodic motions and was tested with a wearable robot to assist hip motion.

From the results it can be concluded that the natural frequency of the legs at the hip is slightly lower than the frequency of normal walking. This is the reason the angle shift of -20 degrees seems to work best. Using the phase oscillator to assist motion can be effective if the right ratio of $c_a$ and $d_a$ is used.
This is the list of contributions of this dissertation:

• Phase based oscillator.

  Showed the relationship with a time based system.**

  Proved the existence of a stable limit cycle using a Lyapunov function.**

  Proved stability with polar coordinates.

• Solution using the Fourier series representation of a periodic trajectory to calculate the best values of $c$ and $d$.**

• Analysis of the behavior of Phase based oscillator.

  with $c$ constant, $d = 0$.

  with $d$ constant, $c = 0$.

  with $c$ constant, $d$ constant.

• Phase based oscillator implementation and testing.

  Oscillating the angular velocity of a DC motor.

  Oscillating a physical pendulum.

• Analysis and comparison of the transition after perturbation of the phase based oscillator and the time based oscillator.**

• Uncertainty analysis on the Phase Based Oscillator.**

  Stability bounds for stochastic perturbations.**

  Uncertainty / noise reduction using Lyapunov redesign.**

  Stability bounds using Small Gain Theorem.**

• Assistance model using the nonlinear phase based forcing function.
• Simulation of knee and hip motion assistance.

• Test of the hip motion assistance approach with the phase based forcing function.

• Also, contributions complementary to this dissertation.

  Built force myography system for robotic ankle control.

  Proportional control of a robotic ankle using force myography.

  Proportional control of a virtual hand using force myography.

  Hand kinematics decoding using force myography.

7.2 Future Work and Research Opportunities

This work can continue to determine the best values of $c$ and $d$ for each person that uses the wearable robot. This can include calculating the frequency components of the trajectory in real time. Also, it could be tested using two different models for the human leg, one representing the stance phase and one for the swing phase. This approach would use two different values of natural frequency for the leg, therefore the values of $c$ and $d$ will be different for each phase.

Additionally, the phase oscillator could be used with other bio-potentials, for example EMG. It only requires the signal to be directly related to the torque magnitude.
REFERENCES


205


[38] M. Holgate, T. G. Sugar, and A. W. B’ohler. A novel control algorithm for


211


2000.

APPENDIX A

FORCE MYOGRAPHY TO CONTROL ROBOTIC DEVICES
A.1 Skeletal Muscles

The nervous system has direct control over the muscles and is in charge of the motion of the body and posture support [124, 128]. Skeletal muscles are made up from fibers connected between each other by connective tissue; these fibers generate the force of the muscle and act together to create a resultant force [128]. The muscles, depending on the type of contraction (isometric, concentric, and eccentric), get longer or shorter. There are three different architectures; the first one is parallel or longitudinal in which the fibers lie along the resultant force axis, like the biceps brachii; the second is the unipennate architecture, where the fibers are aligned with an angle with respect the resultant force axis, e.g. the vastus lateralis; and finally the multipennate architecture where the fibers have multiple directions, like in the gluteus maximus [68]. The contractions are activated by the nervous system in response to chemical, electrical and mechanical stimuli [124] and convert chemical energy to mechanical energy. The contractions are produced by the interaction of two proteins the myosin and actin and it is known as the cross bridge cycle [128]. Lastly, the muscles, that are connected to the tendons, pull on them to rotate the joints[126].

![Diagram of Hill’s Functional Model of the Skeletal Muscles.](image)

Figure A.1: Diagram of Hill’s Functional Model of the Skeletal Muscles.

There are multiple models of the behavior of the skeletal muscles; [125], Hill’s model is the most used given that it has been shown to represent accurately the mechanical behavior of the muscles almost in the entire range of force [124], and recently has been proven that also incorporates insight on the molecular mechanisms that involve the myosin cross-bridges [125]. This model relates the force and velocity of a muscle and it is formed by a contractile element in series with an elastic part, and these two are in parallel with another elastic element as shown in figure A.1. The contractile element represents the active behavior of the muscle; the spring in series is meant to represent the passive behavior of the muscle’s fibers; and the elastic element in parallel characterizes the passive behavior of the connective tissue, elastin fibers and collagen [8, 124, 126, 128]. Mathematically, Hill’s model is a hyperbolic equation of the form:

\[(F + a)(V + b) = (P_0 + a)b\]  \hspace{1cm} (A.1)
where:

\[ F = \text{force exerted by the muscle.} \]
\[ a = \text{the coefficient of shortening heat.} \]
\[ V = \text{velocity of contraction.} \]
\[ b = a \cdot V_0 / P_0 \]
\[ V_0 = \text{contraction velocity of unloaded muscle.} \]
\[ P_0 = \text{maximum isometric tension.} \]

The force-velocity relationship is different for each activation level as seen in Figure A.2, and the force also depends on the length of the muscle. The total relationship is shown in Figure A.3. Given that muscle tissue is quasi-incompressible [124] because it is composed mostly of water, the muscles volume can be assumed constant; therefore, the cross sectional area of the muscle is related to the velocity and the tension.

There are two cross sectional areas in the architecture of the muscles. Force myography is based on the anatomical cross sectional area (CSA) that is normal to the direction of action of the muscle. This is defined as [9]:

\[ CSA = \frac{m}{\rho \cdot l} \tag{A.2} \]

where:

\[ m = \text{mass of the muscle.} \]
\[ l = \text{length of the muscle.} \]
\[ \rho = \text{density of the muscle.} \]

The physiological cross sectional area (PCSA) considers the pennation angle of the fibers; the maximum force of a muscle is proportional to initial PCSA [68]. This area normal to the direction of the fibers is defined as [9]:

\[ PCSA = \frac{m \cdot \sin \theta}{\rho \cdot l \cdot t} \tag{A.3} \]

where:

\[ \theta = \text{angle of the direction of the fibers with respect the line of action of the muscle.} \]
\[ t = \text{is the distance between the muscle’s aponeuroses (muscle width).} \]

The pennation angle of the fibers is always changing with the activation [68], in the case of ankle, the angle increases as the joint angle gets bigger, therefore the PCSA increases as well [9]. See Figure A.4.

A.2 Definition of Force Myography

Force myography is the study of the myograms that measure the force normal to the skin surface produced by the muscular activity. This force changes with the volitional movements in the muscle architecture to generate movement of the human body.

222
As seen in the previous section, during the contractions of the muscles the CSA and the PCSA are dynamically changing, this generates variations on the pressure that the muscles apply to their surroundings. The sum of these variations can be measured on the skin surface as seen in Figure A.5. These measurements contain information about the kinematics of the body, specifically in the case of the limbs, kinematics can be estimated from the measurements.
Figure A.4: Relationship Between the Ankle Joint Angle and the Physiological Cross Sectional Area (PCSA) of the Gastrocnemius Medialis When the Muscle is at Rest [9].

The force myography signals are naturally the low pass filtered muscular activity [94]. When measured with arrays of sensors it is possible to create pressure maps that can be related to the movements of the limbs [89, 94, 96].

Figure A.5: Schematic of the Functioning Principle of the Force Myography. Transverse Section of Right Leg Showing that the Resultant Forces on the Surface is Affected by the Internal Pressures in Each Muscle.

A.3 Force Myography System

A.3.1 Force Sensing Resistors

The force sensing resistors (FSR) are transducers capable of measuring static and
dynamic loads applied to their surface through quantifying the change in the electric resistance [112, 117]. The difference of the resistance value is approximately linear in a log-log plot [98, 112, 117]; the relationship force-resistance is $F(t) \approx c/R(t)$, where $F$ is force, $R$ is resistance and $c$ is a constant.

The main characteristics of these sensors that make them feasible to use in human machine interfaces are [112, 127]:
- Force sensitivity range 0.2 N - 20.0 N.
- Thin cross section, do not require much space (< 1mm).
- Simple circuitry.
- Low cost per unit.
- No sensitivity to electro static discharge (ESD).
- Does not generate electromagnetic interference (EMI).

As the majority of sensors, FSR’s present creep in the output signal [112], meaning that there is change in the output when the applied load is constant. This change is very small compared to the magnitude of the signal < 5% log10(time) [98, 112, 127] and can be eliminated with software.

A.3.2 Measuring System

For this work the FSRs were mounted in rectangular acrylic plates that serve as mechanical support for the sensor and acrylic pads were collocated over the sensors to have an uniform activation of the FSRs, as shown in Figure A.6.

Figure A.6: FSR Mounting System Showing the Acrylic Plate Used as the Main Support and the Pad Used to Distribute the Load on the Sensor Area.

For the testing, the sensors were placed in a medical self adhesive bandages that is thought to be in the socket of the prosthetic limb that is custom made for every patient. The sensors are positioned around the forearm as shown in Figure A.7.

There are two different systems used in this work. Both use four FSRs and are designed to be attached around the residual limb but each use a different connection configuration.

The fist one uses the four sensors in a full Wheatstone bridge configuration as in [98]. The connection diagram is shown in Figure A.9 and the reading is simplified since it fuses four signals into one. See Figure A.8.
Figure A.7: Picture of the Location of the Four FSRs on the Proximal Forearm, and the Elastic Armband Used for the Tests. FSR4 is on the Back of the Forearm in this View.

Figure A.8: Signal Flow Diagram for the System that Uses the Fours FSRs in a Full Wheatstone Bridge Configuration.

Figure A.9: Full Wheatstone Bridge Using FSRs.
In the Wheatstone bridge, the relationship between the resistance value of the FSRs, the input voltage and the output voltage is:

\[ V_O(t) = \left( \frac{R3(t)}{R1(t) + R3(t)} - \frac{R4(t)}{R2(t) + R4(t)} \right) \times V_{In} \]  \hspace{2cm} (A.4)

where

- \( V_O \) = output voltage.
- \( R_i(t) \) = resistance of each FSR.
- \( V_{In} \) = input voltage that in this case is 5 VDC.

The output of the bridge reflects the imbalance on the resistance value of the FSRs. FSR1 and FSR4 are in opposite sides of the limb, taking advantage of the fact that the muscles work in pairs [129], putting one over the flexor muscles and the other over the extensor muscles.

The bridge configuration also reduces the effect of the creep in the output signal of the FSRs, meaning that there is change in the output when the applied load is constant. This change is very little compared to the magnitude of the signal [87, 98, 112].

The second system uses the four sensors independently with the goal of having more information. Then the 4 signals were processed to decode the kinematic motion of limb accordingly. Each FSR in this approach was connected to a voltage divider as shown in Figure A.10. The output of the voltage divider was connected to the data acquisition card. The diagram in Figure A.11 shows the signal flow for the system.

![Figure A.10: Voltage Divider Used in for Every FSR in the Decoding System.](image-url)
Figure A.11: Signal Flow in the System that Uses 4 FSRs Independently.
A.4 Use of force myography from leg to control robotic ankles proportionally

A.4.1 Muscles of the leg

![Muscles of the Leg](image)

Figure A.12: Muscles of the Leg [2].

<table>
<thead>
<tr>
<th>Muscle/Anatomical Motions</th>
<th>Plantar flexion</th>
<th>Dorsiflexion</th>
<th>Foot eversion</th>
<th>Tibia/Fibula flexion</th>
<th>Big toe extension</th>
<th>Toes extension</th>
<th>Leg lateral rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tibialis Anterior</td>
<td>■</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fibularis longus</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extensor digitorum longus</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Fibularis brevis</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extensor hallucis longus</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fibularis tertius</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lateral gastrocnemius</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Medial gastrocnemius</td>
<td>■</td>
<td>■</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table A.1: Anatomical Movements Involving the Muscles of the Leg [2].
The gastrocnemius (medial and lateral), plantaris, and popliteus origin in the femur, therefore during leg rotation, they change positions. Also, they are involved in knee flexing, their cross sectional area changes when the knee is flexed. Tibia and fibula rotate during gait cycle ±10deg. These movements have to be considered for sensor placement given that they can generate undesired changes in the sensors readings.

From the information in Fig. A.12 and table A.1 it was decided to test the system placing the sensors on the fibularis longus, tibialis anterior, lateral gastrocnemius, and on the tibia. The lateral gastrocnemius and the fibularis longus are involved in the plantar flexion movement. The tibialis anterior flexes during dorsiflexion. The idea is to have a signal that reflects the difference in muscular activity between the muscles that work during plantar flexion and the dorsiflexion. And in the case of the tibia region, where minimal activity is registered, it can be used as a reference.

### A.4.2 HERMES ankle

The robotic ankle used is shown in Fig. A.13. This is a robotic ankle for testing from SpringActive Inc. Powered externally with a PWM signal of 24 VDC controlled also externally using a PC-104. It provides the position of the ankle with a continuously incremental encoder.

### A.4.3 Signal processing

The signals are read using a data acquisition card from SENSORAY and processed in a PC 104. The system was programmed using Simulink from MathWorks. The signal flow is shown in the block diagram in Fig A.14. The output signals from the voltage dividers were filtered using a sixth order Infinite Impulse Response (IIR) low pass filter, with a cut off frequency of 3 Hz; then, the initial value is removed from the signals \( S_i(t) = x_i(t) - x_i(0)(t) \), where \( x_i \) is the output of the filters and \( x_i(0) \) is the initial value. Finally, the four channels are fussed using the Wheatstone bridge relation:

\[
S_o(t) = -S_1(t) + S_2(t) + S_3(t) - S_4(t) 
\]

where \( S_o \) is the output signal, \( S_i(t) \) is the transformed signal from each FSR. The output of the bridge reflects the imbalance on the pressure magnitude on the FSRs. FSR1 and FSR4 measure the pressure change from the flexor muscles. FSR 2 and FSR 3 are on the extensor muscles. The processed signal is used as reference in a PID controller to control the position of the robotic ankle as shown in Fig. A.14. For the PID controller the gains used were \( K_d = 2.0, K_i = 0.5, \) and \( K_d = 0.1 \), where these are the proportional, integrative, and derivative gains respectively.

While in operation, the system tracks and keeps the maximum and minimum values of \( S_o \), then, as shown in eq. A.6 the difference is used to transform the range
of operation making $V_{o \text{max}} = 115\%$ and $V_{o \text{min}} = -15\%$. This transformation allows for noise error in the maximum and minimums without over filtering the signal, therefore avoiding to introduce a significant delay to the signal from the muscles.

\[ A(t) = 130 \left( \frac{S_o(t) - S_{o \text{min}}}{S_{o \text{max}} - S_{o \text{min}}} \right) - 15 \]  

(A.6)

where $A(t)$ is the output variable that for this system is the percentage of flexion of the ankle.

**A.4.4 Testing**

![Block Diagram of the FMG System to Control a Robotic Ankle.](https://vimeo.com/151350344)

A video of one of the test is available at [https://vimeo.com/151350344](https://vimeo.com/151350344).

**A.5 Use of force myography from forearm to control robotic hands**

The hands are very dexterous and strong. Many of the muscles controlling the hand are located in the forearm. In an amputation, the wrist and hand are removed
but the muscles used to control them can still be volitionally activated, i.e. flexed and relaxed. Sensing this activation can be used for intuitive control of a prosthesis for transradial amputees. Force sensitive resistors (FSRs) are placed around the circumference of the forearm. When the muscles of the forearm flex, they press the resistors against the socket and a force is measured. The user only needs to open and close the hand fully a single time.

A.5.1 Muscles of the forearm

Figure A.15: Muscles that Move the Forearm. The Muscles that Begin at the Shoulder are in Charge of the Pronation, Supination, Flexion and Extension of the Forearm. The Muscles at the Forearm Contract to Move Fingers, Hand and Wrist. [10]

Table A.2: Anatomical Movements Involving the Muscles of the Forearm.[2]
A.5.2 Processing

As in the ankle system, during operation the maximum and minimum values of voltage across the Wheatstone bridge are tracked. This voltage is an indicator of the state of the hand. When the hand is at different closing angles, the muscles of the forearm have a different shape. The shape of the forearm muscles changes the relative forces on the FSR sensors in the Wheatstone bridge. To calibrate, the voltage is measured with the hand open and with the hand closed. For the bridge configuration, it was found that when the hand is open, the voltage is low. When it is closed, the voltage is high. The maximum and minimum measured voltages are constantly updated and maintained in memory. The percentage the hand is closed is the ratio of bridge voltage less minimum voltage to the maximum voltage less the minimum voltage.

To train the hand, the user opens and closes the hand fully one time. After this simple motion, a minimum and maximum voltage are recorded and indicate the fully closed and open states respectively. Subsequent measurements of the voltage indicate the amount the hand is closed.

A.5.3 Tests

A series of tests were performed with the device as reported in [98]. First, with the palm facing up, in the supine position, the hand was opened and closed slowly,
pausing at fully closed, partially open, comfortably open, and fingers fully extended. The supine hand positions are shown in Figure A.16. In the second test, the hand was opened and closed with a neutral wrist position, as shown in Figure A.17. The calculated percent open from this test are shown in Figure A.18.

![Figure A.16: Supine Wrist.](image)

| 100% closed | 80% closed | 10% closed | 0% closed |

![Figure A.17: Neutral Wrist.](image)

| 100% closed | 80% closed | 10% closed | 0% closed |

![Figure A.18: Output of Bridge During Calibration and Test.](image)

The device was calibrated by opening and closing the hand about 4 times in the first 4 seconds. The hand was then held closed with minimal force from about 4 to
14 seconds (theoretical 100 % closed). Then it was opened to 80 % closed from 14 to 23 seconds. From 23 to 31 seconds, the hand was opened a comfortable amount (10 % closed). For the remainder of the test, the fingers were fully extended, making the hand as flat as possible (0 % closed). Data were then taken to evaluate the ability to detect openness of the hand after rotation of the wrist. With the wrist supine, the hand was opened and closed several times to calibrate. The hand was then held closed for 10 seconds. The forearm was rotated to neutral and the hand was opened and closed several times. The hand was then held closed for about ten seconds before rotating back to supine (38 seconds). The closed, supine hand was held closed for another 10 seconds then opened and closed several times. The results show the system does not work accurately after wrist rotation, see Fig. A.19.

![Graph showing the output of the bridge during calibration and test with wrist rotation. The plot shows four cycles of opening and closing the hand, followed by wrist rotation and subsequent opening and closing. It can be seen that the range of operation changes when the wrist rotates.](image)

**Figure A.19:** Output of Bridge During Calibration and Test with Wrist Rotation. The plot shows four cycles of opening and closing the hand, followed by wrist rotation and subsequent opening and closing. It can be seen that the range of operation changes when the wrist rotates.

However, because the maximum and minimum were changed when the wrist was in the neutral position, the percent closed estimates were not accurate.

Then, a tracking test was done modifying the system using Eq. A.6. The user had to follow a sinusoidal wave and a square wave. The results are shown in Fig. A.20 where it can be seen the system can allow for proportional control of one DOF with Pearson correlation $r=0.8764$ for the sinusoidal case and $r=0.7442$ for the square wave.

### A.5.4 Virtual hand

In order to validate that the signal from the Wheatstone bridge can be used for the proposed purpose, the hand was simulated using the 3D virtual reality simulator of Mathworks for Simulink. The constructed model has 13 degrees of freedom but was used as having only one DOF (close and open hand). The rotation of all the joints of
the model was controlled using the output from the LabView processing. The signal was sent to the model to validate that this method could be used effectively in a robotic prosthetic hand. Fig. A.21 shows the computer running the virtual hand and the hand during the tests.

The user opened and closed the hand with the wrist in the pronate position (palm down). As long as the orientation of the wrist at the time of calibration is maintained, the system accurately estimates the users hand position. The actual and simulated hand positions are shown side-by-side for four hand positions in Figure A.21.

A.5.5 Classification approach using KNN

K Nearest Neighbors (KNN) is a supervised learning algorithm [113, 114] commonly used for the decoding of hand kinematics using electromyography signals [63]. KNN was selected because is one of the most widely classifiers used. The KNN algorithm sorts each unlabeled data point by the majority of its K-nearest neighbors in the training set [113] where K is a constant in the classification scheme that defines
how many nearest data points from the training data are used to classify the testing sample. Therefore, K has an effect on the quality of the classification results since larger values of K usually reduce the influence of noisy points in the training sample but can introduce undesired bias, and small K usually leads to larger variations in the predicted classes but requires less computation [114].

As reference the work from [63] was used to determine the 12 hand positions to be classified; these are shown in Fig. A.22. The 12 positions consist in four degrees of openness in neutral, flexed, and extended positions.

<table>
<thead>
<tr>
<th>Open</th>
<th>Neutral</th>
<th>Closed</th>
<th>Power Grasp</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Neutral</td>
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<td></td>
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<tr>
<td>2</td>
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<td>3 Closed</td>
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<td>4</td>
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<td>5 Flexion</td>
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<td>8</td>
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<td>9 Extension</td>
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<td>10</td>
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<tr>
<td>12</td>
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</tbody>
</table>

Figure A.22: Hand Positions for Classification Using KNN. The Numbers on the Upper Left Corner in Each Picture are the State Number Assigned in the Algorithm.
Tests

Data was collected from an non amputated subject instructing the individual to change the position of the hand when indicated and hold the position until a change was indicated. The order of the positions was in ascending order from 1 to 12 as shown in Fig. A.22.

Fig. A.23 shows the collected data during one test. In light gray it can be observed the state number of the hand and the other 4 signals are the readings from the FSRs. The data was processed off-line in Matlab to determine the classification accuracy of the KNN algorithm. All the data was used, including the data from the transitions.

![Signals from FSRs](image)

**Figure A.23:** Data from FSRs During a Test Where the Hand was Positioned in Each of the 12 Defined States. In Light Green is the Value of the State vs Time.

First, the data was classified varying the value of K from 1 to 20 to find the smaller value of K with good classification results, that in this case, it is important to find the smallest acceptable value of K since in an actual prosthesis this computation will be done in a wearable computer. It was found that K=1 was the best option. In Fig. A.24 can be seen the accuracy of the classification for each value of K. Fig. A.25 Shows graphically the confusion matrix for K=1.
Figure A.24: Off-line Percentage Accuracy for the Same Data Sample Using the KNN Algorithm to Classify the State of the Hand vs Number of K Neighbors Used in the Test.

Figure A.25: Confusion Matrix of One of the Test Done with KNN Algorithm to Classify the 12 Hand States Using Independent FSRs Readings Using K=1. Accuracy = 96%.
VITA

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