RF Convergence of Radar and Communications: Metrics, Bounds, and Systems

by

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ABSTRACT

RF convergence of radar and communications users is rapidly becoming an issue for a multitude of stakeholders. To hedge against growing spectral congestion, research into cooperative radar and communications systems has been identified as a critical necessity for the United States and other countries. Further, the joint sensing-communicating paradigm appears imminent in several technological domains. In the pursuit of co-designing radar and communications systems that work cooperatively and benefit from each other’s existence, joint radar-communications metrics are defined and bounded as a measure of performance. Estimation rate is introduced, a novel measure of radar estimation information as a function of time. Complementary to communications data rate, the two systems can now be compared on the same scale. An information-centric approach has a number of advantages, defining precisely what is gained through radar illumination and serves as a measure of spectral efficiency. Bounding radar estimation rate and communications data rate jointly, systems can be designed as a joint optimization problem.
To Edan
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The problem of spectral congestion is forcing legacy radar band users to investigate methods of cooperation and co-design with a growing number of communications applications. This problem has motivated government entities like the Defense Advanced Research Projects Agency (DARPA) to begin funding and investigating these methods to not only ensure military radar coverage is maintained as spectral allocation is renegotiated, but to potentially improve both military radar and military communications by co-designing the systems from the ground up [1]. However, these issues extend far beyond just commercial communications and military radar, and include a wide variety of applications such as next generation automobiles, medical devices, and 5G wireless backhaul. As a result, researchers have begun investigating not just methods of military radar and communications coexistence, but more fundamentally methods of joint remote sensing and communications.

These two functions, at their core, tend to be at odds with one another. For example, sensing typically sends a known waveform or stimulus and measures a response from the environment, often referred to as the channel. In the case of the radar system, the sent signal is known and the target channel is unknown and is desired to be sensed (estimated). However, a communications system typically sends an unknown signal with the assumption that the propagation channel is known or previously estimated. We can also consider the near inverse of this situation: passive radar. In this case, we must estimate the data as a nuisance parameter to obtain the information we care about (channel estimation). A non-adaptive communications channel, where the channel is stationary or controlled, is the dual of the traditional radar system.
Figure 1.1: Current RF spectrum challenges illustrated in a simplified topology. Only showing two users, a communications and radar user, and some external interference, the challenges facing heterogeneous two-user legacy systems is readily apparent. The users can be operating in the same band and adjacent in space, or co-located and operating in adjacent spectral bands. Regardless, both users present interference to the other and require mitigation to function optimally.

Figure 1.2: Future RF spectrum topology. Users are dynamic and adapt to the environment and one another. There are no dedicated radar or communications resources, but dynamic elements cooperate and are co-designed to meet the instantaneous mission. External interference is reduced inherently, as two-user systems are more easily extensible to multi-user, multi-function nodes capable of adapting and communicating to enable the joint, distributed mission.
Therefore, when considering the general task of jointly sensing and communicating, it becomes immediately apparent that the solution is non-trivial.

A typical two-user topology and the problem of spectral congestion is illustrated in Figure 1.1. With opposing requirements, sensing and communications systems are often designed in isolation. The only consideration for the other user in legacy systems has been in the form of regulatory constraints, such as those imposed by the FCC in the United States. However, governmental regulation does nothing to incentivize either user to minimize interference beyond the required limits or assist each other to mutual benefit. As future systems vie for spectral resources, RF convergence and cooperation is the solution to an increasingly crowded wireless domain. We define the ultimate solution, RF convergence, to be the operating point at which a given bandwidth allocation is used jointly for radar and communications to mutual benefit. This includes but is not limited to multi-function transceivers, in-band full-duplex (IBFD) operation, shared waveforms, and dynamic time allocation.

Shown in Figure 1.2 is where the authors see the future of channel topologies heading. Rather than dedicated radar or communications elements, universal dynamic users are designed to adapt to meet instantaneous mission needs. Bandwidth, data rate, and estimation rate [2] are modulated depending on communications need, targets present, and target dynamics. While one may note both cognitive radio and cognitive radar are both active fields of development, cognitive radio has typically been developed in the context of resource sharing [3], while cognitive radar has traditionally focused more on intelligent radar systems to improve radar performance [4]. Previous surveys have looked at the spectral congestion problem from a dynamic access perspective with a focus on regulatory issues and signal processing [5]. However, the focus in that work is still on dynamic communications users, not necessarily including remote sensing users. Recent work surveyed spectrum sharing methods and
underutilization of RF resources [6], focusing mainly on communications with some mention of sharing with non-communications users. This work focused more on existing spectrum sharing regulation, as opposed to future architectures and limits of coexistence.

In this work, we discuss the general problem of spectral congestion and the future solutions to this problem. For the two-user case, RF convergence is broken down into four topologies: the joint multiple-access channel, the monostatic broadcast channel, the bistatic broadcast channel, and the IBFD channel. These topologies have been explored in recent literature by various researchers as interest in RF convergence has resurged. The joint multiple-access channel problem is addressed in References [2, 7–15], while the bistatic broadcast channel topology is addressed in References [16, 17]. The monostatic broadcast channel is discussed in Reference [18]. Finally, the broad and complicated topic of IBFD is covered in detail in Reference [19]. While we frame the problem for applications concerning radar and communications, the discussion in this work applies to all mediums where sensing and communications are possible. The focus on RF is because the problem arises in the concern for optimizing a precious resource: RF spectrum.

In exploring the problem, we first look at the various applications that can benefit from co-designing systems that require remote sensing and communications in Section 1.1. The special case of two-user topologies for joint radar-communications systems are developed in Section 1.2. In Section 1.3, the different levels of system integration the two users can adopt are presented, ranging from mere coexistence (mitigating mutual interference) to completely co-designed systems. We present the state of the art for joint systems in Section 1.4, and look toward bounds on future systems and solutions in Section 1.5.
1.1 Joint Sensing-Communications Applications

In this section, we discuss the various applications that could benefit from the more general joint sensing-communications paradigm, or are currently being researched from that perspective. These include mixtures of military and commercial users, medical devices, and light based applications, among others. Any systems or industries that have benefited or could benefit from advances in cognitive radio and cognitive radar can most likely greatly benefit from RF convergence, as the problems are closely related. This can also include other resource-limited sensing modalities such as acoustics and sonar-based systems. These applications each have different system goals, constraints, and regulatory issues. As many researchers are finding, marrying remote sensing and communications can be theoretically difficult [20], but the need is readily apparent and increasing in urgency.

1.1.1 Automotive Radar & V2V Communications

The smart car revolution has lead the way in research on intelligent transportation systems (ITS). With self-driving cars on the verge of becoming viable, two clear technological needs emerge: vehicle-to-vehicle (V2V) communications and navigational/avoidance radar. Both RF [21] and light based [22] solutions have been proposed for V2V communications as car technology evolves. Further, vehicle radar is already deployed to consumer vehicles for collision avoidance and self-driving features. However, researchers have already started looking at joint radar-communications systems for V2V applications since these needs are so closely coupled [22, 23].
In commercial flight air traffic control, a joint sensing-communications problem has been present for many decades. Radar-like functionality is desired for locating friendly aircraft, while communications with pilots is paramount to coordinating the flight of multiple aircraft near and around airports. Modern systems employ a Mode S beacon radar system, which combines an interrogating radar with a communications response [24]. In this sense, the Mode S commercial flight system can be thought of as a cooperative radar scheme, where targets respond to radar stimulus with information back to the radar. A related system is the automatic dependent surveillance-broadcast (ADS-B), in which complicit aircraft self-locate via satellite navigation and broadcast their position to allow ground controllers and other aircraft to track their location [25]. New software-defined systems are attempting to integrate these various systems to minimize circuitry and maximize flexibility [26].

1.1.3 Communications & Military Radar

Military radar has been losing spectrum allocation to communications systems, and has subsequently spawned a great deal of research in recent years. The communications user can be commercial or military/governmental, and non-military radar can apply solutions from this thread of research as well. With the accelerating demands of commercial communications, specifically cellular and broadband wireless Internet usage, concerns are growing for the future of military radar allocation [27].

1.1.4 Medical Sensors and Monitoring

Medical devices are often deeply embedded biologically, and therefore operate at lower power. However, sensing is often a primary function of such devices, which
can require significant power to complete effectively. Cloud based approaches have evolved where sensing elements communicate their measurements to external processing structures for further analysis [28]. This topology extends to non-medical applications as well, encapsulating deeply embedded low power sensor applications [29]. The need for combined sensing and communicating naturally arises in these systems, and to do so on a single radio is especially advantageous to minimize the invasiveness and physical footprint of medical implants. Getting a communications signal out of the body has its challenges as well, and researchers have begun looking at devices to estimate and equalize the human body channel prior to transferring data [30].

1.1.5 High Frequency Imaging and Communications

Systems employing upper millimeter-wave have been proposed for high throughput communications and fine resolution sensing. For example, Google’s Project Soli performs precise motion detection using a 60 GHz radar targeted for mobile devices [31, 32]. The ultimate goal is low power, gesture based control for interfaces to the next generation of smartphones and tablets. This complements well with advances in high throughput device-to-device communications using the same frequency bands and wireless backhaul applications [33]. A single radio could handle both sensing for input control, and communications for high bandwidth applications, potentially simultaneously.

1.1.6 Li-Fi and Lidar

Light based systems have been growing in interest as a solution to growing spectral congestion and stressing of 4G systems [34]. There have been numerous research threads investigating wireless communications using infrared and visible light [34].
New standards are being developed for Li-Fi, a light-based equivalent to Wi-Fi [35], and other indoor optical systems to meet growing consumer needs for high throughput, media rich systems [36]. There is an equally fast growing industry researching optics for remote sensing applications [37], such as Lidar for wetland mapping and monitoring [38], and remote sensing using optical systems for coastal resource management (e.g., monitoring sea levels) [39].

1.1.7 RFID & Asset Tracking

Near field sensing technologies like radio-frequency identification (RFID) currently integrate remote sensing and data transfer to some extent. Communications is already the core functionality of RFID technology, as tags communicate back identification, health, and status information [40]. In addition, RFID networks have been modeled as a virtual communications capacity problem [41]. RFID technology has also been researched for radar detection [42] and localization [43]. Far field RFID-like radar systems have also been investigated as a joint radar-communications solution with a cooperative target [44]. Given that a typical RFID requires external stimulus or application of RF energy to initiate the communications link, the parallels to radar naturally arise (sending RF energy to a target for a measurement), and the joint sensing-communications aspects are immediately clear.

1.2 Heterogenous Two-User Topologies

In this section, we explore the various two-user radar and communications topologies seen in the real world and considered for solutions enabling RF convergence. For the topologies discussed here, we assume we have two users signaling in the same band, that are co-located or operating nearby. There are extensions and variations to the models discussed here, but the topologies in the following section represent an
initial attempt at capturing two-user configurations of RF convergence. To start, we
discuss the “problem topology,” the real world heterogeneous multi-user interference
channel.

1.2.1 RF Convergence Model

Here we explore a real world channel that is representative of the spectral con-
gestion problem. For this model, both radar and communications are developed in
isolation and must compete for spectrum in a given space-time. Interference miti-
gation is often managed through regulation. However, provided the respective users
adhere to regulatory requirements, neither user is incentivized to minimize their im-
pact on the other user’s performance. In modern systems, users also employ signal
processing to adaptively minimize interference that remains despite adherence to reg-
ulatory wireless standards.

An example of this topology is shown in Figure 1.1. Even in a simple single
radar, single communications link scenario, there can be significant interference be-
tween systems. Adding communications users and targets, along with strong clutter
reflections typical of real world environments, the various sources of unwanted RF
energy at a given user’s receiver begins to compound. Adhering to regulations often
does not prevent these types of complicated scenarios from occurring, burdening sys-
tems with additional interference mitigation design requirements or limiting system
performance. In fact, representative topologies of the real world problem can be so
complicated, research into joint radar-communications has been focused in some areas
solely on methods of testing cooperative or co-designed techniques [45]. To assist in
analysis and design of joint systems, simplified multi-user topologies are presented in
the following subsections.
1.2.2 Joint Multiple Access Channel (MUDR)

The joint multiple access channel topology includes a common receiver for both radar and communications, but with independent transmitters. In this topology, User 1 is a monostatic radar transceiver (sends and receives radar signal), and simultaneously acts as a communications receiver. User 2 is a communications transmitter. This can easily be extended to $N$ communications users, and $M$ targets. This topology is shown in Figure 1.3. This architecture has numerous advantages. The radar user is sending a known waveform into the channel, and in some cases can obtain equalization data for the communications path. Since the tracking radar is dynamically estimating the target state, the predicted radar return can be subtracted from the signal to minimize radar interference for the communications user. This knowledge also allows the communications user to transmit at a higher rate when it knows the radar is not transmitting or listening for target returns, as opposed to being forced to sense spectrum use.

An example of a system exploiting this topology is known as the multiuser detection radar (MUDR), explored in References [2, 13], and other related works.
Figure 1.4: Monostatic broadcast channel topology. The user on the left acts as a monostatic radar transceiver (transmitter and receiver), while simultaneously functioning as a communications transmitter. The user on the right is a communications receiver.

1.2.3 Monostatic Broadcast Channel

The monostatic broadcast channel uses a shared waveform for both the monostatic radar and the communications link. It is achieved by reversing the roles of the communications link from the previous topology, as shown in Figure 1.4. User 1 is now a monostatic radar transceiver (sends and receives radar signal), and simultaneously acts as a communications transmitter. User 2 is now the opposite role, functioning as a communications receiver. This too can easily be extended to multiple communications users and targets. While a seemingly subtle shift, the two systems are now much more tightly coupled as they must share a common waveform. This means that communications must be parasitic, and that radar performance may be a function of the data being sent.

1.2.4 Bistatic Broadcast Channel

The bistatic broadcast channel also uses a shared waveform for both the communications link and the radar, which now operates bistatically as shown in Figure 1.5. This has the same challenges as the previous topology, but with the added benefit that the bistatic radar inherently performs channel estimation to directly support the
Figure 1.5: Bistatic broadcast channel topology. The user on the left acts as a radar transmitter, while simultaneously functioning as a communications transmitter. The user on the right is a radar receiver, while simultaneously functioning as a communications receiver.

Figure 1.6: In-band full-duplex channel topology. The user on the left acts as a radar transmitter, while simultaneously functioning as a communications receiver. The user on the right is a radar receiver, while simultaneously functioning as a communications transmitter.

communications link equalization requirements. The passive radar fits within this topology.

1.2.5 In-Band Full-Duplex (IBFD) Channel

The IBFD channel is achieved by once again reversing the communications link as seen in Figure 1.6. Now User 2 on the right can provide feedback to User 1, the radar transmitter, regarding target information to support tracking. The topology is named due to the inherent system architecture from this bistatic radar and communications link. That is, both users are effectively attempting to operate in full-duplex mode over the same instantaneous band. Subsequently, the challenges of this architecture are readily understood and are subsumed by the vast literature base of this topic [19].
1.3 Joint System Design & Integration

In this section, we enumerate the various levels of integration of the joint sensing-communications problem. This facilitates exploration of the complex solution space, and helps identify what work falls into what category.

1.3.1 Non-integration (Isolation)

We define non-integration to mean that no attempt at physically integrating the sensing and communications systems is made, as illustrated in Figure 1.7. For example, if each system is completely isolated in spectrum-space-time, then there is no attempt at RF convergence. Realistically, perfect isolation is not achievable, and the various users operate to the limit regulatory laws allow. Often, in real world scenarios, the performance of all users is degraded. This is one of the incumbent solutions, and part of the problem. This architecture, shown in Figure 1.7, may represent the two users being adjacent in space and coincident in spectrum or co-located and adjacent in spectrum. Subsequently, both users are susceptible to interference from the other, and make no attempt to adaptively cancel one another.

1.3.2 Coexistence (Mitigation without Communication)

Coexistence methods burden radar and communications transceivers to treat one another as interferers. This means that any information required to mitigate the other system’s interference is not shared, and must be estimated. For example, cognitive radio blind spectrum sensing techniques are employed to inform space-time duty cycling of spectral access. This is close to where systems are today, but still presumes an attempted level of mutual mitigation instead of assumed isolation in space or time. This is also a legacy solution to the spectral congestion problem, and requires
Figure 1.7: Non-integration block diagram. Radar and communications systems are designed, developed, and manufactured in total isolation.

both systems to consider one another interference. One example is passive radar that takes advantage of communications broadcast to perform radar operation, but must mitigate the source of interference (direct path propagation). As a result, the architecture reflected in Figure 1.8 has added an adaptive canceler function to both users’ paths.

1.3.3 Cooperation (Communication to Mutual Benefit)

Cooperative techniques typically mean that some knowledge is shared between systems in order to more effectively mitigate interference relative to one another. More generally, the two users may no longer consider each other interferers, but may exploit the joint knowledge to improve both systems’ performance. In this regime, the systems may not significantly alter their core operation, but willingly exchange information necessary to mutually mitigate interference. This level of integration is the first step toward joint systems, and is aimed at seriously attempting RF convergence. For example, a passive radar system might receive dynamically updated information
Figure 1.8: Coexistence block diagram. Radar and communications systems are designed to sense one another to some degree and adapt their respective radios to mitigate one another’s interference.

from the communications user, who has knowledge of the passive radar’s intended function, to facilitate with the remote sensing process and assist with direct path and co-channel interference mitigation. This shift in channel architecture is shown in Figure 1.9, where the systems now exchange information to benefit one another and assist in signal mitigation.

1.3.4 Co-design (Joint Design & Optimization)

Co-design is the paradigm shift of considering communications and radar jointly when designing new systems to maximize spectral efficiency. In this regime, systems are jointly designed from the ground up, and now have the opportunity to improve their performance over isolated operation. Note that this does not mean the systems are physically co-located necessarily, but rather describes both systems being designed as a joint system. For example, our passive radar cooperative solution can be improved by co-design of the systems. The communications user can make the
design choice to use codes, modulation schemes, and training sequences that benefit the passive radar operation. In turn, the passive radar user can provide multistatic channel estimation feedback to assist the communications user in the equalization process. This is reflected in Figure 1.10, where the two functions are now represented as a jointly designed system.

1.4 State of the Art

In this section, we investigate the state of the art of joint radar-communications, or more broadly joint sensing-communications. We look at systems that employ cognitive techniques, joint waveform/coding design, user subscription, passive communications, and passive radar, among others.

These solutions can typically be delineated by waveform. This can include what waveforms for each user help reduce mutual interference, and joint waveform designs that accomplish both sensing and communications. Transmitted waveforms are often
central to modern joint radar-communications system design due to the sensitivity of both users to the choice of waveform. An example of this fundamental tradeoff is given in Figure 1.11. This is the autocorrelation function of two types of waveforms: a linear frequency-modulated (FM) chirp typical of radar systems (shown in blue), and an orthogonal frequency-division multiplexing (OFDM) waveform used often in modern communications (shown in dashed orange). In the full autocorrelation, it is immediately clear that the OFDM presents significant ambiguity due to the cyclic prefix. For ranging of targets, this could be catastrophic, as the sidelobes are only 10-12 dB down from the main lobe, and very far away. Further, the autocorrelation is dependent on the data, and can vary significantly. This also affects the peak-to-average power ratio (PAPR) in a data-dependent way, which can cause significant...
Figure 1.11: Normalized autocorrelation function for the linear FM chirp (blue), and an OFDM waveform (dashed orange). The cyclic prefix in the OFDM waveform presents as significant ambiguity at non-zero time delays, presenting a challenge for radar processing in low SNR scenarios. However, the main lobe is narrower by nearly a factor of 2, indicating a significant reduction in range estimation variance in high SNR regimes, due to the noise-like properties of the data-driven OFDM waveform. The linear FM chirp has a wider main lobe, but well-behaved and fast decaying sidelobes. However, they convey no information for arbitrary data transfer.
Figure 1.12: Magnitude of the complex ambiguity function, or CAF, for both the linear frequency-modulated chirp (shown in blue), and an OFDM communications signal (dashed orange). The random data of the OFDM waveform has noise like properties, and so the CAF surface falls off rapidly away from the zero-Doppler, zero-time shift origin. Shown are contours at -3 dB, -6 dB, and -10 dB from the peak at (0,0). The same contours are shown on the LFM ridge. The linear contours are a result of range-Doppler coupling. This can be favorable in some processing conditions, where Doppler shifts do not eliminate range-only matched filters. However, they present a bias in range as a function of Doppler. The near perfect OFDM spike means the cross-ambiguity matched filter may need to be sampled at extremely fine granularity to be detected.

issues for radar power circuitry [46]. The communications user is agnostic to this behavior, as pilot signals are used for alignment, and cyclic prefix is discarded to mitigate channel fading [47]. The linear FM chirp, on the other hand, would make a poor communications signal, as it contains no modulation where data could be encoded. However, it provides superior range estimation for global error scenarios. If we look at the main lobe function up close in the bottom part of the figure, we see the OFDM main lobe is nearly twice as narrow. This means for high signal-to-noise ratio (SNR) tracking scenarios, the variance of the radar measurement is much smaller for the OFDM waveform, at least for this particular bit pattern.
Another way to look at the problem is shown in Figure 1.12, which shows the full complex ambiguity function magnitude for both waveforms as a contour plot. Here, we see that linear FM chirps have range-Doppler coupling that further complicate processing, requiring compensation or the full complex ambiguity function calculation [48]. However, this can be seen as being Doppler resilient, and so may ultimately benefit over OFDM systems. In the case of the OFDM waveform, where the ambiguity function is shown in the dashed orange curves, the noise-like properties of the waveform result in a near delta function correlation shape. This can be beneficial for radar processing when ignoring global error, but may complicate matched filtering by adding significant straddle loss and driving significant processing power [48].

This simple trade illustrates the importance of waveform design, which is why much of the recent work in joint radar-communications has been focused on optimizing a joint waveform. We now discuss some of these systems in more detail and other state-of-the-art methods achieving RF convergence.

1.4.1 Coexistence Methods

Preliminary research looked at the problems posed by spectrum sharing relative to each user. Immediately it was clear that unaddressed or unacknowledged, the effect on radar for even mild interference is large for in-band operation [49]. Therefore, coexistence methods have acknowledged the interference posed by spatially adjacent users to incorporate this interference model into their processing [50]. For example, recent works have looked at modifications to the optimal symbol decision regions for communications based on additive radar interference statistics and the resulting average bit error rate (BER) performance in different SNR regimes [51]. Using the results in Reference [51], we demonstrate an example of the modified detection region for an 8-PSK system in Figure 1.13, with an SNR of 5 dB and a signal-to-interference
ratio of -5.5 dB. This partitioning is determined by employing maximum likelihood symbol detection with a random radar interference phase term:

\[ Y_{rx} = \sqrt{S}X + \sqrt{I}e^{j\Theta} + Z, \quad (1.1) \]

where \( Y_{rx} \) is the received signal, \( S \) is the communications signal power, \( I \) is the interference-to-noise ratio, \( X \) is the communications symbol from constellation \( \mathcal{X} = \{x_1, \ldots, x_N\} \), \( \Theta \) is the radar interference random phase term, and \( Z \) is a standard Gaussian noise source. Under the random radar phase term, the maximum likelihood symbol for a given received point in the constellation is given by [51]

\[ \hat{l}(y) = \arg \min_{i \in [1:N]} |y - \sqrt{S} x_i|^2 - \ln \left[ I_0 \left( 2\sqrt{I} |y - \sqrt{S} x_i| \right) \right], \quad (1.2) \]

where \( y \) is the received signal, \( x_i \) is a hypothesis constellation point, and \( I_0(x) \) is the modified Bessel function of the first kind, order 0. It can be seen in Figure 1.13 that far from the origin, the decision regions are the standard, interference free regions. However, the decision symbol regions close to the origin are distorted by the spatial Bessel function term. This illustrates that even in this simple interference model, the communications decision logic is significantly altered by the presence of the radar.

Conversely, others have looked at the effect of communications interference with significant structure on radar estimation variance bounds [52]. Some researchers have investigated optimization at a circuit level, using Smith Tubes to adaptively manage radar and communications regulatory spectral masks through dynamic impedance matching [53]. Others have investigated radar waveform design in legacy communications bands where the communications systems are rigid and unable to adapt [54], where the radar waveform employs a form of water-filling. Researchers have also looked at spectrally constrained radar waveform design to determine the feasibility of future systems believed to be faced with growing regulatory requirements.
Figure 1.13: Modified maximum likelihood decision region partitioning for 8-PSK communications signaling in the presence of a radar signal modeled as an interference random phase source. The different colors represent different symbol detection regions, one color for each of the 8 points in the constellation. The normal, interference-free constellation is present at radii greater than about 6. Received symbols closer to the unit circle result in a complicated decision region driven by a spatial Bessel function due to the random radar interference phase term.

[55]. Some researchers have looked at radar estimation performance and integration time impacts on a communications user’s ability to cancel radar returns, specifically in high power radar scenarios [56]. Others have developed computationally feasible models for performance impact on meteorological radar from secondary communications users with dynamic frequency selection capability [57]. The link budget of Long-Term Evolution (LTE) systems operating in legacy radar bands at finite stand-off distances in urban environments has been calculated to determine the impact for cellular customers [58]. Others have studied regulatory exclusion zones for radar and cellular users, concluding that the stipulated stand-off range is overly conservative
relative to the impact both users present to one another [59]. Some researchers have fused multiple LTE simulation and radar models to examine the impact of rotating radar users on LTE systems at various distances and cellular configurations [60].

1.4.2 Reconfigurable Systems

As a step toward joint systems, many researchers have focused on developing algorithms, waveforms, and other components to modern radio systems that can be reconfigured for either communications or radar at any given time. For example, OFDM platforms that can execute broadband communications or radar imaging depending on configuration [61, 62]. Others have used fuzzy logic to dynamically allocate bandwidth at the system level to either radar or communications depending on target dynamics [63].

1.4.3 OFDM Waveform Design

Almost immediately since the resurrected interest in joint radar-communications, multiple threads looked at OFDM as a viable option. Specifically, V2V applications were explored [64, 65]. This work was extended to include multiple targets [66], and multipath [67]. Others have worked the joint system into existing software-defined radio (SDR) architectures, using a given illumination to also simultaneously communicate the previous radar image [68]. However, often times results showed conflicting cyclic prefix requirements, data-dependent ambiguities, and trouble mitigating PAPR for typical radar power requirements. As these various problems arose, research shifted to designing joint systems to suppress side-lobes [69], maintain a constant envelope [70], or reduce PAPR [71]. Some methods attempted to remove the dependency of the data from the radar processing [23] but still suffered from conflicting cyclic prefix requirements. Others tried to minimize the effects by only
allocating some subcarriers to the radar operation [72]. In some research, combined
OFDM-multiple-input multiple-output (MIMO) radar and communications is accom-
plished by nonuniformly spacing subcarriers to combat Doppler and range aliasing
issues associated with other OFDM-based joint systems [73]. Improvements in range
estimation were also found by weighting various subcarriers to improve the joint wave-
form root mean square (RMS) bandwidth with constraints to control the PAPR [74].
Some have experimentally demonstrated OFDM-MIMO joint radar-communications
systems employing adaptive interference cancellation [75]. Other researchers have
looked at selection of OFDM phase codes in a tracking context, where prior knowl-
edge of the target state is used to minimize Doppler ambiguity at the expense of
increased PAPR [76]. Finally, work to combine MIMO radar, which itself requires or-
thogonal waveforms, with OFDM to create a joint system that senses radar imagery
and then communicates this to users has been investigated [77]. However, Barker
sequences were required to be overlaid onto the data for radar performance, greatly
reducing the available communications rate. Limits to joint radar detection and data
rate have been explored for OFDM systems with variable data/radar allocation in
prior work as well [17].

1.4.4 Spread Spectrum Methods

Similar to OFDM, spread spectrum waveforms have been proposed for their attrac-
tive, noise-like autocorrelation properties. Some work focused on orthogonal spreading
codes between the radar and communications users employing direct-sequence
spread spectrum (DSSS) [78], while others looked at chirped spread spectrum (CSS)
to avoid jamming between the two users [79, 80]. Ultimately, the performance of
these systems is limited by the degree of orthogonality that can be obtained in the
joint system, theoretically limited to the inverse of the time bandwidth product [81].
1.4.5 Adaptive Spatial Mitigation & MIMO Systems

Others looked at spatial mitigation as a method for enabling joint radar and communications coexistence. Ultimately, the systems adaptively cancel specific users by exploiting system degrees of freedom and performing array processing. For example, some researchers have looked at sharing spectrum between an S-band radar with LTE cellular systems by projecting the radar signal onto the null space of the interference matrix [82–84]. Some researchers have arrived at such solutions by equalizing MIMO radar systems in the presence of MIMO communications in-band interference [85], while others have developed similar results by solving for spatial filters to mitigate communications interference by exploiting MIMO degrees of freedom [86]. However, all of these spatial methods are merely a form of spatial isolation managed by radiation patterns of steered elements, and so once again rely on the classical assumption or driving to requirement of some form of isolation. Nulling interference using spatial degrees of freedom comes at a cost as well, as discussed in Reference [87]. For example, as shown in Figure 1.14, the degradation to radar SNR as the radar attempts to spatially mitigate an interfering communications signal is dependent on the signal strength of the interferer, as well as the beamwidth separation [47]. This unit of separation for two steering vectors $v_1$ and $v_2$ is defined as follows [47, 87]:

$$b = \frac{2}{\pi} \arccos \left( \frac{\|v_1^\dagger v_2\|}{\|v_1\|\|v_2\|} \right)$$

(1.3)

where $(\cdot)^\dagger$ denotes the Hermitian conjugate. This definition is normalized such that $b = 1$ corresponds to orthogonal array responses. Note also that a residual signal-to-interference-plus-noise ratio (SINR) term also remains at high interference SNR levels, as adaptive techniques match the pace of the increasing interferer strength [87].

True MIMO radar techniques have also been proposed, given the independent
transmit elements could enable multiple communications receivers naturally within its framework. Some have looked at an information-based waveform design for MIMO systems that trades detection performance with favorable correlation properties while attempting to minimize communications interference [88]. Others have extended the null space projection methods previously mentioned to MIMO architectures to allow more fine control of the degrees of freedom [89]. Similarly, researchers have extended the interleaved subcarrier OFDM approaches of other work to function as a MIMO radar over traditional phased arrays [90]. Finally, matrix completion MIMO radar techniques that are inherently less susceptible to interference from an in-band MIMO communications user have been investigated in both non-cooperative and cooperative configurations [91]. The matrix completion methodology also has the advantage of requiring less bandwidth to send the radar image data to another site than traditional...
1.4.6 Time & Polarization Orthogonalization

Other methods of isolation include polarization [92] for co-designed systems, where a radar transceiver is on an orthogonal polarization axis relative to the communications receiver, though no performance with respect to isolation is given. Space-time dynamic isolation techniques have also been proposed, such as communications devices duty-cycling carefully to avoid spectral collisions with rotating radars [93–96] assuming knowledge of how the radar is transmitting as a function of time. In the absence of this knowledge, researchers have investigated dynamic communications schemes employing electronic intelligence (ELINT) techniques to augment knowledge aided databases and avoid active radar transmitting interference [97, 98]. As a means to combat radar interference in Wi-Fi systems, some have investigated detecting radar during Wi-Fi quiet times as a means to mitigate the radar signal within the Wi-Fi framework by switching to an interference-free channel [99]. A similar system detects radar pulse trains during quiet time and applies this knowledge to time-division duplexing (TDD) systems like WiMAX [100].

1.4.7 Carrier Exploitation Methods

Rather than cooperate with cellular systems, some have proposed employing the existing cellular framework as a solution to augment dwindling radar spectrum. For example, when radar functionality is required, systems can subscribe as cellular users and allocate bandwidth within the existing cellular framework [101]. We demonstrate an example of the concept outlined in Reference [101], known as radar as a subscriber technology (or RAST), in Figure 1.15. In this plot, the radar allocates 8 subscribers and selects 8 codes from the 32-bit Hadamard matrix to best approximate the ideal
Figure 1.15: Radar as a subscriber technology (RAST) waveform concept. The radar, when needed, subscribes to cellular service as multiple users and selects multiple-access codes optimally to approximate the desired radar waveform. In this example, 32-bit Hadamard codes with 8 subscribers are used with Hadamard codes chosen to maximize the autocorrelation with the ideal chirp waveform.

The radar waveform for this system, a linear FM chirp. The approximation improves with the number of subscribers [101].

This is further enabled by recent additions to the cellular LTE standard that enable carrier aggregation, or user request of multiple carriers to access larger instantaneous bandwidth [102]. While bandwidth is still allocated to isolated users, it is done within a dynamic framework that supports needs-based resource distribution while enabling normal use of high performance cellular infrastructure. Other approaches accepting that cellular infrastructures will dominate aim to design optimal radar waveforms that are robust to in-band, nearby cellular users and also minimize their interference to those users using non-convex optimization techniques [103].
1.4.8 Passive & Parasitic Systems

In the same line as carrier methods, many researchers sought to analyze cellular signals in a radar context should the next generation of radar systems be forced to passively exploit communications illuminations. Some researchers looked at the ambiguity function of OFDM communications transmissions [104], while others analyzed LTE waveforms and how they stack up when used for radar purposes [105]. The ambiguity function and Cramér-Rao lower bound (CRLB) for radar estimation were researched for multistatic passive radar exploiting Universal Mobile Telecommunications System (UMTS) signals in modern cellular architectures [106, 107]. Others have focused on detection for multistatic passive systems in both centralized and non-centralized processing scenarios [108]. Some researchers have investigated alternatives to matched filtering in passive radars exploiting Global System for Mobile Communications (GSM) signals using iterative least squares methods of bounce path estimation [109], while other research groups have employed multiple orthogonal radar waveforms with embedded communications transmissions, where one waveform is the reference and the remaining waveforms exploit the differential phase from the reference to extract the parasitic data transmission [110]. For example, in Reference [111], the authors optimize multiple spatial waveforms to modulate radar sidelobe levels while maintaining a fixed main lobe to keep radar performance constant. The modulation of the sidelobes in amplitude encode a parasitic communications data stream. A simple example of this scheme is shown in Figure 1.16, where two Chebyshev windows are chosen depending on if a 0 or 1 is to be transmitted. Here, there is some difference in the main lobe width since we have only modified the Chebyshev design parameter. More sophisticated optimization schemes such as those in Reference [111] ensure main lobe fidelity. Similarly, another parasitic embedded sidelobe
Figure 1.16: Radar spatial beam pattern for two different element weighting schemes. In this simplified scheme, two different Chebyshev windows are chosen depending on if the embedded communications is sending a 0 or a 1. Thresholding the sidelobes in the receive chain allow an embedded communications user to recover the parasitic information.

Level modulation communications scheme forms a traditional radar main beam from multiple beampatterns with various sidelobe level modulation targeted at the communications receivers [112]. Various extensions in this family of research have looked at other methods of embedding communications, for example an orthogonal waveform for each binary 1 to be encoded for the parasitic amplitude-shift keying (ASK) communications scheme, while omitting waveforms for each binary 0 [113, 114]. Others have embedded communications data streams in MIMO radars by shuffling which antennas transmit which orthogonal waveform at each illumination step in patterns defined by the data [115]. Others have looked at shared waveforms with OFDM-like schemes employing the fractional Fourier transform to modulate data onto chirped subcarriers [116]. Some have looked at performing passive detection by observing the bit error patterns in Wi-Fi protocols [117], while others have exploited the output
of a rake receiver to detect moving targets that present delayed and Doppler-offset reflections of the communications signals [118].

Signal selection relative to radar in contrast to signal selection relative to communications has been a long standing trade, tracing back over five decades [119]. Some approaches to shared waveform outside of coding have been investigated, including multiple threads researching embedded or parasitic communications. For example, some research has investigated phase modulating communications information on top of a linear FM chirp to add channel capacity while improving the ambiguity properties of the waveform with respect to radar [120]. Some have looked at more deeply embedded methods, such as a radar system modulating low-rate communications on the waveform sidelobe levels [121, 122]. Others have looked more explicitly at covert communications by embedding an OFDM waveform into a noise radar spectrum [123]. Some researchers have looked at cooperative targets with known locations that reradiate radar pulses back to the radar into one of many delay-Doppler cells to communicate within the radar physical layer [44, 124–126].

1.4.9 Cognitive Approaches

Advancements in cognitive radios and radar have been proposed as a natural solution to spectrum congestion problems. Traditionally, communications phenomenology has advanced beyond radar in terms of cognitive, bandwidth sharing techniques [3]. This is because radar has enjoyed access to excellent spectral resources and remained unchallenged for many decades [27]. Therefore cognitive techniques in a radar context were primarily for enhanced dynamic behavior in complex environments [127, 128]. Researchers have begun to look at radar scheduling as an application for cognitive systems as the spectral scarcity problem has sparked interest in this area [129]. Some have looked at adapting waveforms to signal-dependent interference from communi-
cations users [130]. Others have employed cognitive techniques to estimate communications channel parameters to reduce the mutual interference between a primary communications user and the secondary radar user [131]. This is analogous to the previously mentioned ELINT assisted techniques where the radar was the primary user. Researchers have looked to develop cognitive radar much more closely resembling cognitive radio by employing similar spectrum sensing techniques, emitter localization, and power allocation to avoid interference with cognitive radio users [132]. Others have developed cognitive techniques to extend prior work in information-theoretic waveform design for radar cross section estimation and detection to include a communications user sharing the same waveform [133, 134].

1.4.10 Information-Centric Systems

Historically, information is well known in the communications phenomenology, but less so in radar. Perhaps surprisingly, radars were looked at in the context of information theory soon after Shannon’s ground-breaking work [135] by Woodward [136]. It was not until Bell’s seminal work on waveform design using information for statistical scattering targets that information theory and radar were seriously looked at [137]. This work has been extended to multiple complicating scenarios and with more advanced features [138–141]. Considering cognitive radar advances, some research has identified a need for more intelligent metrics such as information [142], a need addressed in some research regarding target scheduling and power allocation [143]. Recent results have also found connections between information theory and estimation theory, equating estimation information and the integrated minimum mean square error (MMSE) [144]. In addition, research has started on viewing radar systems as a flow of information [145, 146]. Range estimation grid based on a constant information measure to reduce unneeded computational complexity has also been pro-
posed [147]. Information based techniques have been looked at for intelligent target scheduling in a cognitive framework [148], though similar to classical cognitive radios in that the resource is optimized within the radar phenomenology only. Preliminary work has compared cognitive operation of both users independently and jointly as well [149]. Nonrecurrent, nonlinear frequency-modulated continuous-wave (FMCW) waveforms with hopping spectral gaps with shaping to support range sidelobe roll-off were developed to support access from dynamic communications users and minimize interference [150]. Some researchers have noted that spectrum crowding is an issue within remote sensing allocations itself, and identified cognitive radar as a natural solution to the many user problem [151]. Other researchers have looked at various joint radar-communications mutual information criteria and means to maximize them, and noted that maximizing the mutual information did not always maximize radar probability of detection [152]. Finally, there have been examples of systems employing genetic algorithms to modulate radar spectral access opportunistically, similar to cognitive radio users [153]. A jointly cognitive system state diagram example is shown in Figure 1.17. Both users have to accomplish the spectrum sensing task, and so a joint receiver is used to sense both types of users in gray spectrum allocation. Once active, the dual function of the transmitter and receiver enable enhanced environmental sensing and feedback to adapt system configuration and maximize the joint mission.

1.4.11 Joint Coding Techniques

Joint coding techniques, such as codes attractive from a communications and radar ambiguity standpoint, as well as codes that trade data rate and channel estimation error have been investigated as a solution at the symbol level. Some research has looked at direct relationships between radar estimation sidelobe ambiguity and communica-
Figure 1.17: State diagram for joint radar-communications cognitive scheme. Both users need to perform the spectrum sensing task. However, in the joint case, this task is enhanced due to the dual-function circuitry which is optimized for both communications and radar sensing (instead of one or the other). Further, accomplishing both tasks in the same RF band provides improved overall sensitivity and environmental feedback into the reconfiguration task.

Complementary Golay sequences have attractive correlation properties when their autocorrelations are summed, and also bound the PAPR for OFDM communications waveforms to less than 3 dB. An example of the autocorrelation property from Reference [156] is shown in Figure 1.18. We used an example of the codes provided in the source paper to demonstrate the zero-sidelobe behavior of the sum. The individual autocorrelations for the two complementary sequences are shown in the solid blue line and the dashed red line. Their sum is shown in the thick, solid green line, and has no observable sidelobes to within the quantization noise of our system.

Recent work has looked at trading communications with channel state estimation
Figure 1.18: Complementary Golay sequence autocorrelation properties. The two complementary codes have their individual autocorrelations shown in the solid blue line and the red dashed line. By themselves, they exhibit significant sidelobe activity. However, when the autocorrelations are summed, their complementary nature cancel all sidelobes, making them attractive for using radar ranging. This sum is shown in the solid, thick green line. For communications users, use of these codes has the benefit of bounding the potential peak-to-average power ratio.

[16]. However, in this case, the channel state estimation is a nuisance parameter to increasing communications throughput, not a desired radar or sensing modality. Some research has looked into coding as a means to sharing bandwidth between an OFDM radar and Global Positioning System (GPS) signal [158] to complement both operations. Oppermann sequences have also been proposed as a natural framework for developing radar waveforms with good ambiguity properties and multi-user communications access schemes [159]. Others have applied precoding to both a MIMO communications user and an in-band MIMO radar user operating in clutter, optimizing the radar precoding to maximize joint performance [160].
Modern techniques have proposed co-design and operation as a necessary construct for joint radar-communications [18]. Others have jointly maximized information criterion for radar and communications users to minimize mutual interference by varying radar waveform and communications OFDM parameters in response to dynamic bandwidth allocation [161]. In this work, the operational distance between the two users is proposed as a figure of merit. Other work has looked at employing information exchange to reduce the minimum required standoff range between competing radar and communications users [162]. Similar work has looked at performance as a function of distance, and also the pitfalls of oversimplified models of interference in comparison to experimental results [163]. Others have investigated performance of systems in isolation compared to cooperation to demonstrate that cooperative nodes enjoy a mutual performance enhancement relative to classical isolated operation [164]. Others have begun to investigate joint radar-communications in a similar context to full-duplex communications, focusing on isolation between single hardware operation [165]. Some researchers are looking at highly flexible architectures to support not only radar and communications, but also electronic warfare, and are developing test beds to support future research in these areas [166]. Others have developed a Neyman-Pearson based cooperative metric that captures both radar detection performance and communications data rate in a joint cost metric with parameterizable weighting [167]. Researchers have also developed a more general framework for radar-communications joint resource management through development of joint figures of merits that encapsulate capacity, individual performance, and mutually beneficial performance [168]. Others have employed code division multiple access (CDMA)-like cancellation by decoding, re-encoding, and subtracting signals to mitigate interference for multiple,
heterogeneous users [169]. Some researchers have looked at exploiting energy from communications users to bolster a separate radar user’s probability of detection, and optimizing the radar waveform with the in-band communications system operating as the primary user [170], while others have explored joint channel estimation as a means to measure communications data rate and radar probability of detection in the same band [171].

1.5 Bounds on Jointly Optimized Systems

While state-of-the-art systems have mild elements of co-design, future systems must be co-designed to jointly sense and communicate, maximizing spectral efficiency. Traditionally, communications performance related to spectral efficiency is measured by the channel capacity. This is the achievable, but maximum operating rate of arbitrary data communications for a given channel probability distribution [135]. For example, for a Gaussian, band-limited, power-limited system, the maximum communications data rate is defined as [172]:

$$R_{\text{com}} \leq B \log_2 \left[ 1 + \frac{P_{\text{com,rx}}}{N_0 B} \right],$$

where $B$ is the receiver bandwidth, $P_{\text{com,rx}}$ is the received communications power, and $N_0$ is the noise power spectral density.

Often, prior to recent work on information-theoretic bounds, radar estimation performance limits are dictated by the CRLB. For typical radar range estimation in Gaussian receiver noise, this is given by [173]:

$$\sigma_{\text{CRLB}}^2 = \frac{N_0 B}{8\pi^2 B_{\text{rms}}^2 T_p B P_{\text{rad,rx}}},$$

where $T_p$ is the radar pulse duration, $B_{\text{rms}}$ is the radar waveform RMS bandwidth, and $P_{\text{rad,rx}}$ is the radar receive power.
While these metrics work in isolation for each user, they do not adequately measure joint performance. To address this, researchers have derived fundamental limits on joint radar and communications operation, and were successful in producing several alternative interpretations of joint radar-communications performance and bounds on those resulting joint metrics [174]. Next, we highlight several research threads that investigate bounds on joint radar-communications performance for future systems.

### 1.5.1 Radar Estimation Rate and Joint Bounds

To measure spectral efficiency for radar systems, we look at our recent research quantifying radar information as a function of time: radar estimation rate [2, 7–14]

\[
R_{est} = \frac{I(x; y)}{T},
\]

(1.6)

where \(I(x; y)\) is the mutual information between random vectors \(x\) and \(y\), and \(T\) is the time period between spectral accesses. This can be a pulse repetition interval (PRI) or a target revisit period. This allows construction of joint radar-communications bounds, and allows future system designers to score and optimize systems relative to a joint information metric.

For a simple range estimation problem with a Gaussian tracking prior, this metric takes on the following form [13]

\[
R_{est} = \frac{1}{2T} \log_2 \left[ 1 + \frac{\sigma^2_{proc}}{\sigma^2_{CRLB}} \right],
\]

(1.7)

where \(\sigma^2_{proc}\) is the range-state process noise variance, and \(\sigma^2_{CRLB}\) is the CRLB for range estimation given by Equation (1.5). One immediately notes the similarity to Equation (1.4), where the ratio of the source uncertainty variance to the range estimation noise variance forms a pseudo-SNR term in the Gaussian mutual information.

An example of the joint multiple access channel (MAC) is shown in Figure 1.19 by plotting the communications data rate on one axis and the radar estimation rate
Figure 1.19: Estimation rate enabled MAC diagram with inner bounds. The dashed outer red lines show the isolated communications data rate and radar estimation rate, each system assuming it has the full bandwidth. The goal then is to build systems that get as close to the upper right corner of this manifold as possible. Shown inside these bounds are constructive inner bounds that are discussed in Chapter 4.

on the other axis. Modern systems attempt to get as close to the upper right hand corner of the outer manifold as possible. Here, inner bounds from prior work are shown to see how they compare to the joint theoretical limiting bounding box.

In Reference [15], a bound on radar information is formulated that looks very similar to the bound presented here. However, the mutual information in that work is between a multipath amplitude statistic before and after corruption with receiver noise. As a result, if no multipath is present, the mutual information is null. The bound shown in Figure 1.19 shows radar information sourced from a target tracking prior, before and after measurement. This information is typically desired (learn knowledge of target state), in contrast to the mutual information in Reference [15], which encapsulates the multipath uncertainty, typically a nuisance parameter.
Figure 1.20: Multiple access bound plot for radar capacity defined using MTI and communications. Note, while tempting to compare with Figure 1.19, the notion of radar capacity and estimation rate are incompatible, as they define two different forms of radar information. In this plot, the MTI information limit without communications interference is shown in the vertical dashed red line, while the standard communications-only bound is shown in a similar horizontal dashed red line. The yellow line represents splitting the band between the two users in isolation with different weightings, while the blue line shows time-division multiplexing of the radar and communications systems.

1.5.2 Radar Capacity and MTI with Communications

In Reference [18], radar information capacity is formulated using range-bearing-Doppler binning moving target indicator (MTI). These capacity equations assume a discrete three-dimensional (3D) grid where a target could be present with an implicit probability of 0.5:

$$C_{\text{rad}} = \frac{1}{T} \frac{R_{\text{max}}}{\Delta T_s} \frac{2\pi \text{ PRF}}{\Delta \theta \Delta f_D},$$

where $R_{\text{max}}$ is the maximum range that closes the radar range link budget, PRF is the pulse repetition frequency, $\Delta T_s$ is the sampling rate resolution, $\Delta \theta$ is the bearing resolution, $\Delta f_D$ is the Doppler resolution, and $T$ is the revisit time period. Commu-
communications rate is the same as the previous case. A similar example for this bounding interpretation is given by Figure 1.20. In this bound, each range cell represents a bit of information, with the probability of detection in each cell conservatively assumed to be 0.5. This means each spectral access learns through the multi-bin Bernoulli distribution. In practice, prior information influences the probability of detection for each cell, but the plot is useful as an upper bound on information. While it may be tempting to compare Figure 1.20 with Figure 1.19, the two are inherently incompatible due to the fact they define radar information in two drastically different ways.

1.5.3 Constrained Channel Estimation and Communications

In Reference [16], information-theoretic bounds on a joint capacity-distortion function are developed. In this work, the balance between sending arbitrary data and distortion in estimating the communications channel is explored. Sending less information and more known signals, the channel can be better estimated. A better channel estimate ultimately supports an increased channel capacity. However, the average rate is reduced in sending known symbols to estimate the channel.

It was shown in this work that, for a channel with uniform estimation costs, a system trying to transmit information and simultaneously perform channel state estimation can achieve the following rate-distortion tradeoff [16]:

\[ R \leq B \frac{1}{2} \log_2 \left[ 1 + \frac{\gamma P_c}{P_n} \right], \]  \hspace{1cm} (1.9)

\[ D \geq \sigma_{targ}^2 \frac{\gamma P_c + P_n}{\left(\sigma_{targ} + \sqrt{(1-\gamma) P_c}\right)^2 + \gamma P_c + P_n}, \]  \hspace{1cm} (1.10)

where \( B \) is the bandwidth of the system, \( \gamma \) is a system parameter swept from 0 to 1, \( P_c \) is the received communications power, \( \sigma_{targ}^2 \) is the radar residual variance after radar cancellation, and \( P_n \) is the thermal noise power.
Figure 1.21: Multiple-access bound for joint communications and channel estimation. The communications channel consists of a known direct path channel and an additive radar bounce channel interference term. The monostatic radar signal sends a known waveform, and knows the target channel from the tracking estimation. The communications transmitter sends its own signal direct to the receiver, which has a perfect direct path, but also receives an interfering bounce path radar return from the target. The curve in light blue represents trading radar channel estimation for data rate.

The distortion function, $D$, given by Equation (1.10) can be viewed as the variance of estimation (channel state estimation or a more general parameter estimation) and can be applied to the estimation rate given by Equation (1.6), to obtain a ‘estimation cost’ rate. The communications rate versus the estimation cost rate curve is shown in Figure 1.21.

This example assumes a monostatic radar with an independently transmitted communications signal also broadcast to a destination node, or a modification to the monostatic broadcast channel covered earlier. Due to the monostatic return of the target, the radar knows the radar signal bounce path channel to the destination node, which is modeled as an additional additive Gaussian term after successive interference cancellation (SIC) processing. The joint curve shown represents the tradeoff the
1.6 Our Approach

The remainder of this paper details our approach to the RF convergence problem, previewed in Section 1.5.1. We start by developing information metrics for radar estimation to complement communications data rate. The joint system as a function of bits/seconds can then be considered. We extend this notion to a joint spectral efficiency. Bounds on these figures of merits can be found using channel capacity arguments for the communications user, and the data processing inequality for radar estimation rate. Complex estimation distributions lead to mutual informations that can be difficult to compute. Modeling these problems using Gaussian mixture models (GMMs) allow the mutual information to be bounded using the I-MMSE formula [175]. Finally, constructive inner bounds on joint performance are developed leading to joint systems. The approach here is discussed in part of our published work [7–10, 13, 14, 175–178]. The work here references a larger thread of joint radar-communications research, but we present only content originated by the author due to the scope of this document.
In this chapter, we discuss metrics useful for development of joint sensing and communications systems. The goal is to co-design joint radar-communications systems that can cooperatively share spectrum and even benefit from one another’s existence. To effectively build systems, we need to know the bounds of what is possible and how far a given system is from these limits. Bounds must be constructed from some important metric or figure of merit. Therefore, as a first step to designing systems employing cooperative radar-communications, metrics must be developed.

2.1 Communications Data Rate

As mentioned in Chapter 1, we chose an information-theoretic view of the system to quantify communications data uncertainty and radar channel uncertainty. As such, a natural choice for our communications figure of merit is the communications data rate. This is the rate at which the system is able to communicate arbitrary data. We live in the digital age as of this writing, and so we adopt the overwhelmingly ubiquitous units of bits/second to score the communications system. Therefore, we are interested in the communications data rate, $R_{\text{com}}$, as a measure of arbitrary digital data transfer in bits/second:

$$R_{\text{com}} \quad (2.1)$$

2.2 Radar Estimation Rate

We have expressed the desire to quantify radar information in previous sections. This is important for a number of reasons. First, as opposed to the typical radar fig-
ures of merits such as signal-to-noise ratio (SNR), probability of detection, probability of false alarm, and estimation variance (e.g., the Cramér-Rao lower bound (CRLB)), considering information forces one to identify where the uncertainty is in the system and a means to reduce it. For example, if a target is extremely well modeled dynamically, then the information content of the radar track is low. This means that little to no information is gained through the process of radar illumination, detection, and state estimation. The traditional figures of merit concern themselves only with the performance once it has been determined it is desired to measure the target state, not a quantifiable answer to the question “should one measure the state?”. This is important because to maximize spectral efficiency, it is desired not to measure if the target state is known precisely (or to a precision sufficient to the system). Sending the radar waveform and measuring the state would only serve to confirm the known state, and this reduces the spectral efficiency. The time spent on radar spectral access with insufficient spectral efficiency arguably could have been otherwise utilized by a communications user. Therefore, we are interested in the radar estimation rate, \( R_{\text{est}} \), as a measure of obtainable target tracking information in bits/second [2]:

\[
R_{\text{est}} = \frac{I(x; y)}{T}, \tag{2.2}
\]

where \( T \) is the radar pulse repetition interval (PRI) or target revisit time, \( I(x; y) \) is the mutual information between random variables \( x \) and \( y \), \( x \) is the target state distribution vector (in the measurement domain), and \( y \) is the measurement vector. Note that for frequency-modulated continuous-wave (FMCW) radars or non pulsed radars, \( T \) can be replaced by the coherent processing interval (CPI) [8]. It may also be replaced by the target revisit time if the target’s response is coherent over the full pulse train. The vector \( x \) represents the unknown target state. This could be simply the range for simplified tracking scenarios, or full three-dimensional (3D) Cartesian
position and velocity for more sophisticated systems. The measurement \( y \) represents the state as measured by the radar. This includes any measurement transformation (for example rectangular to polar coordinates and range-rate in lieu of a full velocity state), contributing system thermal noise, phase noise, and distortion terms, as well as clutter amplitude distributions. The mutual information over the processing period gives a measure of obtainable target state information as a function of time; this is radar estimation rate.

Solving for the mutual information depends on the complexity of the problem. In this body of work, we have solved this for pulsed radar (range [2, 10, 13, 177], bearing [7, 9], and Doppler [7, 9]), FMCW radar (range and Doppler [8]), cluttered radar [12], and radar with significant phase noise [14]. However, if a given scenario, no matter how complex, can be represented by an estimation distribution, an estimation rate can be computed. This demonstrates the power of such an approach, as it naturally subsumes many realistic radar scenarios into its framework. Computing the required mutual information in Equation (2.2) for complicated scenarios can be difficult, and so bounds are derived for general additive estimation mutual information in Chapter 9.

### 2.3 Weighted Spectral Efficiency

The communications data rate gives us a measure of data rate in bits/second, while the radar estimation rate provides a measure of data rate the target is uncooperatively communicating with the radar, also in bits/second. Both of these measures can be used to independently score the joint system’s respective performance for the communications and radar components. If we divide both of these measures by the bandwidth, \( B \), they occupy, then we obtain their respective spectral efficiencies in bits/seconds/Hz:

\[
S_{\text{est}} = \frac{R_{\text{est}}}{B}, \quad S_{\text{com}} = \frac{R_{\text{com}}}{B}.
\]  

(2.3)
One way we can obtain the combined weighted spectral efficiency for both components would be to sum the two quantities:

\[ S_{\text{add}}(\gamma) = \gamma S_{\text{est}} + (1 - \gamma) S_{\text{com}} = \frac{\gamma R_{\text{est}}}{B} + (1 - \gamma) \frac{R_{\text{com}}}{B}, \] (2.4)

where \( \gamma \) is a weighting factor between 0 and 1. However, as we see, these quantities can be vastly different orders of magnitude. For example, in Figure 1.19 the communications data rate is 5000 times larger in magnitude than the radar estimation rate, despite a typical radar tracking scenario with non-trivial estimation information. Attempts to optimize spectrum or time access may continually favor the communications system asymmetrically. A more robust method is to take the geometric mean:

\[ S_{\text{geo}} = \sqrt{S_{\text{est}} S_{\text{com}}} = \sqrt{\frac{R_{\text{est}}}{B} \frac{R_{\text{com}}}{B}}. \] (2.5)

This still has the units of bits/second/Hz like the additive metric \( S_{\text{add}} \), but is much more robust to magnitude asymmetries. Note that this imbalance can be corrected using \( \gamma \), but large imbalances may have impacted by numerical precision limitations in processing systems.

Finally, as alluded to earlier, specific applications may weight the radar and communications functions differently, and do so dynamically as a function of the current threat environment. Therefore, we can generalize Equation (2.5) to incorporate this notion as follows

\[ S_{\text{geo}}(\delta) = S_{\text{est}}^\delta S_{\text{com}}^{1-\delta} = \frac{R_{\text{est}}^\delta R_{\text{com}}^{1-\delta}}{B}, \] (2.6)

where \( \delta \) is the geometric weighting factor, also between 0 and 1.

The generalization of both measures to support weighting is key, as not all bits are created equal [177]. For example, a small number of bits from a radar characterizing an incoming missile track is vastly different in priority from many gigabits used to stream funny cat videos.
Now that we have developed information metrics for joint radar-communications systems, we must find the limits of these figures of merits to inform system designers of the obtainable region of performance. This allows us to know how close a given system design is to the theoretical limit on performance, and provides an absolute point of reference for our metrics. The joint metrics derived and used in this work, radar estimation rate and communications data rate, are both measures of information, nominally in bits/second. This means then that bounds on radar estimation rate and communications data rate can be combined as discussed in Chapter 2 to naturally produce bounds on joint radar-communications spectral efficiency.

3.1 Communications Data Rate Channel Capacity

As established in Chapter 2, data rate is our metric of choice for the communications user. Since the main thrust of our effort is to increase spectral efficiency, we measure data rate in bits/second for a given band. This allows for a measure of spectral efficiency for the communications user in bits/second/Hz. It is a well established result derived first by Shannon [135] that the limit on reliable (arbitrarily low bit error rate) communications across a bandwidth $B$ corrupted by additive white Gaussian noise (AWGN) is given by

$$R_{com} \leq B \log_2 \left[ 1 + \frac{P_{com,rx}}{N_0 B} \right],$$

for $N_0$ the channel noise power spectral density in Watts/Hz, and $P_{com,rx}$ the received communications signal power in Watts. This is the channel capacity of a
continuous channel with Gaussian noise [135, 172]. The argument of the logarithm in Equation (3.1) is unitless, and the log$_2$ operation produces a measure of bits. Since bandwidth $B$ is given in Hz (1/seconds), the resulting units are bits/second as we desired. While this does not, in general, quantify the actual communications data rate, it is the achievable limit on communications with arbitrarily low bit error rate, and so it serves to score systems with respect to how close they are to achieving the channel capacity rate. Further, it is a mathematically tractable figure of merit in its own right to determine what a communications user can achieve with various external noise and distortion sources.

3.2 Radar Estimation Rate Bounds

Radar estimation rate is defined as a mutual information between a source distribution measurement without noise, and a measurement of the state with some corruption as discussed in the previous chapter. This quantity represents the maximum amount of information obtainable, but often time processing limits our available information. Further, this can be difficult to compute analytically, and so modeling the estimation problem can provide relaxed bounds that have utility in system design and optimization.

3.2.1 Data Processing Inequality

Since radar estimation rate is a measure of estimation information, it is naturally bounded by the data processing inequality. The bound essentially states that the mutual information between a random variable and a function of that random variable is less than or equal to the original mutual information [172]. If our radar estimation rate is defined as $R_1$, then

$$f(R_1) \leq R_1.$$  \hspace{1cm} (3.2)
Since we typically define our tracking distributions in a continuous space, most systems process this distribution to some extent. For example, quantization at the analog-to-digital converter (ADC) or delay-Doppler binning both contribute to a reduction in source entropy and thus estimation rate mutual information. Therefore, the mutual information sourced as a pure estimation distribution is in fact an upper bound on the radar estimation rate. Thus we always carry the inequality with us when discussing radar estimation rate in the remainder of the paper:

\[
R_{\text{est}} \leq \frac{I(x; y)}{T}. \quad (3.3)
\]

### 3.2.2 Bounds on Mixture Models

The estimation rate formula given by Equation (2.2) is deceptively simple. Computing the mutual information term may not be possible in closed-form, or may be very difficult. To alleviate this computational burden, in Chapter 9 we construct bounds on radar estimation rate obtained through modeling our radar tracking problem as a Gaussian mixture model (GMM). Upper and lower bounds immediately follow in a relatively simple form:

\[
R_{\text{est}} \geq \sum_i \frac{p_i}{2T} \log \left[ |G \Sigma_i G^T \Omega^{-1} \Gamma + I| \right] \quad (3.4)
\]

and

\[
R_{\text{est}} \leq \frac{1}{T} \min \left\{ \sum_k \frac{q_k}{2} \log \left[ |G \Sigma_k G^T \Omega^{-1} \Gamma + I| \right], \right. \\
\left. \sum_{i,k} \frac{p_i q_k}{2} \log \left[ |G \Sigma_i G^T \Omega^{-1} \Gamma + I| + H(p) \right] \right\}, \quad (3.5)
\]

where \(p_i\) and \(\Sigma_i\) are parameters of the source distribution obtained through the modeling process, \(\Sigma\) is the source covariance matrix, \(G\) is an arbitrary transformation matrix, \(q_k\) and \(\Omega_k\) are parameters of the additive noise distribution obtained through
the modeling processing, $\Omega$ is the noise covariance matrix, and $\Gamma$ is a scalar that can be considered an signal-to-noise ratio (SNR) term.

These general bounds were derived in this work, and can be simplified in special cases to univariate estimation problems, bounds on differential entropy, and bounds for Gaussian noise problems. They are called I-MMSE bounds, as they are obtained by exploiting the I-MMSE formula to bridge bounds on minimum mean square error (MMSE) to the information domain.

3.3 Weighted Spectral Efficiency Bounds

Bounds on weighted spectral efficiency are easily found from the bounds on communications data rate and radar estimation rate, since the theoretical framework for the metric were developed in Chapter 2. Quite simply:

$$S_{\text{add}}(\gamma) \leq \gamma \frac{I(x; y)}{TB} + (1 - \gamma) \log_2 \left[ 1 + \frac{P_{\text{com}}}{N_0 B} \right], \quad (3.6)$$

and

$$S_{\text{geo}}(\delta) \leq \frac{I(x; y)^{\delta} \left( \log_2 \left[ 1 + \frac{P_{\text{com}}}{N_0 B} \right] \right)^{1-\delta}}{(TB)^{\delta}}. \quad (3.7)$$

This follows because both Equation (3.1) and Equation (3.3) are inequalities in the same direction, and so given the monotonicity of Equations (2.4) and (2.6) with respect to both rates, the inequality applies to the additive and geometric spectral efficiency as well. These can include any of the I-MMSE bounds also.
Developing joint radar-communications systems is the ultimate goal of research thrusts such as the DARPA shared spectrum access for radar and communications (SSPARC) initiative [1]. To enable this, we have first introduced some information metrics that allow radar and communications to be jointly scored on the same scale. We then bounded these metrics, allowing system designers to score how close the designed systems are to the theoretical limit. We are thus ready to discuss some new and future joint radar-communications systems. The development of inner bounds in previous works [2, 7–14, 177] has provided not just bounds related to specific operation, but bounds that are constructive by their very nature. That is, inner bounds are named as such because they are defined and thus achievable by definition. As a result, these inner bounds provide the processing framework for joint radar-communications systems. For the systems in this introductory chapter, we focus on range estimation in thermal receiver noise for clarity. Many of these systems include a control parameter, denoted $\alpha$ here, that allow the system designer to trade between radar estimation rate and communications data rate by varying parameters like sub-band bandwidth and spectral access time.

4.1 Isolated Sub-Band (ISB) System

The isolated sub-band (ISB) system splits an allocated wireless band between the radar and communications users. Each user, the radar and the communications link, is allocated a contiguous, non-overlapping sub-band of the overall bandwidth $B$. This is a non-cooperative system and a legacy way of thinking, but is included for
Figure 4.1: Isolated sub-band (ISB) system diagram. The two users are completely isolated spectrally, and the overall available bandwidth is split between the two systems. The allocation percentage is varied to construct the inner bound.

Completeness and analysis. The top level diagram is shown in Figure 4.1. The joint bounds for this system are given by:

\[ R_{\text{com,SB}}(\alpha) \leq \alpha B \log_2 \left[ 1 + \frac{P_{\text{com,rx}}}{N_0 \alpha B} \right] \]  \hspace{1cm} (4.1)

and

\[ R_{\text{est,SB}}(\alpha) \leq \frac{1}{2T} \log_2 \left[ 1 + \frac{2\sigma^2_{\text{proc}} (1 - \alpha) 2\pi B_{\text{rms}})^2 T_p P_{\text{rad,rx}}}{N_0} \right] \]  \hspace{1cm} (4.2)

where \( \alpha \) is the blending parameter (takes on values between 0 and 1), \( B \) is the total bandwidth available, \( P_{\text{com,rx}} \) is the communications receive power, \( N_0 \) is the noise power spectral density, \( T \) is the radar pulse repetition interval or revisit time, \( P_{\text{rad,rx}} \) is the radar return power, \( \sigma^2_{\text{proc}} \) is the target process noise variance (range prediction uncertainty), \( T_p \) is the radar pulse duration, and \( B_{\text{rms}} \) is the radar waveform root mean square (RMS) bandwidth. The control parameter \( \alpha \) determines how much of the overall bandwidth \( B \) is allocated to each system. For \( \alpha = 0 \), the radar user is allocated the full bandwidth. For \( \alpha = 1 \), the communications user is allocated the full bandwidth. In between provides a fraction to each (for example, \( \alpha = 0.5 \) being a 50/50 split and each user getting \( B/2 \)). Note that Equation (4.2) is obtained by combining Equations (1.5) and (1.7) and simplifying. This solution serves as a benchmark for spectral efficiency of joint systems, and so is included to inform comparisons with.
Figure 4.2: MUDR successive interference cancellation scheme, or SIC. The communications user exploits the radar prediction to subtract the radar target return, with a residual left behind contributing an additional term to the communications noise floor. The power in this residual depends on how well modeled the target is. The communications signal is then reconstructed and subtracted from the original return, leaving only the radar signal.

4.2 Multi-User Detection Radar (MUDR)

This system employs successive interference cancellation (SIC), and is considered cooperative or co-designed. The full derivation and motivation are given in published works [2, 8, 13] for both pulsed and continuous radars, however we show the final result here for pulsed radar systems. The algorithm is shown in Figure 4.2. The concept is similar to code division multiple access (CDMA) for multiple communications users in the same band, and is called multiuser detection radar (MUDR) for our case which includes heterogeneous users (communications and radar). Since the radar and communications receiver are co-located, the communications receiver has access to the radar target tracking range prediction. As a result, the communications user can subtract the radar waveform, and operate in the same band with only thermal noise and the radar prediction residual remaining. This means the communications data rate is reduced from the full bandwidth potential. The radar receiver can then,
with knowledge of the communications system, reconstruct the communications signal and reapply forward error correction, and subtract this from the original return. The radar then operates unimpeded over the full bandwidth. It is therefore considered a radar-dominant technique. The radar estimation rate is the same as the full rate given by Equation (4.2) with $\alpha = 0$:

$$R_{\text{est,SIC}} \leq \frac{1}{2T} \log_2 \left[ 1 + \frac{2\sigma_{\text{proc}}^2 (2\pi B_{\text{rms}})^2 T_p P_{\text{rad,rx}}}{N_0} \right]. \quad (4.3)$$

The communications user, however, has an additional contributing noise term, and is given by [2]:

$$R_{\text{com,SIC}} \leq B \log_2 \left[ 1 + \frac{P_{\text{com,rx}}}{N_0 B + \sigma_{\text{proc}}^2 (2\pi B_{\text{rms}})^2 P_{\text{rad,rx}}} \right]. \quad (4.4)$$

It should be noted that these two rates have a reciprocal relationship with respect to the radar waveform RMS bandwidth. Ignoring global error, the radar wants to make the RMS bandwidth as large as possible to maximize estimation rate, while the communications user wants to minimize this term to minimize the residual (thus maximizing the data rate). This is discussed further in Chapter 7, where waveform design is explored.

### 4.3 Water-filling (WF) SIC-ISB System

We can combine the notion of MUDR employing SIC, which is a single operating point on the joint information plane, with the ISB system discussed previously. The water-filling (WF) system uses SIC in a co-designed split-band system where the communications user selects the band power allocation optimally by applying water-filling considering radar operation as depicted in Figure 4.3. For some sub-band allocation $\alpha$, we allocate one band to communications only, and the other band to mixed use employing SIC. In the mixed band, after the subtraction of the predicted
Figure 4.3: Water-filling (WF) SIC-ISB system. For a given sub-band allocation, one band is designated for communications only operation, while the other is allocated for joint use using the SIC algorithm. The communications user employs water-filling after the SIC subtraction to determine the optimal power allocation of the communications system to each band.

radar return, the communications user effectively has a two-channel system with different noise floors, and traditional water-filling principles can be applied [172]. The derivation is given in Reference [2], but the resulting communications data rate is given by

\[ R_{\text{com,WF}}(\alpha) \leq \alpha B \log_2 \left[ 1 + \frac{\beta P_{\text{com,rx}}}{N_0 \alpha B} \right] + (1 - \alpha) B \log_2 \left[ 1 + \frac{(1 - \beta) P_{\text{com,rx}}}{N_0 (1 - \alpha) B + \sigma^2_{\text{proc}} (2\pi (1 - \alpha) B_{\text{rms}})^2 P_{\text{rad,rx}}} \right], \]  

for power split factor \( \beta \) given by

\[ \beta = \alpha + \frac{(\alpha - 1)N_0 \alpha B + \alpha \left( N_0 (1 - \alpha) B + \sigma^2_{\text{proc}} (2\pi (1 - \alpha) B_{\text{rms}})^2 P_{\text{rad,rx}} \right)}{P_{\text{com,rx}}} \]  

when

\[ P_{\text{com,rx}} \geq \frac{\alpha \sigma^2_{\text{proc}} (2\pi (1 - \alpha) B_{\text{rms}})^2 P_{\text{rad,rx}}}{1 - \alpha} - \alpha N_0 B. \]  

Since the SIC channel is always degraded more than the communications-only band, when Equation (4.7) is not satisfied, \( \beta = 1 \) and no power is allocated to the mixed
band. The radar estimation rate is equivalent to the split-band case since the in-band communications user ultimately does not affect the radar user:

\[ R_{\text{est,WF}}(\alpha) \leq \frac{1}{2T} \log_2 \left[ 1 + \frac{2\sigma^2_{\text{proc}} ((1 - \alpha) 2\pi B_{\text{rms}})^2 T_p P_{\text{rad,rx}}}{N_0} \right]. \quad (4.8) \]

By varying the parameter \( \alpha \), which controls the sub-band allocation between the communications user and the mixed band as in the ISB system, we can construct an inner bound that trades between the two information metrics.

### 4.4 Constant Information Radar (CIR)

Another way to parameterize the SIC system is to time-share between using SIC in the full bandwidth, and allowing the communications user to operate at full rate without the radar. However, in a tracking scenario, the more time away from a target, the more the uncertainty grows about the target, meaning the estimation information at the next track point will be larger in general [9]. If we assume instead that it does not change, we can think instead about modulating the target revisit time to force the mutual information to be equal. For example, well modeled targets would be revisited less often than highly dynamic targets to achieve the same mutual information per radar measure. This describes the constant information radar (CIR), as shown in Figure 4.4. This is another example of a cooperative, and potentially co-designed system. At the very least, the radar must inform the communications user of the next transmit time. If the communications node is co-located with the radar, this can be done with minimal overhead or data transfer. The result of fixing the information for each radar spectral access means the radar waits until a threshold spectral efficiency for the radar user is achievable before illuminating a target. Therefore, the communications data rate is simply modulated by this time share constant \( \alpha \) that fluctuates with the predicted target dynamics, and weights the overall rate between
Figure 4.4: Constant information radar functional diagram. At the time step $t = k - 1$, the duty cycle between the radar use and free use is indicated by the horizontal bar graph. After this track point is completed, the predicted mutual information $I_{k|k-1}$ is smaller than our constant set value $I_{\text{const}}$ for this illustrative example. The tracking loop period (time between time step $k - 1$ and $k$) is increased to allow for more free spectrum access, as a lower radar emission rate is required to track a lower information target.

the full communications rate and the SIC rate:

$$R_{\text{com,CIR}}(\alpha) \leq \alpha R_{\text{com}} + (1 - \alpha) R_{\text{com,SIC}}$$

$$= \alpha B \log_2 \left[ 1 + \frac{P_{\text{com,rx}}}{N_0 B} \right] + (1 - \alpha) B \log_2 \left[ 1 + \frac{P_{\text{com,rx}}}{N_0 B + \frac{\sigma^2_{\text{proc}} (2\pi B_{\text{rms}})^2}{P_{\text{rad,rx}}}} \right],$$

(4.9)

and similarly for the estimation rate for $1 - \alpha$:

$$R_{\text{est,CIR}}(\alpha) \leq \frac{1 - \alpha}{2T} \log_2 \left[ 1 + \frac{2\sigma^2_{\text{proc}} (2\pi B_{\text{rms}})^2 T_{p} P_{\text{rad,rx}}}{N_0} \right].$$

(4.10)

This scheme is covered in more detail in Chapter 8 and References [7, 9].

4.5 Joint Multiple Access Channel Example

Plotting the communications data rate on one axis and the radar estimation rate on the other axis, a joint multiple access channel bound plot can be shown similar to diagrams employed in multiple communications channels such as CDMA. Plotting the isolated bounds, and all of the constructive inner bound systems in this chapter,
we get the result in Figure 4.5 (same as the plot introduced in Chapter 1 and reprinted here for convenience). The various parameters used to generate this plot are given in Table 4.1. Note the terms here contribute to the standard radar range equation for power and the communications link equation to facilitate the discussion in subsequent chapters. In Figures 4.6 and 4.7, we see the additive and geometric spectral efficiency respectively. In both plots, equal weighting is given to both the communications user and the radar user. The dashed lines in the same color as the inner bounds show the equivalent spectral efficiency if complete isolation for each band was used instead of spectrum sharing.

In the rest of this body of work, we present contributions made by the author in this thread of research. The author has also contributed as a co-author including a more in depth journal expanding on Reference [2] in Reference [13], extensions
### Table 4.1: Parameters for joint multiple access channel bounds example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth</td>
<td>5 MHz</td>
<td>Center Frequency</td>
<td>3 GHz</td>
</tr>
<tr>
<td>Absolute Temperature</td>
<td>1000 K</td>
<td>Communications Range</td>
<td>10 km</td>
</tr>
<tr>
<td>Communications Power</td>
<td>300 mW</td>
<td>Communications Antenna Gain</td>
<td>0 dBi</td>
</tr>
<tr>
<td>Target Range</td>
<td>200 km</td>
<td>Radar Antenna Gain</td>
<td>30 dBi</td>
</tr>
<tr>
<td>Radar Power</td>
<td>100 kW</td>
<td>Target Cross Section</td>
<td>10 m²</td>
</tr>
<tr>
<td>EM Propagation</td>
<td>$3 \times 10^8$ m/s</td>
<td>Pulse Width</td>
<td>25.6 µs</td>
</tr>
<tr>
<td>Radar Duty Factor</td>
<td>1%</td>
<td>Target Process Std. Dev.</td>
<td>100 m</td>
</tr>
</tbody>
</table>

**Figure 4.6:** Additive spectral efficiency example with inner bounds. The solid lines show the joint additive spectral efficiency of the respective inner bounds. The corresponding dashed line of the same color shows the additive spectral efficiency if the same system is replicated using isolated bands for all users instead of sharing bandwidth. The purple dot indicates that there is no radar operation to emphasize the importance of weighting the two users.
Figure 4.7: Geometric spectral efficiency example with inner bounds. The solid lines show the joint geometric spectral efficiency of the respective inner bounds. The corresponding dashed line of the same color shows the geometric spectral efficiency if the same system is replicated using isolated bands for all users instead of sharing bandwidth. Without weighting, we see more balance in this metric over the additive case. The no radar point now results in a null spectral efficiency.

to phase noise complicated scenarios [14], and a survey on joint methods with an emphasis on spectral efficiency and resource prioritization [177]. This work references several other extensions where the author did not contribute directly, such as cluttered scenarios with phase noise [12], and joint bounds using Fisher information [11].
In previous chapters, radar estimation rate is defined mathematically and utilized to derive joint radar estimation and communications Shannon bounds. However, without a more motivated, qualitative derivation, it may appear counter-intuitive to practitioners of information theory and radar engineers alike.

In this chapter we expand on the notion of radar estimation rate introduced in Chapter 1 and attempt to clarify the notion for fields of information theory and radar. A series of simplified scenarios build up the understanding of the communications channel analogy. Intuitive definitions are discussed to supplement the mathematical definition. In addition, we provide a more rigorous statistical motion model and add Doppler processing to the formulation. The resulting metric provides a measure of information the radar can gain by illuminating the target and performing state estimation, giving a notion of spectral efficiency in bits/seconds/Hz for radar systems. The information is gained through uncertainty in the target residual after the Kalman prediction step.

5.1 Radar Estimation Rate Formulation

To start, we look at a series of simplified scenarios to motivate a radar information metric. Since information is defined by the degree to which knowledge of a random variable reduces prior uncertainty [172], we adopt a simple model to convey this transfer of information (or reduction in uncertainty). In each case, the ‘teacher’ is ultimately communicating information to the ‘student.’ We see the information-theoretic construct evolve from a form of channel capacity, to random process source
coding, and finally ending with a form similar to rate-distortion.

If we formulate the radar receiver by using a finite number of range and range-rate bins as is often done in radar phenomenology [48], then the target is communicating from a finite alphabet, and an analogy to standard information-theoretic channel capacity results can be applied [135, 172]. Figure 5.1 shows a typical view of range/range-rate binning, with the target probability distribution shown as Gaussian contours. Note, this is not necessary, and is strictly illustrative to build up the analogy and gain intuition in a format familiar to radar engineers. Note also the use of the term “binning” in the context of this work refers to the common radar receiver signal processing practice, not the Slepian-Wolf binning argument from information theory. While the conditional distribution is typically chosen to maximize the mutual information to determine the channel capacity [172], we show the conditional distribution is ultimately beyond the designer’s control, and is a function of the Markov model, process noise, and estimation noise. The information rate as a channel capacity is therefore constrained. Allowing the range and range-rate bin size to vanish, we can re-derive the estimation rate previously introduced in Reference [2] as a mutual information sourced from differential entropy. Note the focus of this tutorial is on information obtained through radar tracking estimation. There is information to be gained through exploring detection, such as a Bernoulli distributed random variable characterized by probability of detection [172]. More complicated distributions such as track-before-detect could also be used to generalize the detection hypothesis. Detection information has interesting implications from an information-theoretic perspective [179], and discrete information theory results could be applied (such as Fano’s inequality [172]). Radar detection information is explored in Chapter 8, and References [7, 9, 18].

By measuring the target estimation information, we ultimately score how much
Figure 5.1: Radar receiver range and range-rate bins with target prediction estimate (mean) shown as the darkened square with Gaussian contours representing uncertainty of the target estimate. The bins represent the finite alphabet the target is unintentionally communicating from, while the distribution represents the probability spread of the target.

information we stand to gain through measurement. Redundancies in the tracking process, such as target prediction, reduce uncertainty. The extreme example is that a well modeled target is not worth illuminating. Highly dynamic targets that deviate from our predictions drastically, however, contain a large amount of information, and the use of time-bandwidth is critical to update the target state estimate. The signal-to-noise ratio (SNR) plays an equally important role, as noise degrades the estimation rate, even for highly dynamic targets. Simply put, illumination may not yield any “good” information when all desired state information is dominated by thermal noise, clutter, or some other distortion source.

5.1.1 Simplified Scenario 1

The first simplified scenario is shown in Figure 5.2. We can imagine a scenario where the student, indicated by the solid dot, is listening for an emitted pulse from the
Figure 5.2: Situational diagram of Simplified Scenario 1. Student is the solid dot on the left, while the teacher is shown as the circled ‘X’ on the right, with EM waves shown as circular contours. The possible range values are shown below the teacher, with the darkened bin indicating the current range.

Teacher, shown by the circled ‘X’. The teacher emits the pulse at a constant interval, and the student is synchronized to this interval. To communicate with the student, the teacher physically moves in the one-dimensional (1D) space closer to or further from the student, to one of the 8 spaces indicated on the diagram. The student, with knowledge of when the pulse is actually sent by the teacher, measures the time of flight from the unknown teacher location to the student to estimate the range, and makes a hard decision as to which bin the teacher was physically occupying. The range in meters is calculated from the delay as

\[ r = c \tau, \]  

where \( c \) is the speed of the electromagnetic waves in meters/second, and \( \tau \) is the time delay measured in seconds. It should be obvious that, in the noiseless case, the teacher is transmitting 3 bits per interval if the teacher information sequence to be transferred was uniformly distributed at the input. This is because we can assign a \( \log_2[8] = 3 \) bit pattern to each range (for example, a binary count where 000 is closest to the student and 111 is furthest from the student). The teacher can then send an arbitrary message encoded in binary by emitting at the range with the next three
bits of the overall message. Then the capacity of this simple channel is [172]

\[ C_1 = I(s; s) = H(s) - H(s|s) = H(s), \]  

(5.2)

where \( I(x; y) \) is the mutual information between random variables \( x \) and \( y \), \( H(x) \) is the discrete entropy of the random variable \( x \), and \( s \) is the random variable representing teacher’s position. For a uniformly distributed \( s \), the capacity is given by

\[ C_1 = -\sum_{k=1}^{8} \frac{1}{8} \log_2 \left( \frac{1}{8} \right) = 3 \text{ bits,} \]

(5.3)
as stated previously. Since we must wait for the worst case delay, that is when the teacher is in the 8th position furthest from the student, this ultimately bounds the pulse repetition interval (PRI). Consequently, the data rate is given by

\[ R_1 \leq \frac{C_1}{T} = \frac{3c}{r_8} \text{ bits/second,} \]

(5.4)

where \( T \) is the PRI in seconds and \( r_8 \) is the range to the 8th bin in meters. The time to travel from the 8th position to the student is calculated using Equation (5.1) and is easily seen to be \( r_8/c \). If we assume, for example, that \( r_1 = 0 \), then if the teacher is at position 8, and then moves to position 1, to unambiguously convey position, the teacher must wait for \( r_8/c \) seconds for the last pulse to arrive at the student. Note that this is an upper bound, as the rate could be reduced if the PRI is increased, and also due to information-theoretic constraints (see converse to the channel coding theorem [172]).

5.1.2 Simplified Scenario 2

As a next step, we can imagine instead of the teacher actively transmitting a pulse, we can rely on passive reflection off the teacher and have the student send the pulse as shown in Figure 5.3. Since the reflection from the teacher is identical in every way,
except for the time traveled, we simply have to adjust our detector to expect twice the range per each interval, or equivalently, twice the delay:

$$r' = \frac{c \tau}{2}.$$ 

(5.5)

Therefore, the capacity is unchanged:

$$C_2 = C_1 = 3 \text{ bits.}$$

(5.6)

However, it does affect the channel usage rate, since we must wait twice as long to re-transmit the pulse since it has twice the distance to travel. Consequently, the data rate is given by

$$R_2 \leq \frac{3}{T} = \frac{3c}{2r_s} \text{ bits/second.}$$

(5.7)

The achievable rate has been cut in half from Simplified Scenario 1. Note that despite the energy originating from the student, the teacher still controls the flow of information.

### 5.1.3 Simplified Scenario 3

Now we can think of more ways the teacher can communicate information to the student. For example, the teacher can induce a velocity relative to the student at the
time of reflection so that when the pulse reaches the teacher, the teacher is at the proper range and traveling one of 8 velocities (4 toward the student, 4 away from the student). This is illustrated in Figure 5.4. The student can extract this additional information by performing Doppler processing on the returned pulse, which exhibits an apparent shift in frequency relative to the transmitted frequency proportional to the velocity chosen uniformly by the teacher:

$$\dot{r} = \frac{\omega_D c}{2\omega_c},$$

(5.8)

where $\omega_D$ is the Doppler frequency shift and $\omega_c$ is the carrier frequency of the pulse (both in radians/second). The new distribution $s$ is now uniform over this two-dimensional (2D) discrete grid of 8 positions and 8 velocities (64 total states). This would allow an additional 3 bits of information, for a total of 6 bits per channel use in the noiseless case:

$$C_3 = H(s) = 6 \text{ bits.}$$

(5.9)

We therefore double the rate in the previous case, since our PRI limit is unchanged:

$$R_3 \leq \frac{6}{T} = \frac{3c}{r_8} \text{ bits/second.}$$

(5.10)
Once again, though the student sends the pulse physically, the information is controlled by the teacher. Note that since the velocity is along the range vector, it is sometimes referred to as the range-rate.

### 5.1.4 Simplified Scenario 4

In the final simplified scenario, we assume the student would like to know the position and velocity of the teacher (as before), but the teacher is no longer cooperative. Note, this varies from the previous scenarios where arbitrary information is voluntarily communicated by the teacher. The student knows, based on the teacher’s mass and environment, that the teacher cannot move instantaneously. The student can predict where the teacher will be based on the previous measurement. If the teacher was traveling a specific velocity at a specific position, the student can calculate the teacher’s next likely position and velocity. If the student assumes, for example, the teacher is not accelerating, then a simple velocity-position-time formula can provide the student’s best guess. The information the student now stands to gain through measurement is the deviation from this prediction. A realistic distribution could be a correlated Gaussian distribution centered over the predicted range-velocity pair. For example, if the teacher was at range $N$ and velocity $M$ at the previous illumination by the student, the teacher should be more likely to be near range $N + 1$ if $M$ is positive and near range $N - 1$ if $M$ is negative (assuming a positive velocity indicates movement away from the student). Further, since the teacher must accelerate or decelerate, the next velocity is more likely to be near the existing velocity. Since the feasible ranges depend on the previous velocity due to classical motion physics, the range and velocity are now dependent statistics, which is why a correlated bivariate Gaussian is a reasonable model. This is illustrated in Figure 5.5. We now must integrate the 2D Gaussian probability distribution function over the encompassing
Figure 5.5: Final Simplified Scenario 4 with Gaussian input distribution. Student is the solid dot on the left which transmits the pulse, while the teacher shown as the circled ‘X’ on the right acts as a perfect reflector and is in motion. The possible range values are shown below the teacher, with the darkened bin indicating the current range. The direction and speed of the teacher convey additional information. The probability distribution of the range and velocity is a correlated Gaussian centered on the predicted value.

range and velocity bins. Note that probability tails that extend beyond the range and range-rate limits are included in the edge bins constrained by our problem. For example, any time measured beyond the 8th range bin, the student concludes the 8th range bin is the correct one.

The astute information theorist may notice the subtle, but important shift from Simplified Scenario 3 to Simplified Scenario 4. The channel capacity problem is concerned with maximizing a mutual information relative to a conditional distribution [172]. Since this conditional distribution has been forced by our scenario, and is thus beyond the designer’s control, it is no longer a true channel capacity problem. The mutual information is still very much real, and information is being learned by the student (the position and relative velocity of the teacher). However, the distribution is no longer completely in the teacher’s control to convey the information. Interestingly, it depends on how well the teacher conforms to the implicit model the student uses to predict the teacher’s state. If the mutual information is small, little information stands to be gained, as the teacher is behaving according to our prediction. If large, the teacher is deviating drastically from our expectation, and so information is gained
We are now ready to look at a more realistic scenario typical in radar estimation. We assume we observe the delay and Doppler measurements for a pulsed monostatic radar, and we assume we have \(2N + 1\) delay bins and \(2M + 1\) Doppler bins (or cells). The Realistic Radar Scenario is essentially Simplified Scenario 4 with a change of terminology and one additional layer of complexity. The ‘teacher’ is now the ‘target’ and the ‘student’ is now the ‘radar’ transceiver that has noise added to the estimation process. It should now be obvious that even though the radar is emitting and illuminating the target, it is the target that is uncooperatively communicating information to the radar. The noise contributes unwanted entropy to the system, and degrades the estimation rate, or obtainable information at each measurement. The probabilities of each cell are computed by integrating the 2D bivariate Gaussian probability distribution function with covariance matrix

\[
\Sigma = \begin{bmatrix} \sigma^2_\tau & \rho \sigma_\tau \sigma_\omega \\ \rho \sigma_\tau \sigma_\omega & \sigma^2_\omega \end{bmatrix} \tag{5.11}
\]

over a square bounded by the bin coordinates:

\[
P\{i, j\} = \int_{\alpha}^{\beta} \int_{\gamma}^{\delta} \frac{\exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{\tau^2}{\sigma^2_\tau} + \frac{\omega^2}{\sigma^2_\omega} - \frac{2\rho \tau \omega}{\sigma_\tau \sigma_\omega}\right]\right)}{2\pi \sigma_\tau \sigma_\omega \sqrt{1-\rho^2}} d\tau d\omega \tag{5.12}
\]

where the limits are defined as follows

\[
\alpha = \begin{cases} 
-\frac{\Delta \tau}{2} + i \Delta \tau & i \neq -N \\
-\infty & i = -N 
\end{cases}, \quad \beta = \begin{cases} 
\frac{\Delta \tau}{2} + i \Delta \tau & i \neq N \\
\infty & i = N 
\end{cases}
\]

\[
\gamma = \begin{cases} 
-\frac{\Delta \omega}{2} + j \Delta \omega & j \neq -M \\
-\infty & j = -M 
\end{cases}, \quad \delta = \begin{cases} 
\frac{\Delta \omega}{2} + j \Delta \omega & j \neq M \\
\infty & j = M 
\end{cases}
\]
where $\Delta \tau$ is the width of the cell (delay resolution) and $\Delta \omega$ is the height of the cell (Doppler resolution). Note that the edge bins must encompass the remaining tail probabilities as they previously did in Simplified Scenario 4.

The channel model is shown in Figure 5.6. Here we can see the noiseless state $\mathbf{x}$, the true target range and range-rate, is subjected to the channel, where thermal noise and estimation noise introduced by the Doppler processor corrupt $\mathbf{x}$. Under our assumptions that the SNR is reasonable high, we assume this noise is additive white Gaussian noise (AWGN) at the Cramér-Rao lower bound (CRLB) which we define in Appendix A. The state is transformed to a delay-Doppler domain, but the information is unaffected. We then make a hard decision by binning our delay and Doppler measurements as discussed previously, and transform back to the range/range-rate domain. As a result, we can see that our ‘message’ $\mathbf{x}$ is effectively corrupted by discrete noise $\mathbf{n}$ to form our estimate of the range and range-rate:

$$\hat{\mathbf{x}} = \mathbf{x} + \mathbf{n}, \quad (5.13)$$

where $\mathbf{n}$ is the quantized AWGN. Note this effectively incorporates the binning as a part of the channel. We therefore can write the following expression for the desired information the radar can gain:

$$I(\mathbf{x}; \hat{\mathbf{x}}) = I(\mathbf{x}; \mathbf{x} + \mathbf{n}). \quad (5.14)$$
This can be re-written using the standard definitions of mutual information to be [172]

\[ I(\mathbf{x}; \hat{\mathbf{x}}) = H(\mathbf{x} + \mathbf{n}) - H(\mathbf{x} + \mathbf{n} | \mathbf{x}) \geq H(\mathbf{x} + \mathbf{n}) - H(\mathbf{n}) , \tag{5.15} \]

where we have exploited the independence of the source distribution from the additive noise distribution from the Doppler processor. Note the inequality arises due to our hard decision binning in the receiver. This is because the conditional entropy of the corrupted measurement given knowledge of the true measurement, \( H(\mathbf{x} + \mathbf{n} | \mathbf{x}) \), is smaller than the entropy of the noise only, \( H(\mathbf{n}) \), since a non-central distribution would push probability to the edges of our grid. We assume for simplicity equality holds in Equation (5.15).

For our discrete alphabet pulled from our range and range-rate bins, these quantities are easily calculated:

\[ H(\mathbf{x} + \mathbf{n}) = - \sum_{i=-N}^{N} \sum_{j=-M}^{M} P_{\mathbf{x}+\mathbf{n}}\{i,j\} \log_2[P_{\mathbf{x}+\mathbf{n}}\{i,j\}] . \tag{5.16} \]

Note that the cell probability here is for the noisy Doppler measurement, but the structure is identical for the noise only entropy. If we illuminate every \( T \) seconds, then we get an effective bound on our estimation rate

\[ R_{\text{est}} \leq \frac{1}{T} \sum_{i=-N}^{N} \sum_{j=-M}^{M} (P_{\mathbf{n}}\{i,j\} \log_2[P_{\mathbf{n}}\{i,j\}] - P_{\mathbf{x}+\mathbf{n}}\{i,j\} \log_2[P_{\mathbf{x}+\mathbf{n}}\{i,j\}]) . \tag{5.17} \]

Note the inequality in the estimation rate is due to the data processing inequality [172] and the CRLB. This is because realistic systems may have any number of sub-optimal processing elements, which either maintain or decrease the amount of information we stand to gain, as discussed in Chapter 3.

As stated previously, there is no reason to consider the information over a discrete grid. The target measurement can occupy a continuum of states, but the binning was illustrative in our initial example where channel capacity could be readily defined,
and is a familiar construct to radar practitioners. If we allow $\Delta \tau \rightarrow 0$ and $\Delta \omega \rightarrow 0$, we can observe how the quantity becomes continuous. While the following derivation is well known [172], we include it here as an exposition using our notation. We start by invoking the mean value theorem for integration by noting our cell probability can be rewritten as

$$P\{i, j\} = \frac{\exp \left( -\frac{1}{2(1-\rho^2)} \left[ \frac{\tau_i^2}{\sigma^2_x} + \frac{\omega_j^2}{\sigma^2_\omega} - \frac{2\rho \tau_i \omega_j}{\sigma_x \sigma_\omega} \right] \right)}{2\pi \sigma_x \sigma_\omega \sqrt{1-\rho^2}} \Delta \tau \Delta \omega = f(\tau_i, \omega_j) \Delta \tau \Delta \omega, \quad (5.18)$$

where $(\tau_i, \omega_j)$ is a point that lies within our cell that satisfies this equality and $f(\tau, \omega)$ is the bivariate Gaussian distribution with our prescribed covariance. Then for our entropy we have

$$H(x + n) = -\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f_{x+n}(\tau_i, \omega_j) \Delta \tau \Delta \omega \log_2[f_{x+n}(\tau_i, \omega_j) \Delta \tau \Delta \omega], \quad (5.19)$$

where we have made $M$ and $N$ infinite to avoid needing to manipulate them as our cell size vanishes. Applying the same logic to the noise only case, we obtain the following:

$$I(x; \hat{x}) = \Delta \tau \Delta \omega \left( \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f_n(\tau_i, \omega_j) \log_2[f_n(\tau_i, \omega_j)] - \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{x+n}(\tau_k, \omega_l) \log_2[f_{x+n}(\tau_k, \omega_l)] \right) \quad (5.20)$$

Note that the cell size dependent entropies cancel, and that the summations were separated since the equivalent points from the mean value theorem are in general different for each distribution. Since the summations in our equivalent mutual information formula are Riemann integrable [172], then

$$\lim_{\Delta \tau \rightarrow 0, \Delta \omega \rightarrow 0} I(x; \hat{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_n(\tau, \omega) \log_2[f_n(\tau, \omega)] \, d\tau \, d\omega - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x+n}(\tau, \omega) \log_2[f_{x+n}(\tau, \omega)] \, d\tau \, d\omega. \quad (5.21)$$
Both integrals are immediately recognized as differential entropies \( h(n) \) and \( h(x + n) \) of the now continuous distributions [172], and so the estimation rate bound may now be written as

\[
\lim_{\Delta \tau \to 0, \Delta \omega \to 0} \frac{I(x; \hat{x})}{T} = R_{\text{est}} \leq \frac{h(x + n) - h(n)}{T} = \frac{I(x; y)}{T},
\]

(5.22)

which is the formula originally proposed in Equation (2.2) and Reference [2]. This defines the rate of information the target is transmitting unwillingly to the radar via its own entropy or uncertainty relative to the radar, regardless of cell size, removing the dependence of the receiver limitations. Since we quantized the underlying Gaussians in our binning formulation, this expression has a well known closed-form [172]:

\[
R_{\text{est}} \leq \frac{1}{2T} \log_2 \left[ \frac{|\Sigma + J^{-1}|}{|J^{-1}|} \right],
\]

(5.23)

where \(|X|\) denotes the determinant of matrix \( X \), \( \Sigma \) is the source Gaussian covariance, and \( J^{-1} \) is the correlated bivariate Gaussian from the receiver noise (encompassing the scaled inverse Fisher information matrix (FIM) and phase noise terms). Note that we have exploited the well known formula for a correlated bivariate Gaussian entropy [172]. The result looks similar to rate-distortion evaluated at the CRLB, with the distortion function being the estimate covariance. However, since we are not interested in coding the position of the target, but rather learning the position of the target, a communications channel still provides a more intuitive analogy for this metric.

5.2 Intuition

The previous sections and chapters may have hinted or alluded to why this metric has value and some of its intuition. Here, we demonstrate the metric’s utility more concretely.
5.2.1 Signal-to-noise Ratio

Thermal noise and other error sources degrade estimation performance. As a result, target entropy may be completely dominated by poor SNR or high interference signals. Consider freezing our target state and taking repeated random measurements. The true target state may deviate slightly from the prediction. If we had high SNR, we could gain a lot of information by confidently measuring this residual. With poor SNR, small deviations from the prediction are swamped by measurement noise. There is a great deal of uncertainty (in which information is borne), but not the information we care about. As a result, we stand to gain little information (that we care about) through measurement. This reflects recent results investigating range-dependent resolution for compressive sensing applications [147]. In Chapter 8 and References [7, 9], this effect is seen clearly, and radar revisit times modulated by the estimation rate measure follow a similar pattern.

5.2.2 Revisit Time & PRF

One thing worth mentioning is that increasing pulse repetition frequency (PRF) does not, in general, increase estimation rate. The more rapidly the target is illuminated, the less chance it has (given a realistic, physical scenario) to deviate from the motion model. As a result, less information is gained over a shorter period. In fact, estimation rate can be decreased if increasing the PRF results in the system violating the unambiguous range of the target. This is because range ambiguity would force probability that would otherwise be separated to collapse to multiple range bins, decreasing the entropy of the target. Similarly, if the PRF decreased too much, ambiguous Doppler would have a similar information degrading effect. In Chapter 8, we investigate modulating revisit time (or PRI) in order to gain a fixed amount of
information for each target illumination, enforcing a per-visit spectral efficiency.

5.2.3 Model Mismatch

This is the most significant contribution of the estimation rate metric over the legacy figures of merit. It is also the component not present in any of them. In the standard Kalman formulation, a prediction is made based on the underlying dynamical motion model. The amount, on average, the target measurement is expected to deviate from this prediction provides a measure of the information we stand to gain. The better the motion model, the less information the system gains through measurement. As an example, consider a utopia where all aircraft are complicit and file flight plans that they adhere to perfectly. Then no information is gained through measurement. This would be reflected in the Kalman residual, which would be gained to null due to infinite confidence in the motion model. The other extreme is illicit, non-cooperative targets that are attempting to fly covertly and erratically to avoid detection. As a result, the Kalman residual would be large and heavily weighted. Significant information is learned through measurement, since the deviation from our prediction is large. In References [7, 9], targets breaking the non-turning model are visited more frequently than when they are not accelerating.

5.3 Radar Estimation Rate Example

We can look at radar estimation rate over a few parameters such as bandwidth and power, similar to standard information theory literature [172]. To simplify the discussion, the dynamical motion model and measurement model are derived in Appendix A. These detail the underlying mathematics necessary to recreate the results here, but are fairly standard in radar phenomenology and thus do not contribute to the point at hand. We used parameters similar to those used in Reference [2] (where
estimation rate was introduced) for comparison and assumed our waveform is a linear frequency-modulated (FM) chirp with a Gaussian window for simplicity.

To solve for Equation (5.23), we need to define the PRI $T = 2 \text{ ms}$, the predicted source covariance $\Sigma$, and the scaled FIM $J^{-1}$. We assume the predicted source covariance before noise is given by:

$$\Sigma = \frac{4}{c^2} \begin{bmatrix} 100^2 & 5000\omega_c \\ 5000\omega_c & (1000\omega_c)^2 \end{bmatrix}, \quad (5.24)$$

which has been scaled from the range/range-rate domain to the delay-Doppler domain. The measurement covariance is given by

$$J^{-1} = \frac{1}{\text{SNR}} \begin{bmatrix} T_p^2 & -4\pi T_p B \\ -4\pi T_p B & T_p^{-2} + (4\pi B)^2 \end{bmatrix}, \quad (5.25)$$

for $T_p$ the pulse duration which we choose to be 0.02 ms, and $B$ the bandwidth in Hz which we vary. For details on how this was obtained, in addition to detail on other degrading factors such as phase noise, see Appendix A.

In Figure 5.7, we see the radar estimation rate as a function of bandwidth at a few SNRs. We see two regimes in the plot. For low bandwidths, the delay measurement is poor, and the Doppler information dominates. Note the Doppler measurement accuracy is not a function of bandwidth [173], and so this region is flat. As bandwidth increases, the effective thermal noise in our system increases. However, the range resolution is improved, and more accurate measurements may be made. As we saw in Section 5.2, this allows more “good” information to be obtained through measurement.

5.4 Summary

We derived the previously defined radar estimation rate quantity using a more rigorous approach to justify the original formulation. We modeled radar estimation as an
uncooperative communications channel between the target and the radar transceiver. To provide some intuition behind this model, we built up a series of simplified scenarios that grew increasingly complex, until this uncooperative channel was ultimately revealed. We then started with a fixed alphabet using a finite discrete range and range-rate grid typical in radar phenomenology. From the fixed Markov process distribution and estimation noise derived from the CRLB, the mutual information over the PRI defined the rate of information transferred in this radar channel. Allowing the bin size to vanish, the continuous entropy based quantity used in previous works was revealed: the radar estimation rate. The resulting metric provides a useful quantification of information about the target relative to the radar as a function of time. Much of the difficulty in this derivation is the conceptual notion of radar information theory. Where there is a probability distribution, there is entropy, and thus information. Even in radar scenarios where the target is well-detected, or even cooperative
(such as IFF [46]), there is still an implicit probability distribution derived from the uncertainty that the target may not be well modeled the next time it is illuminated.
In this chapter, we solve for the specific case of frequency-modulated continuous-wave (FMCW) radar, and include Doppler estimation. In previous chapters, joint radar estimation and communications Shannon bounds were found in an attempt to define the attainable region for cooperative pulsed radar and communications operation.

Here we extend this work by allowing for continuous radar signaling. To facilitate this extension, a Markov motion model and extended measurement model with Doppler estimation are included in the formulation. The advantages of FMCW radar are numerous, but the range resolution afforded is typically one of the strongest motivators behind its use [180]. While many works in FMCW radar extract the target delay and Doppler via a beat frequency, we assume the more general matched filter. We also introduce another parameter, the coherent processing interval (CPI), which should be tuned to the maximum value to maximize estimation performance, ensuring the period encompasses a stationary target. Due to the complex statistical interdependencies, it is necessary to also include a dynamical Markov model and Doppler estimation into our extended formulation. A detailed derivation of the radar mutual information model and challenges therein are given in Appendix A.

6.1 Joint Radar-Communications Model

Much like the previous cases, here we assume a radar system functions both as a radar transmitter and receiver to perform target detection and tracking in a monostatic configuration, as well as a communications receiver. The joint system scenario is illustrated in Figure 6.1. This scenario is covered under the joint multiple access
Figure 6.1: Main FMCW system scenario; target shown in red, communications transmitter node in blue, and radar transceiver pattern in green. The radar acts as the radar transceiver and communications receiver. The uncertainty in the target position is indicated by Gaussian contours. This model falls under the joint multiple access channel topology depicted in Figure 1.3.

channel topology shown in Figure 1.3. The goal is to derive information-theoretic bounds akin to the multiple access bound in communications phenomenology, as discussed in Reference [2] (see Figure 1 within). The information-theoretic rates for this model are well known for two communications users [172, 181]. However, it serves only as an introduction and analogy, since one of our “users” is not a communications system attempting to communicate from a countable dictionary, but rather a radar system attempting to illuminate and track a target. For the communications user, this is still a valid model, and the resulting rate bound is given by [172]

\[ R_{\text{com}} \leq B \log_2 \left[ 1 + \frac{b^2 P_{\text{com}}}{k_B T_{\text{temp}} B} \right], \] (6.1)

where \( B \) is our available bandwidth in Hertz, \( b \) is the combined communications antenna gain (antenna sidelobe gain and propagation loss), \( P_{\text{com}} \) is the total communications transmit power in Watts, \( k_B \) is the Boltzmann constant in Joules/Kelvin and \( T_{\text{temp}} \) is the absolute temperature in Kelvin. The combined communications gain is given by \( b^2 = G_{\text{com}} c^2/(2 R_{\text{TX}} \omega_c)^2 \), for \( G_{\text{com}} \) the antenna sidelobe gain, \( c \) the speed
of electromagnetic waves in meters/second, $R_{TX}$ the range to the communications transmitter in meters, and $\omega_c$ the center frequency in radians/second. This is equivalent to Equation (1.4), with more system level detail to facilitate development of this special case.

Once again, for the radar, we employ the radar estimation rate to splice the radar system into what has traditionally been thought of as a multi-communications user model. However, for this scenario, we need to derive the mutual information for an FMCW scenario which can result in a complicated distribution. To simplify the analysis, we linearize about a nonlinear tracking model using the extended Kalman filter (EKF).

### 6.1.1 Dynamical Model & Source Entropy

While previous work [2] focused only on range measurements for pulsed radar, we extend these results for FMCW radar. However, several underlying simplifications in the previous work no longer apply. For example, with large CPIs typical of FMCW radar [180], range-Doppler coupling and Doppler mismatch become significant factors [48]. As a result, Doppler estimation must be included in the formulation. In addition, previous works started with the delay process modeled as a Gaussian. However, we now have to consider the joint delay-Doppler process, which are in general correlated as discussed in Chapter 5. This is easy to see, since range as a function of time can be used to estimate the range-rate. Therefore, we must start our formulation at the source, and work through the measurement model to capture the interdependencies.

The most natural formulation to that end is to use a dynamical Markov model as a part of a tracking process. We therefore assume a constant velocity, two-dimensional (2D) linear motion model with a Gaussian perturbation acceleration model for our
target [182]

\[
\mathbf{s}_{k+1} = \begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \\ y_{k+1} \\ \hat{y}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \\ y_k \\ \hat{y}_k \end{bmatrix} + \mathbf{w}_k, \tag{6.2}
\]

where the \( \mathbf{w}_k \) is the process noise with covariance defined as

\[
\mathbf{Q} = \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix}, \tag{6.3}
\]

\( x_{k+1} \) is the target position along the x-axis at discrete time step \( k + 1 \), \( \hat{x}_{k+1} \) is the target velocity projected on the x-axis, \( y_{k+1} \) is the target position along the y-axis, \( \hat{y}_{k+1} \) is the target velocity projected on the y-axis, \( T \) is the discrete time step duration of the system (duration between time steps \( k \) and \( k + 1 \)) as well as the CPI, \( \mathbf{s}_k = [x_k \ \hat{x}_k \ y_k \ \hat{y}_k]^T \) is the state vector, and \( q_k \) is the process model error intensity. While the measurement space reduces our state dimensionality, the Cartesian model allows motion dynamics to advance uncoupled from the coordinate system [182].

Note that although we assume the velocity is constant (and therefore acceleration is zero), in reality the small perturbations of the velocity modeled by the dynamical error covariance matrix allow for small variations in acceleration. This is the stochastic state estimation approach that assumes, by construction, that the motion model contains some error [46]. In this sense, changing velocity is tracked assuming there is uncertainty in the motion model. Other models can be used if the target exhibits higher order dynamics [182, 183].
6.1.2 Measurement Model & Measurement Entropy

The observed parameters, after Doppler processing [46], are the range $r_k$ and range-rate $\dot{r}_k$. We assume a narrowband environment such that only a frequency shift (Doppler shift) is induced in the returned waveform, not the more general Doppler time scaling. This is possible under the assumption [173]:

$$\frac{T B}{c} \ll \frac{1}{2\dot{r}}. \quad \text{(6.4)}$$

This means that the range and range-rate which offset the peak of the narrowband cross-ambiguity function at the measurement processor relate to the position and velocity state as follows [173, 184]:

$$r = \sqrt{x^2 + y^2}, \quad \dot{r} = \frac{x \dot{x} + y \dot{y}}{\sqrt{x^2 + y^2}}. \quad \text{(6.5)}$$

This model may be extended to include bearing measurements, as it is in Chapter 8.

For the measurements, we must solve for the transformed entropy of our source Gaussian state. The range, in general, is Rice distributed [47]. Unfortunately, there is no closed-form solution for the differential entropy of a Rician [185]. However, we can compute this entropy using numerical integration or Monte Carlo methods [183]. The range-rate is seen to be the sum of products of two correlated Gaussians, normalized by a correlated Rician. It is not obvious what the marginal distribution is. To find the joint entropy, we would need to solve for range-rate entropy conditioned upon the range [172]. We then have to consider measurement noise. If we assume additive white Gaussian noise (AWGN), the already complicated distributions are now convoluted with the additive Gaussian distribution. The delay and Doppler measurements originate from the same signal source, the matched filter receiver, and therefore their noise is, in general, correlated [173]. So we must consider the delay and Doppler entropy jointly. An analysis of the source distributions and entropy that exists in
closed-form is presented in Appendix A.

Here, the difficulties that are mounting warrant an alternate formulation. We can simplify the analysis by linearizing the measurement model about the predicted state, much like the EKF [183]. This results in a time-varying, but linear transform on our Gaussian state vector. An affine transform on a multi-variate Gaussian like our source vector results in another multivariate Gaussian [186]:

\[
s_{k+1} \sim \mathcal{N}(\mu, \Sigma) \implies C s_{k+1} \sim \mathcal{N}(C \mu, C \Sigma C^T),
\]

(6.6)

for \( C \) the linearized transform. Our predictive state \( s_{k+1} \) is defined in Equation (6.2) with covariance matrix \( \Sigma \). Here, for simplicity, we assume \( \Sigma = Q \) defined in Equation (6.3). Note this implicitly assumes we know the prior state of the target. However, we have not lost generality, as one can easily replace \( \Sigma \) with the predicted covariance advanced by the Kalman equations [183]. To construct our linear measurement transform, we consider \( R \) and \( \dot{R} \) the measurement functions for the range and range-rate respectively, and the partial derivatives are all to be evaluated at the predicted state. These are evaluated in Appendix A, wherein we have made use of the fact that the time delay is given by \( \tau = 2r/c \), and the Doppler shift is given by \( \omega_D = 2\omega_c \dot{r}/c \). In addition, our measurement is corrupted by AWGN, \( n_{k+1} \), from the estimation process (encompasses thermal noise and waveform-dependent ambiguity [173]):

\[
C s_{k+1} + n_{k+1} \sim \mathcal{N}(C \mu, C \Sigma C^T + J^{-1}).
\]

(6.7)

\( J^{-1} \) is the scaled inverse Fisher information matrix (FIM), which is given for a single sample, and so we scale this by the integrated signal-to-noise ratio (SNR) to get the post-processing bound [48]. Since the mean of a multivariate Gaussian does not affect the differential entropy [172], we need only to solve for \( |C \Sigma C^T + J^{-1}| \), or the determinant of our noisy transformed covariance matrix. Then the differential
entropy for the complete linearized measurement model is found to be [172]:

\[ h(\tau, \omega_D) = \frac{1}{2} \log_2 \left( (2\pi e)^2 |C \Sigma C^T + J^{-1}| \right). \] (6.8)

We now must find the estimation noise of our measurements. We only consider a single target scenario to compare with other special cases in this work and because computing the performance with multiple targets in FMCW radars can be complicated [187]. While traditionally in FMCW radar, the beat frequency is extracted to obtain the range estimate by mixing the radar returns with the continuous transmitter reference [188], we assume correlation with a general waveform, in which case we only are affected by the root mean square (RMS) bandwidth \( B_{\text{rms}} \) and RMS envelope \( T_{\text{rms}} \) of the signal [173]. We are assuming the Cramér-Rao lower bound (CRLB) is achievable, and sufficiently high SNR such that we can define the FIM to be [173]

\[
J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \tag{6.9}
\]

where

\[
J_{11} = \frac{\text{ISNR}}{\pi} \int_{-\infty}^{\infty} \omega^2 |S_{\text{rad}}(j\omega)|^2 d\omega = 2 T s_{\text{rms}}^2 \text{ISNR} (2\pi B_{\text{rms}})^2, \tag{6.10}
\]

\[
J_{22} = 2 \text{ISNR} \int_{-\infty}^{\infty} t^2 |s_{\text{rad}}(t)|^2 dt = 2 T s_{\text{rms}}^2 \text{ISNR} (T_{\text{rms}})^2, \tag{6.11}
\]

\[
J_{12} = J_{21} = 2 \Im \left\{ \text{ISNR} \int_{-\infty}^{\infty} t s_{\text{rad}}(t) \frac{\partial s_{\text{rad}}^*(t)}{\partial t} dt \right\}, \tag{6.12}
\]

for \( s_{\text{rad}}(t) \) our radar waveform, \( S_{\text{rad}}(j\omega) \) the Fourier transform of our radar waveform over the CPI, ISNR the integrated SNR and the waveform finite RMS defined as

\[
\text{ISNR} = \frac{T B a^2 P_{\text{rad}}}{\sigma_{\text{noise}}^2}, \quad s_{\text{rms}} = \sqrt{\frac{1}{T} \int_{-T/2}^{T/2} |s_{\text{rad}}(t)|^2 dt}, \tag{6.13}
\]

respectively. For the integrated SNR, \( a \) is the combined radar antenna gain (antenna main lobe gain, radar cross section, and two-way propagation loss), \( P_{\text{rad}} \) is the total...
radar transmit power in Watts, and the receiver thermal noise power in Watts is defined as $\sigma_{\text{noise}}^2 = k_B T_{\text{temp}} B$. The combined radar gain is given by

$$a^2 = \frac{G_{\text{rad}}^2 \text{RCS} c^2}{16\pi R_{\text{targ}}^4 \omega_c^2}, \quad (6.14)$$

for $G_{\text{rad}}$ the main lobe antenna gain relative to an isotropic radiator, RCS the radar cross section, and $R_{\text{targ}}$ the range to target. Note we have assumed the baseband waveform is constructed on an axis such that the mean frequency is 0 [173]. In addition, we are assuming a continuously transmitted FMCW waveform that is non-repeating over our CPI, well modeled by a random process. This allows arbitrary selection of the CPI, while still attaining a matched filter. Overlapping CPIs are possible, but this would not affect the estimation rate, since it only amounts to staggered processing at the radar receiver. Note we are also assuming sufficient isolation of the transmit and receive waveforms, and that successive CPI waveforms do not impact the matched filter process significantly. By representing our radar signal in polar form, we can only rewrite the third term as:

$$J_{12,21} = -4\pi \text{ISNR} \int_{-\infty}^{\infty} t \mathbb{E}[f(t)] |s_{\text{rad}}(t)|^2 dt,$$

(6.15)

where $f(t)$ is the instantaneous frequency function of our FMCW waveform assumed to be a random process. Since the average of the instantaneous frequency should be 0 (unmodulated signal) for our previous simplification (noting we can always choose the center frequency to satisfy this), this term subsequently is on average 0. Then the scaled inverse FIM, or CRLB becomes:

$$J^{-1} = \begin{bmatrix} \sigma_r^2 & \rho \sigma_r \sigma_\omega \\ \rho \sigma_r \sigma_\omega & \sigma_\omega^2 \end{bmatrix} = \frac{1}{2T s_{\text{rms}}^2 \text{ISNR}} \begin{bmatrix} (2\pi)^{-2} B_{\text{rms}}^{-2} & 0 \\ 0 & T_{\text{rms}}^{-2} \end{bmatrix}. \quad (6.16)$$

Finally, we can complete our radar estimation rate

$$R_{\text{est}} \leq \frac{1}{2T} \log_2 \left[ \frac{|C \Sigma C^T + J^{-1}(B)|}{|J^{-1}(B)|} \right], \quad (6.17)$$
where we have parameterized the scaled inverse FIM $J^{-1}$ by the bandwidth $B$ since we vary this parameter later.

6.2 Inner Bounds

As in Chapter 4, we solve for a series of inner bounds on joint radar and communications operations to test our new outer bounds for this special case of an FMCW radar.

6.2.1 Isolated Sub-band

As before, the simplest inner bound we can derive is the isolated sub-band (ISB) inner bound. Recall, this is where we allocate sub-bands from our overall bandwidth $B$ to the communications user and radar user separately. This is traditionally how these systems operate, but it serves to build a basis for comparison with other inner bounds. The bounds are easily seen to be

$$R_{\text{com,ISB}} \leq \alpha B \log_2 \left[ 1 + \frac{b^2 P_{\text{com}}}{k_B T_{\text{temp}} \alpha B} \right],$$

$$R_{\text{est,ISB}} \leq \frac{1}{2T} \log_2 \left[ \frac{|C \Sigma C^T + J^{-1}(\overline{\alpha} B)|}{|J^{-1}(\overline{\alpha} B)|} \right],$$

where $\alpha$ is our mixing parameter which may be varied between 0 and 1 to allocate the complementary sub-bands of $B$, and $\overline{\alpha} = 1 - \alpha$.

6.2.2 Successive Interference Cancellation (SIC)

As in Chapter 4, we now look at the possibility of predicting and subtracting the radar return, decoding the communications signal to remove from the original return, and performing radar processing. However, the successive interference cancellation (SIC) bound in this case is much more complicated than given in the previous chapters.
(and originally published in Reference [2]). For the FMCW radar case, the composite radar/communications received signal minus our predicted radar return is given by:

$$\tilde{z}(t) = \sqrt{P_{\text{com}}} b s_{\text{com}}(t) + n(t) + \sqrt{P_{\text{rad}}} a s_{\text{rad}}(t - \tau) e^{j\omega_D(t-\tau)}$$

$$- \sqrt{P_{\text{rad}}} a s_{\text{rad}}(t - \tau_{\text{pre}}) e^{j\omega_{D,\text{pre}}(t-\tau_{\text{pre}})}, \quad (6.20)$$

where $\tau_{\text{pre}}$ and $\omega_{D,\text{pre}}$ are the predicted time delay and Doppler shift of our radar return respectively, and $s_{\text{com}}(t)$ is the communications waveform we wish to decode.

Exploiting independence in the receiver noise $n(t)$, we can write the interference plus noise from the communications receiver’s perspective as:

$$\sigma_{\text{int+n}}^2 = \mathbb{E}[\|\tilde{z}(t) - \sqrt{P_{\text{com}}} b s_{\text{com}}(t)\|^2] = \sigma_{\text{noise}}^2 + P_{\text{rad}} a^2 \mathbb{E}[\|\Delta s\|^2], \quad (6.21)$$

where $\Delta s$ is the interference residual from the predicted subtraction. Note that while this residual is not Gaussian distributed, we treat it as such moving forward to make the final bounds more tenable.

We can assume for simplicity that our predicted Doppler shift is approximately equivalent to the actual Doppler shift. For slowly accelerating targets in the middle of a track, this is a reasonable assumption. Therefore

$$e^{j\omega_{D,\text{pre}}(t-\tau_{\text{pre}})} \approx e^{j\omega_D(t-\tau+n_{\tau,\text{pre}})} = e^{j\omega_D(t-\tau)} e^{j\omega_D n_{\tau,\text{pre}}} \quad (6.22)$$

since $\tau = \tau_{\text{pre}} + n_{\tau,\text{pre}}$ where $n_{\tau,\text{pre}}$ is the measurement process noise for the delay. However, the first term is the correct Doppler shift with the correct delay. Since we can pull this out of the residual, the expected residual is unaffected. Therefore:

$$\mathbb{E}[\|\Delta s\|^2] = \mathbb{E}[\|s_{\text{rad}}(t - \tau) - s_{\text{rad}}(t - \tau + n_{\tau,\text{pre}}) e^{j\omega_D n_{\tau,\text{pre}}}\|^2]. \quad (6.23)$$

The $L^2$ norm in this expectation produces three unique products, each of which we
integrate over our CPI and take their expected value:
\[
\|s_{\text{rad}}(t - \tau)\|^2, \|s_{\text{rad}}(t - \tau + n_{\tau,\text{pre}})\|^2, \\
- 2\Re \left\{ s_{\text{rad}}(t - \tau)s_{\text{rad}}^*(t - \tau + n_{\tau,\text{pre}})e^{-j\omega_D n_{\tau,\text{pre}}} \right\}
\] (6.24)

The first two terms, integrated over our coherent processing interval, produce the signal RMS squared times our CPI
\[
\|s_{\text{rad}}(t - \tau)\|^2 = s_{\text{rms}}^2 T.
\] (6.25)

We thus must solve for
\[
A = \int_{-T/2}^{T/2} s_{\text{rad}}^*(t - \tau) \mathbb{E} \left[ s_{\text{rad}}(t - \tau + n_{\tau,\text{pre}})e^{j\omega_D n_{\tau,\text{pre}}} \right] dt,
\] (6.26)
where we have pulled out the deterministic radar signal cross-term from the expectation and defined \(A\) to be the integral expectation of the last term in Equation (6.24).

Note we have a random process governed by a Gaussian time delay and a Gaussian phase shift. To solve for this expectation, we can use Parseval’s theorem:
\[
A = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S_{\text{rad}}(j\omega)|^2 \mathbb{E} \left[ e^{j\omega_D n_{\tau,\text{pre}}} e^{j\omega n_{\tau,\text{pre}}} \right] d\omega.
\] (6.27)

We note that the argument of the exponential is Gaussian. Therefore, the exponential term has a log-normal distribution [186]. The expectation is then the mean of a log-normal, which is given by [186]
\[
\mathbb{E} \left[ e^{j n_{\tau,\text{pre}}(\omega_D + \omega)} \right] = e^{-\frac{1}{2}(\omega_D + \omega)^2 \sigma_{\tau,\text{pre}}^2},
\] (6.28)
for \(\sigma_{\tau,\text{pre}}^2\) the variance of the delay process noise. If we look carefully, we see this is a Gaussian function. Therefore, we see that the statistical mismatch in delay due to the model process noise of our delay results in a Gaussian spectral mask \(G(j\omega)\) centered at our Doppler frequency. Therefore
\[
\mathbb{E} [\|\Delta s\|^2] \approx 2s_{\text{rms}}^2 T - \frac{1}{\pi} \int_{-\infty}^{\infty} |S_{\text{rad}}(j\omega)|^2 G(j\omega) d\omega.
\] (6.29)
Note that we can make the simplifying assumption that our spectrum is flat, in which case we are simply integrating a non-normalized, truncated Gaussian. Using this knowledge we get
\[
E[\|\Delta s\|^2] \approx 2s_{\text{rms}}^2 T \left(1 - \frac{Q(\omega_D \sigma_{\tau,\text{pre}} - B \pi \sigma_{\tau,\text{pre}}) - Q(\omega_D \sigma_{\tau,\text{pre}} + B \pi \sigma_{\tau,\text{pre}})}{B \sqrt{2\pi \sigma^2_{\tau,\text{pre}}}}\right),
\]
(6.30)
where \(Q(x)\) is the complementary cumulative distribution function of the standard normal distribution. The communications rate is therefore defined for SIC as
\[
R_{\text{com},\text{SIC}} \leq B \log_2 \left(1 + \frac{b^2 P_{\text{com}}}{k_B T_{\text{temp}} B + P_{\text{rad}} a^2 E[\|\Delta s\|^2; B]}\right)
\]
(6.31)
where we have parameterized the residual power \(E[\|\Delta s\|^2]\) by the bandwidth. That is, the rate must be decreased due to the interference of the radar subtraction residual, which is a function of the target entropy. Since we can only predict the return on average as well as it is modeled, we have an average interference term. Assuming the communications system reduces its rate to handle the increase in noise and interference from the subtraction residual, we assume we can then decode the communications signal perfectly and remove it from the composite return so that the radar signal can then be processed as if it were the only system in the band. Therefore
\[
R_{\text{est},\text{SIC}} \leq \frac{1}{2T} \log_2 \left(\frac{\| C \Sigma C^T + J^{-1}(B) \|}{\| J^{-1}(B) \|}\right).
\]
(6.32)

6.2.3 Water-filling

We can combine the concepts of both the ISB and SIC inner bounds to form another inner bound using water-filling (WF). The technique is described in detail in References [2, 13], and so we only present the final equation here. As before, we allocate one sub-band of our full bandwidth \(B\) for communications only, and another for joint radar-communications use (using SIC). The sub-band allocation is parameterized by \(\alpha\) as with the ISB, and for each allocation, an optimal communications
power allocation to the communications only band and the mixture band is found through the general solution in References [2, 13]. The WF inner bound rates are therefore

\[ R_{\text{com,com}} = \alpha B \log_2 \left( 1 + \frac{b^2 \beta P_{\text{com}}}{k_B T_{\text{temp}} \alpha B} \right) \] (6.33)

\[ R_{\text{com,mix}} = \overline{\alpha} B \log_2 \left( 1 + \frac{b^2 \overline{\beta} P_{\text{com}}}{k_B T_{\text{temp}} \overline{\alpha} B + P_{\text{rad}} a^2 E[\|\Delta s\|^2; \overline{\alpha} B]} \right) \] (6.34)

\[ R_{\text{com,WF}} \leq R_{\text{com,com}} + R_{\text{com,mix}} \] (6.35)

\[ R_{\text{est,WF}} \leq \frac{1}{2T} \log_2 \left[ \frac{|C \Sigma C^T + J^{-1}(\overline{\alpha} B)|}{|J^{-1}(\overline{\alpha} B)|} \right] \] (6.36)

where \( \overline{\beta} = 1 - \beta \). The optimal power distribution \( \beta \) is given by [47]

\[ \beta = \begin{cases} \alpha + \frac{1}{P_{\text{com}}} \left( \frac{\pi}{\mu_{\text{com}}} + \frac{\alpha}{\mu_{\text{mix}}} \right) & P_{\text{com}} \geq \frac{\alpha}{\overline{\alpha} \mu_{\text{mix}}} - \frac{1}{\mu_{\text{com}}} \\ 1 & P_{\text{com}} < \frac{\alpha}{\overline{\alpha} \mu_{\text{mix}}} - \frac{1}{\mu_{\text{com}}} \end{cases} \] (6.37)

for

\[ \mu_{\text{com}} = \frac{b^2}{k_B T_{\text{temp}} \alpha B}, \text{ and } \mu_{\text{mix}} = \frac{b^2}{k_B T_{\text{temp}} \overline{\alpha} B + P_{\text{rad}} a^2 E[\|\Delta s\|^2; B]} \] (6.38)

6.3 Examples

In Figure 6.2, we see an example of all the bounds we have extended in this chapter. We used parameters similar to those used in Chapter 4 for comparison, as shown in Table 6.1. The communications SIC vertex (orange dot) is shown linearly interpolated to the radar-free communications bound. This is achievable because we can simply utilize time-division multiplexing (TDM) between operating at the SIC vertex with joint radar and communications, and simply operating the communications by itself [172] by employing the constant information radar (CIR) discussed in Chapter 8. It appears both the ISB and WF inner bounds approach an asymptote for the radar estimation rate. This is the tonal bound, defined as the point
Table 6.1: Example parameters for joint communications and FMCW radar performance bounds

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth</td>
<td>5 MHz</td>
<td>Center Frequency</td>
<td>3 GHz</td>
</tr>
<tr>
<td>Absolute Temperature</td>
<td>1000 K</td>
<td>Communications Range</td>
<td>10 km</td>
</tr>
<tr>
<td>Communications Power</td>
<td>100 W</td>
<td>Communications Antenna Gain</td>
<td>10 dBi</td>
</tr>
<tr>
<td>Previous Target Range</td>
<td>40 km</td>
<td>Radar Antenna Gain</td>
<td>30 dBi</td>
</tr>
<tr>
<td>Radar Power</td>
<td>100 kW</td>
<td>Target Cross Section</td>
<td>100 m²</td>
</tr>
<tr>
<td>CPI</td>
<td>250 ms</td>
<td>Process Noise Intensity</td>
<td>250000</td>
</tr>
<tr>
<td>RMS Bandwidth</td>
<td>$B/\sqrt{12}$</td>
<td>RMS Envelope</td>
<td>$T/\sqrt{12}$</td>
</tr>
<tr>
<td>EM Propagation</td>
<td>$3 \times 10^8$ m/s</td>
<td>Previous Bearing</td>
<td>22.5°</td>
</tr>
<tr>
<td>Previous $x$ Speed</td>
<td>125 m/s</td>
<td>Previous $y$ Speed</td>
<td>125 m/s</td>
</tr>
</tbody>
</table>

at which the bandwidth vanishes, and only a tone is used for the radar. As a result, only the range-rate measurement contributes to the joint mutual information. As we can see, the WF bound outperforms all of the other inner bounds for this parameter set, though it may not in general. The estimation rate also appears much lower than in the example in Chapter 4 in Figure 4.5, but this is due to the long CPI with a similar source entropy. Note also we have started the estimation rate axis near the tonal bound to emphasize detail of the inner bounds.

6.4 Summary

In this chapter we extended previous results for producing joint radar and communications performance bounds to accommodate FMCW radars. This extension necessitated a more thorough development of a Markov motion model, as well as an extended measurement model to include Doppler estimation. This in turn required
Figure 6.2: Communications data rate and radar estimation rate bounds. The multiple access channel (MAC) outer bound is shown in the dashed red lines, indicating the independent rates for each system. The isolated sub-band (ISB) inner bound is shown by the yellow line. The successive interference cancellation (SIC) operating point is indicated by orange dot, and the joint performance achieved by time division multiplexing (TDM) of the radar and the communications operating with SIC is shown by the solid blue line (CIR). The water-filling (WF) inner bound is shown in solid green. Finally, the range-rate only (tonal radar) estimation rate is shown by the dashed purple line.

A reformulation of the SIC inner bound statistical interference term due to Doppler phase mismatch. The EKF formulation provided a straightforward framework for the otherwise complicated distributions that result from these extensions.
Here we extend the notion of multiuser detection radar (MUDR) further by designing a jointly optimal waveform for radar-communications systems employing successive interference cancellation (SIC). In previous chapters, estimation rate was defined and used to derive joint radar-communications information bounds in a multiple access channel (MAC) scenario. These bounds showed that the radar estimation rate, which is a measure of the amount of information the target is uncooperatively communicating with the radar as a function of time, was dependent primarily on the root mean square (RMS) bandwidth when considering local error. In addition, the SIC inner bound, which determined the limit of communications data rate when canceling radar returns in-band, was also dependent on the RMS bandwidth of the radar waveform. As a result, a fixed RMS bandwidth was used in deriving the joint performance, and this value resulted in a direct calculation of both the radar estimation rate and communications information rate operating using SIC. In addition, the waveform was not considered a degree of freedom for optimizing the joint inner bound performance. Therefore, the radar waveform is a significant design choice for joint systems. Further, the waveform affects global ambiguity, and so the notion of global radar estimation rate becomes significant when considering radar waveform design. An example of a simple spectral weighting scheme for radar waveform modification is shown in Figure 7.1.
Figure 7.1: Radar waveform spectral weighting using a constrained 2nd order polynomial. The base waveform is a linear frequency modulated chirp, and the spectral weighting emphasizes or de-emphasizes components of the otherwise flat spectrum symmetrically (approximately shown in purple). The radar user wants to maximize RMS bandwidth when ignoring global error (shown in blue). The communications user, however, can more easily mitigate the radar return when the radar RMS bandwidth is at a minimum (shown in red). A continuum of waveforms exist between these extremes, and trade between radar estimation rate and communications information rate.

7.1 Joint Information Bounds

Here we briefly review the communications Shannon bounds and a more general radar estimation rate to motivate the radar waveform design for joint radar-communications optimization.

7.1.1 Communications Data Rate

As discussed in previous chapters, a well known result for a communications user with bandwidth $B$ is that the theoretical limit of communications in bits/second is given by [135]

$$R_{\text{com}} \leq B \log_2 \left[ 1 + \frac{b^2 P_{\text{com}}}{k_B T_{\text{temp}} B} \right],$$

(7.1)
where \( b \) is the channel gain accounting for path loss [2], \( P_{\text{com}} \) is the communications transmit power, \( k_B \) is Boltzmann’s constant, and \( T_{\text{temp}} \) is the absolute temperature of the receiver. This is identical to the bounds given in Chapter 3, but with parameters that more verbosely describe the communications link.

### 7.1.2 Radar Estimation Rate

Previous chapters cast radar tracking as an information problem and derived the notion of estimation rate, a measure of radar tracking information as a function of time:

\[
R_{\text{est}} \leq \frac{1}{2T} \log_2 \left[ 1 + \frac{\sigma^2_{\text{proc}}}{\sigma^2_{\text{est}}} \right],
\]

(7.2)

where \( T \) is radar pulse repetition interval (PRI) or revisit time, \( \sigma^2_{\text{proc}} \) is the target process noise variance (range prediction uncertainty), and \( \sigma^2_{\text{est}} \) is the range estimation noise variance. This shows that the estimation rate is increased by “good” uncertainty (reduction of uncertainty from measurement of previously unknown/uncertain range), and degraded by “bad” uncertainty (estimation noise). In previous examples, only local error was considered, and so the Cramér-Rao lower bound (CRLB) contributed solely to the estimation error:

\[
\sigma^2_{\text{est}} = \sigma^2_{\text{CRLB}} = \frac{k_B T_{\text{temp}} B}{8\pi^2 B_{\text{rms}}^2 T_p B a^2 P_{\text{rad}}}.
\]

(7.3)

where \( B_{\text{rms}} \) is the radar waveform RMS bandwidth, \( P_{\text{rad}} \) is the radar transmit power, \( T_p \) is the radar pulse duration, and \( a \) is the channel gain accounting for two-way path loss and radar target cross section [2]. In practice, the estimation noise depends not only on the CRLB, but also global estimation ambiguity error as well, especially at low signal-to-noise ratios (SNRs). This is a generalization to the bounds given in Chapter 3 to include global error, but with parameters that more verbosely describe the radar range equation.
It should be obvious from Equation (4.4), which calculates the reduced communications rate for the MUDR operating with in-band radar and SIC, the communications user would like to minimize the radar RMS bandwidth, as it directly contributes to the additional noise source. Immediately, considering only local error, the joint system has conflicting requirements. However, we can use the RMS bandwidth as a parameter to sweep between radar estimation rate and communications data rate preference by designing a waveform that varies the RMS bandwidth in a well-behaved way.

7.2 Waveform Design

The radar waveform can be designed to maximize radar estimation rate, communications data rate, or some weighting therein. Without consideration of global error, waveform design can be simplified to tuning $B_{\text{rms}}$ [47]. As a first order look into waveform design, we consider a closed-form, parameterized spectral weighting. To extend previous results, we subsequently allow global error to contribute to estimation rate degradation and solve for unconstrained spectral weighting using numerical methods. This more general error depends on radar correlation ambiguity in addition to the RMS bandwidth.

7.2.1 Spectral Weighting

As discussed in Chapter 1, many modern approaches to RF convergence have looked at waveform design in the context of a single, unified waveform for radar and communications, such as orthogonal frequency-division multiplexing (OFDM). Rather than fight the conflicting system-level waveform requirements of a dual-purpose waveform, we instead design a single radar waveform that performs well for global radar estimation rate and is more easily canceled for in-band communications users.
performing SIC at the joint receiver. The result jointly maximizes both the radar and communications users’ information rate for heterogeneous multiple-access scenarios like those shown in Figure 1.3.

The problem is set up as follows. We assume we have a monostatic pulsed radar employing a linear frequency-modulated (FM) chirp which spans from $-B/2$ to $B/2$ in time $T_p$. We then apply a frequency domain spectral mask weighting to the chirp, $W(f)$. In pursuit of a closed-form solution, we can first represent this spectral mask with a simple quadratic polynomial:

$$W'(f) = x + y f + z f^2, \quad |f| \leq \frac{B}{2}, \quad (7.4)$$

where $x$, $y$, and $z$ are waveform design parameters. To assist in optimization, we can force symmetry for the positive and negative frequencies. This amounts to setting $y = 0$. Thus,

$$W(f) = x + z f^2, \quad |f| \leq \frac{B}{2}. \quad (7.5)$$

Finally, we must constrain this polynomial such that $W(f) > 0$ (we allow it to touch 0 at a single point, however). This ensures we do not induce any phase shifts in the spectrum, or null out the entire waveform. Ultimately, this amounts to constraining the roots of the polynomial. For $z \geq 0$, the only requirement is $x \geq 0$. Note that if $z = 0$, $x$ must be strictly $> 0$ (we discuss normalization next). For $z < 0$, we need to ensure the roots of the polynomial are $\geq B/2$ in magnitude. The ultimately amounts to the following constraint:

$$x \geq \frac{z B^2}{4}. \quad (7.6)$$

The weighting must be normalized to preserve waveform energy. Since normalization is required, there is redundancy in this parametrization space. For example, \{x, z\} = \{0, 1\} is normalized to the same polynomial mask as \{x, z\} = \{0, 2\}. We can reduce the dimensionality by constraining the polynomial coefficients to the perimeter
of the unit circle \([189]\):

\[
x = \sin(\phi), \quad z = \cos(\phi),
\]

and then the constraints become

\[
0 \leq \phi \leq \arctan\left(-\frac{B^2}{4}\right) + \pi.
\]

Note we have effectively reduced our waveform design parameter to a singleton dimension over a finite support.

The RMS bandwidth of the resulting weighted chirp is found by assuming the chirp spectrum is approximately flat using the principle of stationary phase (PSP) [48]. As a result, the RMS bandwidth is easily calculable in closed-form for the polynomial:

\[
B'_{\text{rms}} = \sqrt{\frac{x^2 B^3}{12} + \frac{x z B^5}{40} + \frac{z^2 B^7}{448} + \frac{x^2 B^3}{6} + \frac{z^2 B^5}{80}}.
\]

To make this more numerically stable with the constraints, we can instead assume \(B = 1\), turning this equation into the shaping parameter, and find the RMS bandwidth by multiplying by \(B\):

\[
B_{\text{rms}} = \sqrt{\frac{\sin(\phi)^2}{12} + \frac{\sin(\phi) \cos(\phi)}{40} + \frac{\cos(\phi)^2}{448} \frac{1}{6} + \frac{\cos(\phi)^2}{80} } B,
\]

with new constraints

\[
0 \leq \phi \leq \arctan\left(-\frac{1}{4}\right) + \pi.
\]

For the closed-form solution, we then must choose \(\phi\) to jointly maximize the radar estimation rate and communications data rate operating using SIC. In this case, we can simply evaluate the RMS bandwidth for various \(\phi\) chosen uniformly over the range provided and evaluate performance.

In addition to the closed-form solution for local error, we are interested in more general results optimizing with respect to global radar ambiguity. For numerical results considering global error, we can redefine the spectral mask using 32 nonnegative
independent values in frequency that can be varied. This number was chosen to be tractable for simulation purposes, but provide enough granularity to enable meaningful inspection and interpretation of the results. The 32 weights are chosen using differential evolution (DE) [190], with the cost function driven by global and local error, as discussed next. The autocorrelation of the resulting weighted chirp typically has a main lobe, and multiple sidelobes. The main lobe is captured by the CRLB, which is a function of the RMS bandwidth [173] as shown in Equation (7.3). The peak sidelobe is also considered to characterize global error in the transition region.

7.2.2 Global Estimation Rate

We use the method of interval errors [191] to calculate the effect of non-local errors on time-delay estimation performance. For the sake of simplicity, we assume that only the largest sidelobe can be confused for the main lobe. The values and locations of the largest sidelobe peaks are found through simulation. A closed-form solution of the probability of sidelobe confusion, \( P_{s.l.} \), is obtained in terms of the values and locations of the sidelobe peaks, SNR, and the Marcum Q-function \( Q_M \) [47]. The method of intervals time-delay estimation variance is then given by

\[
\sigma^2_{MOI} = [1 - P_{s.l.}(ISNR)] \sigma^2_{CRLB}(ISNR) + P_{s.l.}(ISNR) \phi^2_{s.l.},
\]

where \( \phi_{s.l.} \) is the offset of the sidelobe peak from the main lobe in delay measurement units, and ISNR is the integrated SNR [13]. The probability of sidelobe confusion is given by [47]

\[
P_{s.l.}(ISNR) = 1 - Q_M \left( \sqrt{\frac{ISNR}{2}} \left( 1 + \sqrt{1 - \|\rho\|^2} \right), \sqrt{\frac{ISNR}{2}} \left( 1 - \sqrt{1 - \|\rho\|^2} \right) \right) + Q_M \left( \sqrt{\frac{ISNR}{2}} \left( 1 - \sqrt{1 - \|\rho\|^2} \right), \sqrt{\frac{ISNR}{2}} \left( 1 + \sqrt{1 - \|\rho\|^2} \right) \right),
\]

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where \( \rho \) is the ratio of the main lobe to the peak sidelobe of the autocorrelation function. Finally, the global estimation rate is then given by Equation (7.2) with \( \sigma^2_{\text{est}} = \sigma^2_{\text{MOI}} \). Since the communications data rate and radar estimation rate may be largely disparate depending on the target dynamics \([9]\), we use the geometric spectral efficiency as our cost function as described in Chapter 2. This provides a more numerically stable error term, even when \( R_{\text{com,SIC}} \gg R_{\text{est,SIC}} \), without requiring strict definition of the relative spectral efficiency weighting factors. To construct our optimal inner bound based on global error waveform design, we vary \( \delta \) in the cost function from 0 to 1:

\[
S_{\text{geo}}(\delta) = \frac{R^\delta_{\text{est}} R^{1-\delta}_{\text{com}}}{B}, \quad (7.13)
\]

We can use bounds on Marcum’s Q function \([192]\) to speed up the process since we are using an evolutionary method of optimization that benefits greatly from a large number of trials \([190]\).

### 7.3 Results

The parameters for all examples shown here are given in Table 4.1 from Chapter 4 for comparison. To start, we look at the quasi closed-form polynomial spectral mask. First, we vary the spectral shaping parameter \( \phi \) between 0 and \( \arctan(-1/4) + \pi \). The resulting radar RMS bandwidth is shown in Figure 7.2.

This plot defines the achievable RMS bandwidth for the radar waveform with this variable shaping. The shapes that are optimal for radar and communications are shown with the blue circle and red triangle respectively. The unweighted chirp RMS bandwidth is indicated by the purple square. These three waveform extremes are shown in Figure 7.1. Using these values for the RMS bandwidth, Equations (7.1) and (7.2) are jointly solved and the resulting MAC bound plot is given in Figure 7.3. The same markers denote the accompanying joint performance on this plot. The
Figure 7.2: Achievable RMS bandwidth using the constrained quadratic polynomial spectral mask applied to a linear frequency modulated chirp radar waveform. The optimal points for radar (blue dot) and communications (red triangle), are shown at the two extremes maximizing and minimizing the RMS bandwidth respectively. For reference, the unweighted chirp performance is indicated by the purple square.

completely isolated bounds are shown in orange, while the solid green line is achieved by varying the shaping parameter in the polynomial spectral mask, which emphasizes radar when “smiling” (blue dot), and communications when “frowning” (red triangle), with the unweighted waveform indicated by the purple square for reference. The yellow line shows the same points but considering local error. Note we have optimized with respect to local error, but are showing (on this curve) the actual global error performance. Once can see that a Pareto optimal point exists somewhere in between the inflection optimal for the radar and the unweighted spectrum.

For the numerical methods using DE and the 32 weight mask, we see the resulting MAC joint bounds in Figure 7.4. The weighting that is optimal for radar estimation rate maximization is shown in Figure 7.5. The resulting autocorrelation function is shown in Figure 7.6. The ratio of the sidelobes contribute to the global error term.
Figure 7.3: Multiple-access joint radar estimation rate and communications data rate bounds. Isolated bounds are shown in orange. The solid green line is achieved by varying the shaping parameter in the polynomial spectral mask, which emphasizes radar when “smiling” (blue dot), and communications when “frowning” (red triangle), with the unweighted waveform indicated by the purple square for reference. The green line is considering only local error for the radar, while the yellow line includes global error as well.

As can be clearly seen, the overall sidelobe content is very high. This is because only the peak sidelobe is considered, which is typically adjacent to the main lobe, resulting in a small global error accompanied by a very small main lobe error.

The optimization engine took advantage of the single sidelobe simplification, and so the resulting waveform has significant ambiguity. For high SNR scenarios, this waveform performs very well, but most likely limits radar performance in degraded target configurations.

To maximize the communications rate, and therefore minimize radar residual interference, the frequency mask in Figure 7.7 is applied to the radar spectrum. This ultimately, after removal of the phase information in matched filtering, represents a sinusoidal tone. It is clear, given the theoretically 0 RMS bandwidth why the com-
Figure 7.4: Multiple-access joint radar estimation rate and communications data rate bounds, now considering global radar error. Isolated bounds are shown in orange. The solid green line is achieved by varying the weighting parameter $\delta$ in the optimizer, which varies emphasis between the radar (blue dot) and communications user (red triangle), with the unweighted waveform indicated by the purple square for reference.

Figure 7.5: Normalized frequency weighting mask optimal for the radar estimation rate maximization. The spectrum of the linear FM chirp is weighted by this mask to maximize the estimation rate under global error consideration.
Figure 7.6: Normalized autocorrelation for optimal radar waveform (maximizes radar estimation rate). The correlation axis is given in dB with respect to the zero-lag autocorrelation offset. The ratio of the peak to the first sidelobe in linear scale is considered when determining global error estimation rate. 

munications optimized mask would be a Dirac delta as shown. After applying this mask, the autocorrelation that results for the radar processor is given by Figure 7.8, which is a sinc function. The unweighted chirp autocorrelation function is shown in Figure 7.9 for reference.

7.4 Summary

In this chapter we demonstrated promising results for waveform design for joint radar-communications systems. By jointly considering communications data rate and radar estimation rate, we can design a waveform that performs well for both radar global estimation error and minimization of radar cancellation residual limiting communications interference. We derived the global error estimation rate, and SIC bound for the weighted linear FM chirp, and provided a parameterized spectral weighting to tune the RMS bandwidth. We then presented an example of the optimization
Figure 7.7: Normalized frequency weighting mask for radar waveform optimal for communications interference mitigation using successive interference cancellation.

Figure 7.8: Normalized autocorrelation optimal for minimizing radar interference relative to an in-band communications user. The spectrum of the radar's linear FM chirp is weighted by this mask to maximize the in-band communications rate, effectively minimizing radar interference.
Figure 7.9: Normalized autocorrelation for unweighted chirp for reference.

using the quasi closed-form polynomial and unconstrained numerical methods. Opti-
mal spectral masks for radar estimation rate maximization and communications rate
maximization were shown.
Chapter 8

THE CONSTANT INFORMATION RADAR

The constant information radar (CIR) is introduced as a method of radar target scheduling that is optimal for a specified spectral efficiency. It was noted in the previous chapters that the radar estimation rate could be time-division multiplexed with the communications user or a mixed band employing successive interference cancellation (SIC). This is achieved by fixing the mutual information value for the target tracking scenario, while modulating the target revisit time, and the CIR naturally arises from this method of joint operation. The resulting radar scheduling algorithm ensures a fixed spectral efficiency is achieved for each target spectral access, enabling more time for dynamic communications users or equivalently more targets to be visited in the same band allocation.

8.1 Motivation for the CIR

As discussed in Chapter 1, radar and information theory have a surprisingly long history. However, applications have fallen short in terms of intelligent metrics for control [127, 142], and algorithms outside of waveform design for statistical radar cross-section (RCS) mutual information maximization [137, 138]. Some works have investigated variable radar resolution cell size to keep the Fisher information constant [147]; not in an effort to modulate radar spectral access, but rather to vary the computational resolution for compressive sensing applications. Similar concepts exploiting sparsity with respect to tracking have been explored [193], but not in an information-theoretic sense. Recent research explored using data-driven techniques to supplement estimated clutter distributions to adapt to changing detection statis-
tics [194]. Some research looked at modulating revisit time based on predicted state covariance error thresholding [195]. However this does not take into account the measurement transform, and thus the true information available through target revisit. Tracking information has been used in the context of radar waveform design, but for limited applications and one-dimensional (1D) scenarios [196]. Others have attempted to modulate dwell time on target based on RCS to maintain a fixed signal-to-noise ratio (SNR) as well [197].

In this chapter, we formally develop the CIR and the corresponding scheduling concept, which is depicted at a high level in Figure 8.1. The radar is in track-mode, and performs the typical Kalman filtering prediction steps. From the predicted state, the estimation rate is also predicted. The mutual information between the noiseless and noisy target state is estimated using the predicted SNR and the previous Kalman residual (a measure of state prediction error). The operator sets a constant information value, and the radar modulates the revisit time of the target track to attempt to keep the information measure constant. If the previous Kalman residual is large in magnitude, this indicates the target is deviating from the predicted motion model, and the radar sets a shorter revisit time. If the predicted SNR is low then the information content is also low, even for dynamic targets. The CIR subsequently sets a longer revisit time to allow the state uncertainty to grow large enough to overcome the measurement noise variance. This modulation of revisit time maintains a fixed radar spectral efficiency instead of suboptimally sampling the target at a regular interval. For well-modeled or low-SNR targets, this scheme allows for increased spectral access time for cognitive communications users or scheduling of additional radar targets.
Figure 8.1: Top level view of the constant information radar (CIR) scheduling concept. At the Kalman prediction step, the predicted SNR and model covariance are used to predict the tracking mutual information. If the predicted information is smaller than some specified value, then the revisit time is increased. Conversely, the revisit time is decreased if the predicted information exceeds this value, indicating greater than desired uncertainty. This modulation ensures a fixed spectral efficiency for the radar, and precludes unnecessary sampling of well-modeled targets or targets that are obfuscated by noise or clutter beyond target state uncertainty.

8.2 Radar Tracking & Measurement Model

The radar tracking model is critical to the CIR framework. The model forms a hypothesis for the target state based on prior physical knowledge and the previously estimated state, typically involving a position and velocity component. Based on classical physics, we can predict where the target will be after time $T$ later. For more advanced models, this can include compensation for acceleration and higher order motion components. Here we assume, for simplicity and illustration, a constant velocity model. We start our formulation at the source, and work through the measurement model to capture the interdependencies. The most natural formulation to that end is to use a dynamical Markov model as a part of a tracking process, followed by a range, range-rate, and bearing measurement model.
8.2.1 Target Motion Model

For the target motion, we assume a constant velocity, linear two-dimensional (2D) motion model with a Gaussian perturbation acceleration distribution [182]:

\[
s_k = \begin{bmatrix} x_k \\ \dot{x}_k \\ y_k \\ \dot{y}_k \end{bmatrix} = A(T)s_{k-1} + w_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ \dot{x}_{k-1} \\ y_{k-1} \\ \dot{y}_{k-1} \end{bmatrix} + w_k, \quad (8.1)
\]

where the \( w_k \) is the process noise with covariance defined as

\[
Q_k(T) = \begin{bmatrix}
q_{x,k} \frac{T^3}{3} & q_{x,k} \frac{T^2}{2} & 0 & 0 \\
q_{x,k} \frac{T^2}{2} & q_{x,k} T & 0 & 0 \\
0 & 0 & q_{y,k} \frac{T^3}{3} & q_{y,k} \frac{T^2}{2} \\
0 & 0 & q_{y,k} \frac{T^2}{2} & q_{y,k} T
\end{bmatrix}, \quad (8.2)
\]

\( x_k \) is the target position along the \( x \)-axis at discrete time step \( k \), \( \dot{x}_k \) is the target velocity projected on the \( x \)-axis, \( T \) is the revisit time of a particular target (duration between time steps \( k \) and \( k - 1 \)), \( s_k \) is the state vector, and \( q_{x,k} \) is the process model error intensity for the \( x \)-axis (similarly for the \( y \)-axis). We assume the \( x \)-position and velocity have an independent process noise power from the \( y \)-position and velocity. These powers are estimated to track model mismatch along each dimension. We have parameterized the linear motion model matrix \( A(T) \) and the process noise covariance \( Q_k(T) \) by the revisit time \( T \), as this is our dynamic parameter. An illustration of the motion model and target prediction is shown in Figure 8.2. Though this model is specified in Appendix A, we repeat it here to emphasize the parameterization of the revisit time. The revisit time is a key parameter for tracking radar systems, as it specifies the amount of time between illuminations for a specific target. Between track points, the radar predicts the next target location \( T \) seconds later based on the
Figure 8.2: Illustration of target motion model. The previous measurement is indicated by the solid gray plane. The positional covariance contour is shown surrounding the plane, while the velocity vector leading the plane has a covariance as well to indicate degree of confidence in the current target state. Using a constant velocity model, the prediction shown by dashed lines is made by advancing the target in time as if the last state were truth, and the plane is not accelerating. Since the plane can, in fact, accelerate and because we had an initial uncertainty about the state, the prediction covariance shown in the dashed contours is increased.

previous target state. If $T$ is set too large, the time between target illuminations may be too great, and the target may not be within the beamwidth of the radar beam steered to the predicted angular target location. If $T$ is too small, ambiguous range measurements, high average power, and heavy spectral use can occur. Depending on the Swerling target model [48], some tracking systems may define $T$ to be the pulse repetition interval (PRI) instead of target revisit time.

This model can easily be extended to three-dimensional (3D) space as needed, and may include more advanced motion dynamics such as acceleration. A more advanced predictive model results in a better tracker yielding a smaller Kalman residual, assuming the prior knowledge of the target is accurate. This theoretically reduces the true measurement deviation, an important point we discuss later.

8.2.2 Target Measurement Model

As in many of the previous cases, we assume a monostatic pulsed radar system. The observed parameters, after cross-ambiguity processing [46, 173], are the range $r_k$ and range-rate $\dot{r}_k$. We assume a narrowband environment such that only a Doppler shift is induced in the returned waveform, not the more general Doppler time scaling (though the model can easily be extended to encompass this). We also obtain a bearing measurement from our antenna array. We assume we have a phased array with
half-wavelength spacing, and that the platform is steerable, so the beam can be formed normal to the array at the predicted target bearing. The actual distributions for the range, range-rate, and bearing are very complicated assuming an underlying Gaussian source model noise distribution [8], as outlined in Appendix A. We can simplify the analysis by linearizing the measurement model about the predicted state, much like the extended Kalman filter [8, 183]. This results in a time-varying, but linear transform on our Gaussian state vector, resulting in another multivariate Gaussian, as we derived in Chapter 6. This source to measurement transformation is illustrated in Figure 8.3.

We only consider a single target scenario to compare with prior work and the previous chapters for simplicity. We are assuming the Cramér-Rao lower bound (CRLB) is achievable, and sufficiently high target SNR such that only local, main lobe errors contribute to performance degradation [173]. Subsequently, our range and range-rate are corrupted by correlated Gaussian noise (correlated between the range and range-rate measurement, but independent in time). The bearing measurement is corrupted by additive white Gaussian noise (AWGN) at the CRLB, independent of the Doppler noise [47]. The results here can easily be extended to include global ambiguity using the method of interval errors [10, 47, 198], as shown in Chapter 7. Extending the tracking model to multiple targets with more complicated error models in theory only amounts to more complicated probability distributions. This is important, as our predicted mutual information depends solely on the Markov tracking distributions. Therefore, as long as distributions can be formulated for a given scenario, the mutual information can be computed, or at least bounded [175], enabling the CIR to modulate revisit time.
Figure 8.3: Measurement model showing Cartesian to polar transformation. While we track the target in Cartesian space to avoid coordinate-coupled physics, we measure waveform echo delay, Doppler shift, and bearing angle. From the delay and bearing, a direct polar transformation can recover the 2D position. However, the velocity vector is only projected along the radial axis through the Doppler effect (range-rate), meaning multiple measurements and position history are needed to unambiguously recover velocity.

8.3 Target Tracking Information

In this section, we augment the classical Kalman filtering model to include target state information. We extend radar estimation rate to include a notion of detection, as any probability distribution can be used to derive an entropy. For the linearized Kalman tracking problem here, radar estimation rate is a function of the model covariance and measurement covariance. Both of these are included in the standard Kalman formulation, and so predicted information is easily incorporated into the framework.

8.3.1 Radar Estimation Rate

We leverage the radar estimation rate defined in previous chapters to measure the information of the target. The estimation rate is defined as the mutual information between the noiseless and noisy target tracking state per unit time. The noise can encompass any perturbative distribution, such as clutter distributions. The target state can include position and velocity components as described in the previous section. The mutual information can be calculated over the PRI or the target revisit.
period, as we do here. The result is an information flow for the radar channel.

For our scenario, we assume a Gaussian state distribution and linearize the measurement using the extended Kalman filter corrupted by an independent and additive Gaussian distribution. Therefore, the estimation rate of our target is given by [9]

$$R_{\text{est}} \leq \frac{1}{2T} \log_2 \left[ \frac{|C P_{k|k-1} C^T + \Sigma|}{|\Sigma|} \right],$$  \hspace{1cm} (8.3)

where $|\cdot|$ is the determinant function, $P_{k|k-1}$ is the predicted model covariance, $C$ is the linearized measurement transform, and $\Sigma$ is the inverse Fisher information matrix (FIM) [199], scaled by the integrated SNR given by [8]

$$\Sigma = \frac{J^{-1}}{\text{ISNR}},$$  \hspace{1cm} (8.4)

where $J$ is the FIM and the integrated SNR for our scenario is given by [8]

$$\text{ISNR} = \frac{N_p T_p B P_t G^2 c^2 \sigma}{f_c^2 (4\pi)^3 r^4 k_N T_{\text{temp}} B},$$  \hspace{1cm} (8.5)

where $N_p$ is the number of pulses over the coherent processing interval (CPI), $T_p$ is the pulse duration, $B$ is the pulse bandwidth, $P_t$ is the radar transmit power, $G$ is gain of the radar antenna, $c$ is the speed of light, $\sigma$ is the RCS, $f_c$ is the carrier frequency of the radar waveform, $r$ is the range to the target, $k_N$ is the Boltzmann constant, and $T_{\text{temp}}$ is absolute temperature of the receiver.

The form of Equation (8.3) is well known, and arises due to the Kalman filtering formulation. The source uncertainty is our desired information, and is modeled as a multivariate Gaussian with covariance $P_{k|k-1}$. This covariance is obtained by taking the last target distribution at time step $k-1$, and advancing the prediction in time using the linear motion model $A$:

$$P_{k|k-1} = A(T) P_{k-1} A(T)^T + Q_k(T).$$  \hspace{1cm} (8.6)

The covariance of the measured distribution, after linearizing about the predicted state, is given by $C^T P_{k|k-1} C$, where $C$ is the Jacobian of the measurement matrix
linearized about the predicted state as defined in Appendix A (see Equation (A.44)).

The linear transform of the multivariate Gaussian results in another multivariate
Gaussian. Finally, this source distribution is corrupted at the receiver by Gaussian
noise with covariance $\Sigma$ independent from the source uncertainty. In general, the
range and range-rate noise are correlated as they are coupled at the matched filter,
while the bearing measurement noise is independent. The mutual information be-
tween the corrupted measurement and the noiseless measurement is then presented
in the form given by Equation (8.3) [172].

For our case of 2D tracking using range, bearing, and Doppler measurements, the
scaled inverse FIM is given by (assuming Gaussian time window and flat spectrum)
[48, 173]

$$\Sigma = \frac{1}{\text{ISNR}} \begin{bmatrix}
\frac{T_p c^2}{4} & -\frac{T_p B c^2}{2f_c} & 0 \\
-\frac{T_p B c^2}{2f_c} & \left(\frac{c}{T_p 4\pi f_c}\right)^2 + \left(\frac{B c}{f_c}\right)^2 & 0 \\
0 & 0 & \frac{6}{\pi^2 \cos^2(\Delta \theta) N_A(N_A-1)}
\end{bmatrix}, \quad (8.7)
$$

where $\Delta \theta$ is the difference between the predicted bearing and the true target bearing
and $N_A$ is the number of antenna array elements. Finally, the linearization matrix is
given by computing the Jacobian of the measurement matrix [8, 183]:

$$C = \begin{bmatrix}
\frac{2 \partial R}{c \partial x} & \frac{2 \partial R}{c \partial y} & \frac{2 \partial \dot{R}}{c \partial y} & \frac{2 \partial \dot{R}}{c \partial y} \\
\frac{2 \omega_c \partial R}{c \partial x} & \frac{2 \omega_c \partial R}{c \partial y} & \frac{2 \omega_c \partial \dot{R}}{c \partial y} & \frac{2 \omega_c \partial \dot{R}}{c \partial y} \\
\frac{\partial \Theta}{\partial x} & \frac{\partial \Theta}{\partial y} & \frac{\partial \dot{\Theta}}{\partial y} & \frac{\partial \dot{\Theta}}{\partial y}
\end{bmatrix}, \quad (8.8)
$$

where $R$, $\dot{R}$, and $\Theta$ are the measurement functions for the range, range-rate and
bearing respectively, and the partial derivatives are all to be evaluated at the predicted
state. The terms in this matrix are solved for in Appendix A. Note that because this
linearization occurs about the predicted state, $C$ implicitly depends on the target
revisit time $T$ as well, as this modifies the predicted state.
If the information changes for each target revisit, then $R_{\text{est}}$ fluctuates. Instead, we can focus on the mutual information, and attempt to keep it constant by modulating the revisit time $T$:

$$I'_{\text{const}} = \frac{1}{2} \log_2 \left[ \frac{|CP_{k|k-1}C^T + \Sigma|}{|\Sigma|} \right].$$  

(8.9)

This way, each time we visit the target, we stand to gain the same amount of information. It is important to note that $P_{k|k-1}$, $C$, and $\Sigma$ all depend on the revisit time for which we are solving: $C$ as previously discussed, $P_{k|k-1}$ through the motion model $A(T)$ and process noise covariance $Q_k(T)$ (both of which contain $T$ as a term), and $\Sigma$ through the predicted SNR.

In addition to the target process noise, another source of information can be accounted for: probability of detection. This can be done by assuming we have two channels, the detection channel and the empty channel. The detection channel is used by the target to communicate with the radar (unwillingly) with probability $P_D$, while the empty channel is used with probability $1 - P_D$. The detection channel communicates at a rate $R_{\text{est}}$ defined in Equation (8.3), while the empty channel has zero rate. It can be shown [172] that the general capacity is defined as

$$C = H(P_D) + P_D C_{\text{det}} + (1 - P_D) C_\emptyset,$$

(8.10)

where $H(P_D)$ is the entropy of a Bernoulli distribution parameterized by $P_D$ [172], $C_{\text{det}}$ is the capacity of the detection channel, and $C_\emptyset$ is the capacity of the empty channel. The proof is as follows. If we let $D$ be an indicator random variable where $D = 1$ if we detect the target and $D = 0$ if we do not, then the mutual information
is derived using basic information theory identities [172]

\[ I(x; y) = h(y) - h(y|x) \]  \hspace{1cm} (8.11)

\[ = h(y, D) - H(D|y) - h(y|x, D) - I(y; D|x) \]  \hspace{1cm} (8.12)

\[ = h(y, D) - h(y|x, D) \]  \hspace{1cm} (8.13)

\[ = H(D) + h(y|D) - h(y|x, D) \]  \hspace{1cm} (8.14)

\[ = H(D) + I(x; y|D) \]  \hspace{1cm} (8.15)

\[ = H(P_D) + P_D I(x; y|D = 1) + (1 - P_D) I(x; y|D = 0) \]  \hspace{1cm} (8.16)

\[ = H(P_D) + P_D C_{det} + (1 - P_D) C_\emptyset \]  \hspace{1cm} (8.17)

We can now define the more general estimation information as follows:

\[ I_{\text{const}} = P_D I'_{\text{const}} - P_D \log_2[P_D] - (1 - P_D) \log_2[1 - P_D] \]  \hspace{1cm} (8.18)

Making a hard decision about target detection is sub-optimal, and so the form given by Equation (8.18) can be thought of as a worst case bounding the potential performance. For example, track-before-detect (TBD) methods could be used with an augmented state space to include detection or with a discrete Markov chain layered on top of the normal tracking filter [200]. The extension of this work to include these distributions requires only solving for the mutual information of the more complicated densities.

The goal is now to select the target revisit time \( T \) such that the predicted information is given by the value calculated in Equation (8.18).

### 8.3.2 Target Predicted Information

We now parameterize the predicted information and solve for the estimated target revisit time to maintain our constant information, \( I_{\text{const}} \). This amounts to solving Equation (8.18) for \( T \). Solving for this in closed-form is very difficult, and may not be possible. The term \( T \) appears in \( A(T) \) and \( Q(T) \), both of which drive \( P_{k|k-1} \) in
Equation (8.9). It also appears indirectly in $C$, since the linearization is about the predicted state which depends on $T$ as well. Finally, the predicted range depends on $T$, which drives a predicted SNR given in Equation (8.5), and ultimately a prediction for $\Sigma$. Therefore, all three matrices in Equation (8.9) depend on $T$ in a highly nonlinear way. Further, the dual channel form including probability of detection in Equation (8.18) includes the term $P_D$, which also depends on $T$ through the predicted SNR.

To simplify solving for $T$, we evaluate the information for each entry in a table of $M$ revisit times $T$ to determine the best new revisit time among the table values, $T_{\text{next}}$. The steps for predicting the information and picking the nearest revisit time in the table are summarized in Algorithm 1, where $r_{k|k-1}$ is the predicted range (obtained from the predicted state $s_{k|k-1}$ which contains the predicted position $\{x_{k|k-1}, y_{k|k-1}\}$), $\text{ISNR}_{k|k-1}$ is the predicted integrated SNR, $r_0$ is the reference range where the integrated SNR is unity on a linear scale, $\Sigma_{k|k-1}$ is the predicted scaled inverse FIM, $R_{zz, k|k-1}$ is the predicted measurement covariance, and $P_{D, k|k-1}$ is the predicted probability of detection for a fixed probability of false alarm, $P_{FA}$. Note these are the standard Kalman formulae tailored to our radar tracking problem and augmented to include the predicted information.

First, the state is predicted using the linear motion model applied to the previous state estimate. After advancing the mean, the state covariance is also modified by this transform and added to the model covariance. We can predict the range measurement from the predicted state, which directly relates to predicting the SNR and thus the scaled inverse FIM. The state prediction covariance is advanced through the linearized measurement model and added to the scaled inverse FIM. We have parameterized the Jacobian matrix $C$ by the predicted state to emphasize the dependence of the linearization on the predicted state which varies for each choice of $T_i \in T$. Note
all these statistics are multivariate Gaussians with linear modifiers, so Gaussianity is
maintained. The probability of false alarm, a configurable system parameter, is then
used to predict the probability of detection. Finally, all of the predicted quantities
from the normal Kalman filtering steps can be plugged into Equation (8.18) to predict
the mutual information for this particular value of \( T_i \in T \). We perform the prediction
step \( M \) times to determine the revisit time \( T_{\text{next}} \) in the table \( T \) that yields a predicted
mutual information that is closest to \( I_{\text{const}} \). This can easily be extended to interpolate
between table values to provide a more accurate revisit period selection. We do
not motivate or derive the standard Kalman formulae, as it is readily accessible in
literature and extensively covered in prior work [183].

**Algorithm 1** Revisit Time Modulation (Solving for \( T \))

\[ T = \{ T_1, T_2, \ldots, T_M \} \quad \text{(Revisit Time Table)} \]

for \( i = 1 : M \) do

Pick \( T_i \in T \) (Hypothesis Revisit Time)

\[ s_{k|k-1} = A(T_i) s_{k-1} \quad \text{(State Prediction)} \]

\[ P_{k|k-1} = A(T_i) P_{k-1} A(T_i)^T + Q_k(T_i) \quad \text{(Source Covariance Prediction)} \]

\[ r_{k|k-1} = \sqrt{x_{k|k-1}^2 + y_{k|k-1}^2} \quad \text{(Range Prediction)} \]

\[ \text{ISNR}_{k|k-1} = \frac{r_0^4}{r_{k|k-1}^4} \quad \text{(SNR Prediction)} \]

\[ \Sigma_{k|k-1} = J^{-1} / \text{ISNR}_{k|k-1} \quad \text{(Scaled Inverse Fisher Information Prediction)} \]

\[ R_{zz,k|k-1} = C(s_{k|k-1}) P_{k|k-1} C(s_{k|k-1})^T + \Sigma_{k|k-1} \quad \text{(Measurement Covariance Prediction)} \]

\[ P_{D,k|k-1} = P_{FA}^{1/(1+\text{ISNR}_{k|k-1})} \quad \text{(Probability of Detection Prediction)} \]

\[ I_{T_i,k|k-1} = P_{D,k|k-1}^{1/2} \log_2 \left[ \frac{|R_{zz,k|k-1}|}{\Sigma_{k|k-1} P_{D,k|k-1}} \right] - (1 - P_{D,k|k-1}) \log_2 [1 - P_{D,k|k-1}] \quad \text{(Predicted Information)} \]

end for

\[ T_{\text{next}} = \arg \min_{T_i \in T} |I_{T_i,k|k-1} - I_{\text{const}}| \quad \text{(Calculated Revisit Time)} \]
The new predicted information step of the Kalman formulation used by the CIR falls in line naturally after the normal predicted quantities, as it depends on them. Any method of model-based filtering involving distribution propagation (for example, particle filtering for highly nonlinear applications [183]) can add the information prediction step to the corresponding algorithm and subsequently apply the CIR scheduling algorithm. The tracking estimation can also be augmented with transient features like automatic target recognition, where the information from the classification distribution would drive an initially higher sampling rate to reduce classification uncertainty, or wait to interrogate the target more rapidly when the predicted state indicated recognition would be more favorable.

The distributions used in this work are linear and Gaussian, and so closed forms for the predicted information exist and are mathematically tractable. More general tracking solutions can have complex distributions where the mutual information can be difficult to compute or estimate. In these cases, the CIR scheduling algorithm may still be used by applying reasonably tight bounds that can be formulated by modeling the filtering distributions as Gaussian mixture models (GMMs) [175] if the perturbative distributions are independent and additive as outlined in Chapter 9.

In addition to the normal Kalman recursion, the process noise power is also recursively computed after each track point. To start, the process noise is estimated by subtracting the predicted state from the current state estimate [201]:

\[ \hat{w}_k = s_k - s_k|k-1. \]  

Since both are unbiased statistics of the true state, the result is a zero-mean Gaussian with covariance given by the sum of the covariances of the prediction and the estimate [183]. This is effectively the Kalman weighted residual [183] or innovation. We then
compute the likelihood over a 2D table of hypothesis noise intensities for $q_x$ and $q_y$:

$$q_k = \arg \min_{\mathbf{q} = [q_x, q_y]} \ln[|\Omega(\mathbf{q})|] + \mathbf{\hat{w}}_k^T \Omega(\mathbf{q})^{-1} \mathbf{\hat{w}}_k,$$  
(8.20)

where

$$\Omega(\mathbf{q}) = 2\mathbf{P}(\mathbf{q}) - \mathbf{P}(\mathbf{q}) \mathbf{C}^T \mathbf{R}_{zz,k|k-1}^{-1} \mathbf{C} \mathbf{P}(\mathbf{q}),$$  
(8.21)

and

$$\mathbf{P}(\mathbf{q}) = \mathbf{A}(T) \mathbf{P}_{k-1} \mathbf{A}(T)^T + \begin{bmatrix} q_x \frac{T^3}{3} & q_x \frac{T^2}{2} & 0 & 0 \\ q_x \frac{T^2}{2} & q_x T & 0 & 0 \\ 0 & 0 & q_y \frac{T^3}{3} & q_y \frac{T^2}{2} \\ 0 & 0 & q_y \frac{T^2}{2} & q_y T \end{bmatrix},$$  
(8.22)

and $\mathbf{R}_{zz,k|k-1}$ denotes the measurement covariance prediction as defined in Algorithm 1. The joint value that minimizes Equation (8.20) is chosen as an estimate for the noise intensities at that time step and used for the next time step prediction. These values are thus used in Algorithm 1 to select the next target revisit time, as they modify the predicted information.

Note the authors expect the novel predicted information step to subsume the convergence properties of the standard Kalman filter state variables, but this extension is not explicitly covered here. This is due in part to the fact that the CIR is designed to modulate $T$ to force the track to remain sufficiently dynamical, and therefore convergence may not be possible for the time-varying statistics.

### 8.4 Revisit Time Modulation

Here we discuss our motivation for modulating the target revisit time. The revisit period is modulated to maintain a constant measure of information at each radar illumination of a given target. If we predict the mutual information to be larger than the previous measurement information, then there is more uncertainty predicted, and
Figure 8.4: Illustration of prediction compared to measurement with a low entropy, and ultimately low information target. Predictions at each track point are represented by the dashed outlines. Actual measurements are shown in the filled-gray planes. The last track point shows the prediction and model mismatch covariance or uncertainty contour. In this illustration, our constant velocity model is well matched to the benign target. As a result, the deviation from prediction at each measurement is small, and for a fixed revisit period as illustrated here, little information is learned through spectral use.

therefore more information to be gained from knowledge of the radar return. If we predict the mutual information will decrease from the previous measurement, then less information is predicted to be measured if we maintain a constant revisit time.

8.4.1 Model Mismatch

We start by looking at modulation with respect to model mismatch. As discussed in the previous sections, the radar targets have a physical motion model. When targets adhere to this model, then the prediction step is more accurate, and the true measurement offset from this prediction can be small. This is illustrated in Figure 8.4. The dashed outlines represent the target state prediction, while the gray planes are the actual measurements. In this work, we assume a constant velocity model, and since the plane is not accelerating appreciably, the prediction is quite accurate. As a result, the information gained through measurement may be small for this fixed revisit time.

In Figure 8.5, we illustrate a highly dynamic target. Instead of maintaining a constant velocity, as in our model hypothesis, the plane is maneuvering in a serpentine fashion. As a result, the difference between the prediction and measurement is larger at each track point. This increased model error uncertainty is represented in the final point shown by a larger prediction covariance contour. In this scenario, for
Figure 8.5: Illustration of prediction compared to measurement with a high entropy, and ultimately high information target. Predictions at each track point are represented by the dashed outlines. Actual measurements are shown in the filled-gray planes. The last track point shows the prediction and model mismatch covariance or uncertainty contour. In this illustration, our constant velocity model is poorly matched to the dynamic target executing a serpentine maneuver. As a result, the deviation from prediction at each measurement is larger, and for a fixed revisit period as illustrated here, significant information is learned through spectral use.

A fixed target revisit time, more information is gained through measurement. It should be clear that the true information is in the measurement residuals, lending credence to the name “innovations.” A larger residual results in more information. A more advanced and accurate model generates a better estimator, but measurement is required less often to learn the same amount of information.

To quantify model mismatch, the model noise powers $q_{x,k}$ and $q_{y,k}$ used in Equation (8.2) are recursively computed as given in Equation (8.20). As the Kalman fusion process is completed, significant model mismatch produces a higher process noise estimate. As a result, the process noise that predicts the covariance, and thus information, for the next time step is increased. Since the target is exhibiting larger variation from the model, more information stands to be gained through measurement, and the revisit time should be decreased to maintain constant information.

Note that the model covariance is modified by the measurement transform as shown in Equation (8.3). Therefore, uncertainty in the state distribution may be removed or obfuscated after the measurement transforms the state. For example, only the radial projection of the target velocity vector is measured when exploiting the Doppler shift of the returned waveform. Uncertainty of the target velocity
orthogonal to the radius vector of the target is not captured in the final mutual information. Therefore, uncertainty in the measurement domain with respect to the noisy measurement is the true source of information that can be gained.

8.4.2 Signal-to-Noise Ratio

Examining Equation (8.3), the other key factor affecting the modulation of the revisit period of a particular target is the SNR. Since the example shown here is a special case, this can be generalized to include signal-to-interference ratio, or signal-to-clutter ratio. Any degrading distribution captured in the mutual information relationship can be used in calculating the estimation rate. Here, we are assuming the thermal noise in our system is fairly consistent, so only the signal affects the SNR. If the signal level is degraded (for example, due to increased range), then the estimation rate decreases.

The estimation rate of the target may be interpreted as the minimum number of bits/second needed to encode or compress the target information. If we attempt to encode the noisy target range and range-rate from consecutive draws, noise could dominate the target dynamics. This reduces the number of required bits needed to encode the residual, since additional bits would only serve to encode unwanted noise. Subsequently, the revisit time is increased to compensate. This relationship to SNR can be thought of as an estimation analogy to joint typicality encountered in the channel capacity problem [172].

Probability of detection also affects the revisit time. For a fixed false alarm rate radar, probability of detection is dependent only on SNR. The closer $P_D$ is to 0.5, the more information is contained in the Bernoulli distribution. However, a far more powerful relationship is the multiplication of the radar estimation rate by $P_D$ shown in Equation (8.18). Therefore, probability of detection affects estimation rate very
Figure 8.6: Illustration of measurement variance in contrast to mean residual. The prediction is indicated by the blue-filled plane with the dashed outline. The swarm of gray planes with solid outline represents multiple measurements at the same track point. Each time a measurement is made over the ensemble, it is perturbed uniquely by either clutter, noise, or interference. In the high SNR case on the left, the perturbations are small and the grouping is tight. As a result, the mean offset from the prediction is significant and appreciable. For the low SNR case to the right, the offset of the mean to the prediction is insignificant compared to the variance of the measurement indicated by the degree of spread in these samples. As a result, low information gain is possible compared to the high SNR case.

These concepts are illustrated in Figures 8.6 and 8.7. The first figure shows an example of high SNR compared to low SNR. The prediction is shown in blue with dashed outline, while the gray planes with solid outline represent a sampling of measurements. This assumes we can essentially freeze the state of the track, and measure the target multiple times. Each measurement is perturbed slightly around the true mean due to noise, clutter, or interference. In the high SNR case, we can easily see the mean of the measurements offset from the prediction, and the residual contains meaningful information. For the low SNR portion of Figure 8.6, we see the residual from the prediction in blue to the mean of the samples is insignificant compared to the spread of the samples. In this case, the information is low because the noise entropy is dominating the source entropy.

In Figure 8.7, we illustrate an example of how to mitigate the low SNR scenario.
Figure 8.7: Motivation of delayed revisit time in response to poor SNR. The prediction is indicated by the blue-filled plane with the dashed outline. The swarm of gray planes with solid outline represents multiple measurements at the same track point. Each time a measurement is made over the ensemble, it is perturbed uniquely by either clutter, noise or interference. For low SNR, the offset of the mean to the prediction is insignificant compared to the variance of the measurement indicated by the degree of spread in these samples shown on the left. By increasing the revisit time, the model uncertainty grows larger, and the offset from the prediction can grow to become significant relative to the variance of the measurement. As a result, significant information is recovered, and spectral efficiency is increased.

by modulating the revisit time. When the residual is insignificant relative to measurement variance, we can delay revisiting the target for a while. At this point, our uncertainty grows, and we are less confident in our prediction since we have not measured the target in a long time to correct the target track. Measuring at this later time means we are more likely to have a measurement mean deviating more significantly from our prediction relative to the measurement spread.

8.5 Examples

We tested the newly derived CIR against multiple track scenarios. The main parameters used are given in Table 8.1. The CRLB was formulated assuming a linear frequency-modulated (FM) waveform with Gaussian window [173], which depends on the root mean square (RMS) envelope and RMS bandwidth, as well as the chirp rate.
\( P_{FA} \) is the set false alarm probability, and \( f_c \) is the carrier frequency. All predicted SNRs and resulting measurements assume no angle error. For a given probability of detection, a Bernoulli random variable is drawn to determine if the target is observed at any given time step. In addition, targets falling outside of the 3 dB beamwidth of our array are excluded from processing (using a shaping factor of 0.89). For the process noise, only the last innovation is used to quickly respond to model mismatches. A total of 6 values for the process noise intensity logarithmically spaced from \( 10^2 \) to \( 10^7 \) were available in the lookup table, independent for \( q_x \) and \( q_y \). The tracking revisit time table is constructed from 100 evenly spaced values from \( T = 500 \) µs to 5 s. The minimum revisit time is assuming a 10% duty cycle PRI, and is the time required to send the full 10 pulses for a given CPI. The number of bits in the fixed information, \( I_{\text{const}} \), was set to 10 for the first two scenarios and increased to 15 for the final scenario to show how the tracker behaves and more clearly observe trends in revisit time modulation. In all three examples, a traditional radar employing a fixed revisit time is simulated to compare with the CIR. The traditional radar’s fixed revisit time was determined by running multiple trials and selecting the revisit period that resulted in nearly the same tracking mean square error (MSE) as the CIR. The average spectrum availability for both radars is computed by averaging the revisit times minus the CPI for each track point. The CIR ultimately allows for more free spectrum time compared to the traditional radar, allowing more access for cognitive communications users, or enabling the radar to track more targets. These results are summarized in Table 8.2.

### 8.5.1 Looping Track (Model Mismatch Modulation)

The first track is shown in Figure 8.8. Note the plot in Figure 8.8 is not intended to strictly convey radar tracking performance, but rather to demonstrate the variable
Table 8.1: CIR example simulation parameters, varied $I_{\text{const}}$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth</td>
<td>5 MHz</td>
<td>Absolute Temperature</td>
<td>1000 K</td>
</tr>
<tr>
<td>Pulse Duration</td>
<td>5 $\mu$s</td>
<td>Probability of False Alarm</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>Radar Antenna Gain</td>
<td>30 dBi</td>
<td>Radar Transmit Power</td>
<td>100 kW</td>
</tr>
<tr>
<td>Target Cross Section</td>
<td>10 $m^2$</td>
<td>Chirp Rate</td>
<td>$B/T_p$</td>
</tr>
<tr>
<td>Window Variance</td>
<td>$T_p$</td>
<td>Wave Speed</td>
<td>$3 \cdot 10^8$ m/s</td>
</tr>
<tr>
<td>Number of Pulses</td>
<td>10</td>
<td>Number of Array Elements</td>
<td>10</td>
</tr>
</tbody>
</table>

Figure 8.8: Example of looping track with realization overlay. The green circles indicates the true target position at each time step (indicated by the markers). The blue crosses indicates the CIR’s estimated target position. In the straight portions of the track, the target is not accelerating and is adhering to the predicted model well. During the curved parts of the track, the target is necessarily accelerating and as a result the CIR revisits the target more frequently over an equivalent time period.
Figure 8.9: Spectral efficiency at each track point given in bits/s/kHz for looping track. On the left, in blue, is the CIR, which modulates the radar target revisit period in response to information. The goal was to maintain 10 bits of information, or 4 bits/s/kHz with this radar configuration. Initial uncertainty during track acquisition resulted in a larger spike. Subsequently, the revisit time is modulated to maintain 10 bits. On the right, in red, is a traditional radar with a fixed revisit time. Spectral efficiency fluctuates in response to changing target dynamics.

revisit time in response to target acceleration. The true track at each observation is shown by the green circle, and the CIR’s estimated track position is shown by the blue crosses. The track was designed to have portions well modeled by the linear, non-accelerative motion model, and portions that deviate from this that are captured statistically by the white Gaussian perturbative accelerative noise [182]. In Figure 8.9, we show the information measure for each track point. The blue plot on the left is the CIR, while the red plot on the right is a traditional radar with fixed revisit time set at 100 ms. For the CIR, the goal is to maintain a constant 10 bits for each target revisit, which provides the minimum number of bits to encode the target position [172]. Taking into account the radar pulse duration, number of pulses, radar duty cycle, and radar waveform bandwidth, this corresponds to a spectral efficiency of 4
bits/s/kHz. Since significant redundancy exists from the inclusion of a motion model, only deviations from the model contain information that would need to be encoded. The initial spike larger than 4 bits/s/kHz is due to track acquisition, where the radar detected the target for the first time, and obtained a raw measurement with no prior.

In Figure 8.10, we see the time to revisit increase during portions of the track where the target is well modeled. When the target accelerates to turn, the CIR estimates a higher process noise power, and so more information is predicted to be gained through measurement. As a result, the time between target revisits is decreased. To compare, the same track was simulated using a fixed revisit period of 100 ms. This achieved the same tracking MSE, but with a spectral free time of 99.500% compared to the CIR which achieved 99.870%. Given the CPI of the radar, this means the CIR can track nearly 800 targets with similar dynamics compared to the fixed radar’s 200 targets. The spectral efficiency of the fixed radar shown in Figure 8.9 show how
Figure 8.11: Example of an approaching radial track with realization overlay; radar position represented as an orange block. The green circles indicates the true target position at each time step, while the blue crosses indicates the CIR estimated target position. Since the track is linear and the target is not accelerating along this line, the constant velocity model is a good match to the target dynamics. However, the radius from the radar is decreasing linearly with time, and so the SNR is increasing with time. As a result, the track is sampled more frequently by the CIR as time progresses.

spectral efficiency fluctuates with changing target dynamics.

Note that the range was chosen to be approximately the same throughout this track so that SNR did not contribute to the modulation of $T$ significantly. We explore the effect of SNR next by fixing the process noise.

8.5.2 Approaching Radial Track (SNR Modulation)

To test a constant process noise but variable SNR, we simulated an approaching radial track with virtually no acceleration that travels from very far from the radar to very close as seen in Figure 8.11. The target revisit time between each track point is shown in Figure 8.13. When the target is far from the radar, the process noise is swamped by measurement noise. As the target moves toward the radar, much more
Figure 8.12: Spectral efficiency at each track point given in bits/s/kHz for the approaching radial track with increasing SNR. On the left, in blue, is the CIR, which modulates the radar target revisit period in response to information. The CIR was set to maintain 10 bits, or 4 bits/s/kHz for this scenario. Ripples are from quantization due to the finite number of available revisit times in the lookup table described in Algorithm 1 derived in Section 8.3.2. On the right, in red, is a traditional radar with a fixed revisit time. Spectral efficiency increases steadily in response to increasing SNR as the target approaches the radar location.

Information stands to be gained from the high SNR. As a result, the actual target information is increased over the same interval, and the revisit time decreases. This curve is similar to one displayed in Reference [147], where compressive sensing based on range is explored.

The information plot is shown in Figure 8.12, where a constant 10 bits was set to be maintained on average (corresponding to a spectral efficiency of 4 bits/s/kHz). Once again, the CIR is on the left in blue, with the fixed radar on the right in red. Note that the quantization of the revisit time lookup table is reflected by the quantization in Figure 8.13, and results in a sawtooth modulation of the information in Figure 8.12 for the CIR. To compare, the same track was simulated using a fixed revisit period of 50 ms. This achieved the same tracking MSE, but with a spectral
**Figure 8.13:** Target revisit time at each track point given in seconds for the approaching radial track with increasing SNR. Since the SNR is increasing as the radial distance decreases, the information content is increasing for a fixed revisit time. As a result, the revisit time is decreased as the track progresses to keep the information constant.

free time of 99.000% compared to the CIR which achieved 99.834%. Given the CPI of the radar, this means the CIR can track around 600 targets with similar dynamics compared to the fixed radar’s 100 targets. As the SNR increases, for a fixed revisit time, the spectral efficiency slowly climbs as shown.

Next we look at a track that incorporates both global trends (variation in SNR), and local fluctuations due to process noise variance changes.

### 8.5.3 Evasive Track (Global and Local Trending)

Evasive targets adapt automatically within the entropy framework, as the information content grows with target track uncertainty. A track of an evasive target is shown in Figure 8.14. In this case, the fixed information value was set at 15 bits to highlight detail in revisit time modulation.

The target revisit time is shown in Figure 8.16. During the initial period where the
Figure 8.14: Example of evasive track with realization overlay; radar position is indicated by the orange block. The green circles once again indicate the true target position at each time step, and the blue crosses show the CIR’s estimated target position. Initially, the target executes an evasive serpentine maneuver, and the CIR reacts appropriately by revisiting the target more frequently. As the target moves significantly far away from the radar, the decaying SNR forces the CIR to sample less often.

When the target is closer to the radar and attempting to break lock, the radar samples the target much more frequently. The average revisit time is small because of the proximity to the radar, while the small dips in revisit time are during the accelerative turns of the winding behavior. As the target enters linear portions of the track, the radar samples less frequently. Finally, as the target circles around and gets closer to the radar again, the illumination rate is once again starting to increase given the increase in resolution from the growing SNR. The information plot is given by Figure 8.15, where now we maintain a constant 15 bits of information, or 6 bits/s/kHz. To compare, the same track was simulated using a fixed revisit period of 200 ms. This achieved the same tracking MSE, but with a spectral free time of 99.750% compared to the CIR which achieved 99.940%. Given the CPI of the radar, this means the CIR can track almost 1700 targets with similar dynamics compared to the fixed radar’s 400 targets.
Figure 8.15: Spectral efficiency at each track point given in bits/s/kHz for an evasive track. On the left, in blue, is the CIR, which modulates the radar target revisit period in response to information. The CIR was set to maintain 15 bits, or 6 bits/s/kHz. On the right, in red, is a traditional radar with a fixed revisit time. Spectral efficiency fluctuates in response to changing target dynamics, as well as SNR varying slowly in response to target location relative to the radar.

In Figure 8.16, the envelope represents the change in revisit time due to SNR, which is a function of radial distance to the target. Fast variations are due to the process noise variance calculation which responds instantaneously to deviations in the residual. If averaging were used on the estimation of the process noise to indicate model mismatch, this variation would be less rapid with respect to track point. The varying spectral efficiency for the fixed radar shown in red in Figure 8.15 also show the local and global variation in response to the two target regimes. The results on availability for a traditional radar compared to the CIR for all three track examples are shown in Table 8.2.
Figure 8.16: Target revisit time at each track point given in seconds for an evasive track. During the initial maneuver, the target is in nearly constant acceleration and so the revisit time is very short on average. The target then slowly eases into a linear constant velocity straight-away and the CIR revisits the target less frequently. As the target rounds a turn, there are periods where the target is sampled more often before it is again well modeled. There is a clear global trend overlaid on the local trend. The global envelope of the revisit time plot is due to SNR, and fluctuates with radial distance. Locally, the dips and peaks comprising this envelope are due to periods of acceleration or turning since the process noise variance is calculated every track point.

8.6 Summary

We derived the CIR, an information driven radar that employs a novel algorithm to limit radar spectrum use for dynamic spectrum access. Modulation of the target revisit time as a function of measurement information was discussed to explain the radar behavior. A constant SNR example was given where portions of the track conformed to the motion model, conveying little information. In this same example, sharp turns prompted the CIR to revisit the target more quickly, as target uncertainty increased relative to the tracking model. A radial track with increasing SNR was also presented to illustrate modulation as a function of noise degradation with fixed
model mismatch. With SNR increasing as the target traveled closer to the radar, more information from the target measurement was available for a fixed revisit time, and so the target was revisited more rapidly with the CIR. Finally, a hybrid track with global SNR changes and local process noise variations from accelerative turns showed how the two key dependencies of the CIR scheduling algorithm influence revisit time. This dynamic time allocation allows for dynamic cooperative sharing with a growing contingent of communications users operating in legacy radar bands. In all examples, the CIR out-performed a traditional radar with a fixed revisit time (chosen such that the average tracking MSE achieved was nearly the same as that achieved by the CIR). The CIR’s increased availability time means more spectrum access time is possible for in-band cognitive communications users, or more targets can be tracked by the CIR. The CIR concept subsumes the vast model-based tracking phenomenology, so it can be easily generalized to include multiple targets, alternative sensing modalities, complicated clutter and interference distributions, and general target tracking distributions. The result is a mathematically-controlled radar with a target scheduling scheme that fixes the radar spectral efficiency for a particular target, ensuring that time-bandwidth is used only when truly needed.
Using radar estimation rate, as demonstrated in the previous chapters, requires compu-
tation of the target tracking mutual information. However, even for a simple esti-
mation problem, the source entropy can be difficult or impossible to derive analytically
[185], let alone the more complicated mutual information between the noiseless and
noisy measurement. Beyond radar estimation rate, there is broad utility in estimation
information or estimation mutual information. For example, computing estimation
information is desired to inform channel capacity limits [135, 172] and radar waveform
design information metrics [10, 136, 137, 179].

While other methods exist to estimate or bound mutual information for esti-
mation problems, many shortfalls exist that preclude widespread or universal use.
The method presented here requires modeling the estimation problem as a Gaussian
mixture, and reasonably tight (under typical conditions) upper and lower bounds im-
mEDIATELY follow in a relatively simple form [175]. Deriving these bounds using the
I-MMSE formula provide additional insight into bounding assumptions on mutual in-
formation and minimum mean square error (MMSE) for mixture distributions. The
overall concept is depicted in Figure 9.1.

9.1 Challenges with Estimation Information

Computing, estimating, or deriving mutual information analytically is challenging
despite a growing desire to do so in signal processing applications. For example,
estimation information is growing in popularity as a metric of optimization for radar
waveform design for statistical target cross sections [133, 137], cognitive networks
Figure 9.1: Block diagram depicting I-MMSE bridge for cross-domain bound results. The arbitrarily distributed source vector, $x$, is first modeled by a Gaussian mixture model. The measurement, $y$, contains the source scaled by $\sqrt{\Gamma}$, which can represent an SNR term, an arbitrary linear transform $G$, and whitening matrix $W$, added to arbitrary noise, $n$, also modeled by a GMM. When bounds on the minimum mean square error are found, using the I-MMSE formula, bounds on the estimation information are solved via integration. Results extended in the information domain can be bridged back to give insight and results in the MMSE domain.

[134, 148], and joint maximization of communications data rate and radar estimation rate [10]. Though mutual information has been applied to signal processing for nearly as long as it has been in existence [135, 136, 179], a ubiquitous and widely adopted means of computing or estimating signal processing information has yet to emerge. The closed-form integrals are surprisingly difficult to derive, even for simple estimation problems. This can be seen by looking at the very basic mutual information equality for estimating a random variable $x$ through observation $y = x + n$ where $n$ is an additive noise term independent of $x$. This simple mutual information can be written as [172]:

$$I(x; x + n) = h(x + n) - h(n),$$

(9.1)

where $I(x; y)$ is notation for the mutual information between random variables $x$ and $y$, and $h(x)$ is notation for the differential entropy of a continuous random variable $x$. The noise entropy $h(n)$ is easy to compute if the Gaussian assumption is made.
However, entropy alone has a limited set of closed-form solutions for popular distributions [185]. Even if the distribution of $x$ is fairly standard, and $n$ is Gaussian, $h(x + n)$ rarely exists in closed-form [175]. In the probability distribution domain, the new distribution of $x + n$ is obtained via the convolution of $f_x(x)$, the probability density function of $x$, and $f_n(n)$, the noise probability density function [199]. Often times, this distribution lacks a name, let alone a closed-form entropy.

In lieu of analytical derivation, there are many other methods of estimating mutual information such as the $k$-nearest neighbors ($k$-NN) method [202], Edgeworth method [203], local Gaussian method [204], maximum likelihood density ratio method [205], and numerous histogram-based methods [206]. Undersmoothed kernel estimators [207] and other sample-impoverished methods [208] are suitable when only data is available, but not enough to estimate a proper histogram. Histogram methods for estimating mutual information are useful when a large amount of data is already available from the distribution. If not, the data may need to be simulated using Monte Carlo methods before computing the histogram [183]. However, a large sampling of data may not be available for all estimation problems. In addition, even with large sample sizes, significant biases in the mutual information estimation can occur [209]. The $k$-NN method avoids estimating the density at all, and uses the data to estimate the mutual information directly [202]. However, not all applications have access to a sufficient data set as in the case of the histogram estimator. In addition, the $k$-NN method is very sensitive to the parameter choice of $k$ while lacking reliable methods to estimate this parameter [206]. While Edgeworth methods have been applied to estimate the entropy of Gaussian mixtures [203], the approximation requires sufficient separability of the mixture components, limiting its modeling capability. In addition, non-Gaussian-like distributions can suffer large biases using these methods [206]. The maximum likelihood density ratio method also avoids estimating the density, and
directly models the density ratio employing convex optimization to approximate the maximum likelihood estimate of this ratio [205]. This is also a data-driven method, requiring samples from the distribution. The local Gaussian approximation to mutual information estimation attempts to improve the $k$-NN method by making the local clustering region a Gaussian [204]. However, results show this method tends perform poorly for circular source distributions [204].

The I-MMSE formula provides an integral bridge between the MMSE and mutual information for estimation problems [144]. In fact, in this body of work [69, 144], it was hypothesized that the I-MMSE may allow for computation of the mutual information for an estimation problem, specifically when the equivalent integration of the MMSE was more easily computed or bounded. Some bounds exploiting these relationships have also been derived [210]. Of course these methods require calculation or estimation of the same problem’s MMSE, something that may be equally challenging or mathematically intractable. In Reference [211], this specific challenge was addressed by modeling the estimation problem using Gaussian mixture models (GMMs) and then bounding the MMSE using side knowledge assumptions or assumptions on the estimation problem. Taking these bounds in the MMSE domain, we can bridge these results to the mutual information domain to form a lower and upper bound on the estimation information of arbitrary source distributions in additive Gaussian noise, where the noise is independent of the source [175].

9.2 The Gaussian Mixture Model

The problem setup is reviewed here to lay the technical background before extending the bounds and their analysis. We start with our estimation model employing GMMs.
9.2.1 Estimation Model

A typical signal processing estimation problem is formulated as our base model. A random vector $\mathbf{x}$ is the desired source of information (the parameter or parameter set we wish to estimate). Naturally, we cannot observe $\mathbf{x}$ directly, but often can model our observation $\mathbf{y}$ as our desired signal corrupted by an additive noise vector $\mathbf{n}$. Note $\mathbf{x}$ and $\mathbf{n}$ are independent with respect to one another. In general, both vectors $\mathbf{x}$ and $\mathbf{n}$ may have correlated components. Mathematically, our model is given by

$$
\mathbf{y} = \sqrt{\Gamma} \mathbf{G} \mathbf{x} + \mathbf{n},
$$

(9.2)

where $\mathbf{G}$ is an arbitrary modifying linear transform, and $\Gamma$ is a positive scalar which can be interpreted as an signal-to-noise ratio (SNR) term. While this may not represent SNR depending on the noise distribution and the system context, we refer to it as SNR in this work. The desired source $\mathbf{x}$ can be arbitrarily distributed, making a general solution to this problem difficult. To facilitate the analysis, we model the source distribution using a GMM:

$$
\mathbf{f}_x(\mathbf{x}) = \sum_i p_i \mathcal{N}(\mu_i, \Sigma_i),
$$

(9.3)

for $\mathcal{N}(\mu, \Sigma)$ the multivariate Gaussian density function with mean vector $\mu$ and covariance matrix $\Sigma$. The distribution is constructed by drawing from the discrete distribution $p$, and then drawing from the mixture component selected, $\mathcal{N}(\mu_i, \Sigma_i)$. Similarly, the noise is modeled using a GMM:

$$
\mathbf{f}_n(\mathbf{n}) = \sum_k q_k \mathcal{N}(\nu_i, \Omega_i).
$$

(9.4)

Often times, the mutual information between the noiseless source $\mathbf{x}$ and the corrupted observation $\mathbf{y} = \mathbf{x} + \mathbf{n}$ is a desired quantity in signal processing applications:

$$
I(\mathbf{x}; \mathbf{x} + \mathbf{n}) = h(\mathbf{x} + \mathbf{n}) - h(\mathbf{n}),
$$

(9.5)
where we have exploited the signal and noise independence. This tells you how much information is still contained in the observation after distortion from the additive noise. To exploit the I-MMSE formula as we do in this work, we can whiten the noise through a linear transform:

$$y = \sqrt{\Gamma} H x + W n,$$

(9.6)

for $H = W G$, and whitening matrix $W$. After whitening, the noise vector covariance is the identity matrix of appropriate dimensionality, but the mutual information does not change [144].

It is convenient for analysis to rewrite $y$ in terms of additional discrete random variables $u_x, u_n$ which specify the signal and noise mixture components, respectively:

$$y = \sqrt{\Gamma} H x_{u_x} + W n_{u_n},$$

(9.7)

where $u_x \sim p$ so that $P(u_x = i) = p_i$ and $u_n \sim q$ so that $P(u_n = k) = q_k$, and $x_k \sim N(\mu_i, \Sigma_i)$, and $n_k \sim N(\nu_k, \Omega_k)$.

Gaussian mixtures permit modeling of general distributions to an arbitrary level of precision (in a distribution sense) given a high enough order. A formal proof of this remark is given in the appendix of Reference [212]. Another benefit to bounding GMMs is that the discrete mixture distribution enables relaxing assumptions such as genie knowledge of the drawn distribution, which lends itself to bounding quantities involving mixtures.

In Figure 9.2, we see an ideal Weibull distribution, a common statistic used in cluttered radar amplitude returns. We can see based on the shape presented that the density is not very Gaussian. It is asymmetric and has only positive support. Figure 9.3 shows the individual Gaussian components obtained using differential evolution [190] to curve fit the GMM to this density. Each of these components has a mean, variance, and mixture ratio (the weight or discrete probability) solved for
Figure 9.2: Ideal Weibull distribution, a common radar clutter return statistic. The distribution is not Gaussian, notably due to the asymmetry and positive-only support.

Figure 9.3: Approximated Weibull mixture components. Each component shown in gray is a Gaussian density with a mean, variance, and weighting factor determined by the model fitting process. Summing these weighted components approximates the Weibull distribution.
Figure 9.4: Sum of approximated Weibull mixture components overlaid onto ideal Weibull distribution. The ideal distribution is given by the dashed purple line, while the approximated Gaussian mixture sum is shown in the orange overlay. There is modeling error present at the peak, and the Gaussian components that comprise the sum have infinite support that contribute to a small negative support in the sum. During the optimization process. Finally, in Figure 9.4 we see the sum of these components in orange overlaid onto the true distribution in dashed purple. A few observations are immediately apparent. First, there is some modeling error which can be seen at the peak. In addition, the mixture sum has a negative support, though it is small. To remedy this, we can increase the number of Gaussian components in our model, but this can be a detriment to the bounds (discussed later) and burden the curve fitting process prohibitively.

9.2.2 Minimum Mean Square Error (MMSE)

The MMSE for this estimation model is the minimum error variance achievable for our constrained problem, a widely used figure of merit in signal processing [47]. This metric can be difficult to compute in general. Recent results in Reference [211]
have upper and lower bounded the MMSE for estimation problems formulated as in Equation (9.2). In their most general form, they are given by [211]

\[
\sum_{i,k} p_i q_k \text{Tr}\left( (G \Sigma_i G^T \Omega_k^{-1} + \Gamma I)^{-1} \right) \leq \text{mmse}(H x, \Gamma) \leq \text{Tr}\left( (G \Sigma G^T \Omega^{-1} + \Gamma I)^{-1} \right)
\]

(9.8)

where $\Sigma$ is the covariance matrix of the source mixture distribution (similarly for $\Omega$ for the noise), $\text{Tr}(\cdot)$ the trace function, and $\text{mmse}(H x, \Gamma)$ is the MMSE for arbitrary source $H x$ given observation of $\sqrt{T} H x$ corrupted by unit variance independent additive noise. The MMSE is defined as [144]

\[
E[||x - E[x|y]|^2],
\]

(9.9)

where $E[\cdot]$ is the expectation operator.

If the distribution is well modeled, $\Sigma$ is the covariance matrix of $x$. We assume this is true, as it only improves our bound accuracy under modeling error. The upper bound is derived by assuming the linear MMSE estimator is used [211]. On the other hand, the lower bound is derived under the assumption that the estimator has knowledge of which mixture component is being drawn from at any point in time [211]. This is called the “genie bound,” as it purports some omnipotent or magical knowledge about the problem not known in practice. Note one subtly here that must be addressed. The MMSE bounded in Reference [211] is the MMSE for $x$ for observation $\sqrt{T} H x + n$, while here we assume the MMSE for $\sqrt{T} H x$ for the same observation to be consistent with the assumptions in I-MMSE literature.

Note that mixture models are convenient because they can be used in numerous ways. For example, the estimation mixture distribution can be modeled in real time during data processing, such as in classification machine learning problems [213]. From live data, mixture models can be solved for using the expectation-maximization algorithm [214]. Alternatively, they may be used in offline scenarios where the prob-
ability density function is known, but the mutual information does not have a closed-form integral. In this case, GMMs are used for curve fitting, and access to large sets of data is not necessary (even if possible). There are other methods of density mixture modeling that may be used, however, GMMs are proven to capture all densities with arbitrary precision (see appendix of Reference [212]) and are well known in statistics, in addition to naturally existing in some processing frameworks like the aforementioned classification problem. Specifically, recent work in nonlinear classification for compressive sensing applications has exploited bounds between mutual information and mean square error (MSE) [210] to optimize classification accuracy and has tested results on GMMs [215].

9.3 I-MMSE Formula

In this section, we review the I-MMSE formula, a connection between estimation information and the MMSE for arbitrary sources in Gaussian noise. This gives us a connection to take bounds on the MMSE and create bounds on the mutual information.

9.3.1 Univariate Source, Gaussian Noise

The authors in Reference [144] show a fascinating connection between information and estimation measures. For the simplest case, a univariate random variable $x$ perturbed by an independent, additive Gaussian random variable $n$ with unit variance, the following relationship holds [144]:

$$I(x; \sqrt{\Gamma} \frac{x}{\omega} + n) = \frac{1}{2} \int_0^{\Gamma} \text{mmse}(\frac{x}{\omega}, \gamma) \, d\gamma,$$

(9.10)

where $\omega$ is the noise standard deviation, and the equivalent one-dimensional “whitening” normalization is shown. It should be noted that the assumption that $\Gamma > 0$ is
Figure 9.5: Mutual information as integration of MMSE interpretation of the I-MMSE formula. The MMSE curve versus SNR is shown in the solid orange line. As expected, the error decreases as SNR increases. The area under the MMSE curve is highlighted in light blue for an example SNR demarcated by the dashed purple line. The area shown is equivalent to twice the mutual information at this estimation SNR, as indicated by the I-MMSE formula.

made. The function \( \text{mmse}(\cdot) \) computes the achievable minimum mean square error of the estimate based on deriving the MMSE estimator. A visualization of this is given in Figure 9.5. In this example, a Gaussian source is embedded in Gaussian noise. It can be easily seen that one interpretation of this relationship is the diminishing return of information with increasing SNR, since error vanishes asymptotically. There is an alternative form that offers further insight [144]:

\[
\frac{d}{d\Gamma} I(x; \sqrt{\Gamma} \frac{x}{\omega} + n) = \frac{1}{2} \text{mmse}(\frac{x}{\omega}, \Gamma).
\]

This is truly remarkable, as it relates the rate of change of information as a function of SNR to the MMSE at a point SNR. This interpretation is shown in Figure 9.6. We exploit this connection to solve for bounds on mutual informations that otherwise have no closed-form or are difficult to estimate. That is, we can find estimation scenarios where the mutual information calculation is difficult, but where bounds on
Figure 9.6: MMSE as slope of mutual information interpretation of the I-MMSE formula. In this case, the mutual estimation information is shown with respect to SNR in the light blue line. At an example SNR, demarcated by the dashed purple line, the derivative slope is shown in the orange tangent line. This slope is the MMSE of this estimation problem at this SNR to within a scale factor.

the MMSE are readily calculable, and solve for bounds on the mutual information accordingly from the above relation.

9.3.2 Multivariate Source, Gaussian Noise

The previous result for the scalar can be generalized to a real Gaussian vector [144]:

\[
I(x; \sqrt{\Gamma} H x + n) = \frac{1}{2} \int_0^\Gamma \text{mmse}(H x, \gamma) \, d\gamma,
\]

where the source \( x \) is now a vector, \( H \) is an arbitrary deterministic matrix, and \( n \) is now a vector of independent unit variance Gaussian random variables after the whitening operation.

Note that while this assumes uncorrelated additive noise, the authors note in Reference [216] that whitening the noisy measurement can allow one to make use
of the preceding formula. Since the resulting transform is linear, it is distributive, and results in another Gaussian noise source that is now decorrelated. We can thus subsume the whitening transform into the matrix $H$ as needed. Now that we have a mixture model distribution with bounded MMSE, and a connection between the MMSE and the mutual information, we need only to integrate to solve for bounds on the mutual information.

### 9.3.3 Vector General Additive Noise

The I-MMSE is limited for estimation in general additive noise. In Reference [144], only the scalar version of the general additive noise was provided. In Reference [217], a vector channel model was presented, but only for identically distributed noise. Due to the lack of a simple form, like the integral for Gaussian noise, we handle general noise in the information domain by making similar assumptions on genie knowledge and entropy maximization.

### 9.4 Information Bounds

With the background in Section 9.2, we now use the I-MMSE formula to bring MMSE bounds for GMMs to the information domain to bound the mutual information defined by Equation (9.5).

#### 9.4.1 Bounds in Gaussian Noise

To start, we review the principle contribution from Reference [175]. For an estimation problem as described in Section 9.2.1 where signal $x$ is a vector Gaussian mixture as in Equation (9.3) and noise $n$ is a Gaussian vector with covariance matrix $\Omega$, then the following bounds on mutual information between signal and observation
\[ \sum_i p_i \frac{1}{2} \log \left[ |G \Sigma_i G^T \Omega^{-1} \Gamma + I| \right] \leq I(x; y) \leq \frac{1}{2} \log \left[ |G \Sigma G^T \Omega^{-1} \Gamma + I| \right]. \] (9.13)

Note that the mutual information terms in this chapter are unspecified to be consistent with the source literature [144], but any base may be used (such as base 2 for binary encoding as used throughout this dissertation).

To show this, we start with the simplified form of the MMSE given in Equation (9.8). Integrating each bound in the inequality from 0 to \( \Gamma \) and using some simple manipulation [47], we can interchange the trace operator and the integral to yield:

\[ \sum_i p_i \frac{1}{2} \mathrm{Tr} \left( \int_0^\Gamma \left( G \Sigma_i G^T \Omega^{-1} + \gamma I \right)^{-1} d\gamma \right) \leq I(x; y) \leq \frac{1}{2} \mathrm{Tr} \left( \int_0^\Gamma \left( G \Sigma G^T \Omega^{-1} + \gamma I \right)^{-1} d\gamma \right). \] (9.14)

To solve for the trace, we use the eigenvalue decomposition of the argument of the inverse as outlined in Appendix B, which leads to the claim.

Next we show a simplified case for univariate estimation problems, as it is insightful to gain an intuition for what these bounds represent when the source and noise are scalar. For an estimation problem as described in Section 9.2.1 where the signal \( x \) is a scalar Gaussian mixture with the \( i \)th component having variance \( \sigma_i^2 \), and noise \( n \) is a Gaussian scalar with variance \( \omega^2 \), the bounds from Equation (9.13) collapse to:

\[ \sum_i p_i \frac{1}{2} \log \left[ \frac{\sigma_i^2}{\omega^2} \Gamma + 1 \right] \leq I(x; y) \leq \frac{1}{2} \log \left[ \frac{\sigma^2}{\omega^2} \Gamma + 1 \right]. \] (9.15)

The upper bound in Equation (9.15) is seen to be the mutual information between a Gaussian source distribution with variance \( \sigma^2 \Gamma / \omega^2 \) perturbed by unit variance Gaussian noise [172]. Though we arrived at this through bounds on the MMSE, it is noted we can arrive at this result by exploiting the maximum entropy property.
of Gaussian estimation problems [172]. This offers immediate insight by using the I-MMSE bridge. Assuming the linear MMSE estimator to bound the MMSE results in bounding mutual information under the Gaussian source assumption. A related result may be found in Reference [218], where the multivariate MMSE “single crossing point” and its properties are explored, relating to the maximum entropy assumption.

The lower bound is the weighted sum of the individual Gaussian mixture component mutual informations, which is derived from the “genie” bound for the MMSE [211]. This bound assumes the estimator has perfect knowledge of which component of the mixture is being drawn from at any given time [211]. Again the I-MMSE bridge offers interesting insight: the reduction of error in the MMSE estimation process from source mixture component knowledge reduces uncertainty and thus mutual information. Interestingly, this improves the MMSE (reduces the MMSE), as we have exploited additional knowledge of the problem. However, we have removed uncertainty from our source, and so less information is learned through measurement.

9.4.2 Bounds in General Noise

We now treat the case where both the signal and noise are Gaussian mixtures. In Reference [211], the lower bound on MMSE assumed perfect knowledge of the chosen source mixture component and noise mixture component. While gaining knowledge of each mixture component decreases MMSE, learning the noise component increases mutual information while learning the signal component decreases it.

Instead, we can apply a similar assumption in the mutual information domain to the noise in our upper bound. We assume that we know with prior knowledge which noise component is chosen. As a result, the conditional mutual information is simply the previous estimation case. This decrease in noise entropy increases the mutual information, and so if we could perfectly solve the previous problem, we would have
an upper bound. Therefore, we can apply this notion to the previous upper bound to get a general noise upper bound. Similarly, we can create a hybrid lower bound. We can take the lower bound from the Gaussian noise case, which assumes the worst case noise degradation for unit variance (Gaussian noise). Interestingly, this approach does not alter the form of the lower bound from the previous case.

For an estimation problem as described in Section 9.2.1 where signal $x$ is a vector Gaussian mixture as in Equation (9.3) and noise $n$ is a vector Gaussian mixture as in Equation (9.4), then the following bounds on mutual information between signal and observation hold:

$$\sum_{i} \frac{p_i}{2} \log \left[ |G \Sigma_i G^T \Omega - I + I | \right] \leq I(x; y) \leq \sum_{k} \frac{q_k}{2} \log \left[ |G \Sigma G^T \Omega_k^{-1} - I + I | \right] . \quad (9.16)$$

To show this, we start by using the convexity of mutual information when conditioned on noise in Equation (9.7),

$$I(x; y) \leq \sum_{k} q_k I(x; y|u_n = k) \quad (9.17)$$

$$= \sum_{k} q_k I(x; \sqrt{\Gamma} H x + W n_k). \quad (9.18)$$

Applying Equation (9.12) then Equation (9.8) to each term in Equation (9.18),

$$I(x; y|u_n = k) \leq \frac{1}{2} \int_{0}^{r} \text{E} \left[ \| x - \text{E}[x|\sqrt{\gamma} H x + W n_k]\|^{2} \right] d\gamma$$

$$\leq \frac{1}{2} \int_{0}^{r} \text{Tr} \left( (G \Sigma G^T \Omega_k^{-1} + \gamma I)^{-1} \right) d\gamma. \quad (9.20)$$

The trace and integral sign can be interchanged using the procedure in Reference [47] and evaluated as in Appendix B:

$$I(x; y|u_n = k) \leq \frac{1}{2} \log \left[ |G \Sigma G^T \Omega_k^{-1} - I + I | \right] . \quad (9.21)$$

Substituting back into Equation (9.18) gives the upper bound:

$$I(x; y) \leq \sum_{k} \frac{q_k}{2} \log \left[ |G \Sigma G^T \Omega_k^{-1} - I + I | \right] . \quad (9.22)$$
For a fixed covariance matrix, Gaussian noise maximizes mean square error. Then letting $\tilde{n} \sim \mathcal{N}(0, \Omega)$:

$$I(x; y) \geq I(x; \sqrt{\Gamma} H x + W \tilde{n}).$$

(9.23)

Applying Equation (9.13),

$$I(x; y) \geq \sum_k \frac{p_k}{2} \log \left[ |G \Sigma_i G^T \Omega^{-1} \Gamma + I| \right] + H(p).$$

(9.24)

as desired.

In fact, Equation (9.16) generalizes Equation (9.13). Since the upper bound assumes the linear MMSE estimator and noise mixture genie knowledge, we denote this bound the Linear-Genie bound. Similarly, the lower bound assumes genie knowledge of the source mixture component and the Gaussian noise assumption, and is denoted the Genie-Gauss bound.

9.4.3 Mutual Information Isolated Upper Bound

If we make the assumption the integration support of the mutual information of the individual mixture components are isolated, treating them as independent, then we get a new upper bound on the mutual information. Using the log sum inequality and assuming the noise is Gaussian, we get the following result:

$$I(x; y) \leq \sum_i \frac{p_i}{2} \log \left[ |G \Sigma_i G^T \Omega^{-1} \Gamma + I| \right] + H(p).$$

(9.25)

Considering mixture noise, the resulting bound is as follows:

$$I(x; y) \leq \sum_{i,k} \frac{p_i q_k}{2} \log \left[ |G \Sigma_i G^T \Omega_k^{-1} \Gamma + I| \right] + H(p).$$

(9.26)
To show this, we can follow a series of identities for the mutual information:

\[
I(x; y) \tag{9.27}
\]

\[
= \int \log \left( \frac{p(x, y)}{p(x)p(y)} \right) p(x, y) d(x, y) \tag{9.28}
\]

\[
= \int \log \left( \frac{\sum_{u_x} p(x, y, u_x)}{\sum_{u_x} p(x, u_x)p(y)} \right) \left( \sum_{u_x} p(x, y, u_x) \right) d(x, y) \tag{9.29}
\]

\[
\leq \sum_{u_x} \int \left( \log[p(u_x)p(x, y|u_x)] - \log[p(u_x)p(x|u_x)p(y)] \right) p(u_x)p(x, y|u_x)d(x, y) \tag{9.30}
\]

\[
\leq \sum_{u_x} \int \left( \log[p(x, y|u_x)] - \log[p(u_x)p(x|u_x)p(y)] \right) p(u_x)p(x, y|u_x)d(x, y) \tag{9.31}
\]

\[
= H(u_x) + \sum_{u_x} p(u_x)I(x_{u_x}; y) \tag{9.32}
\]

where Equation (9.30) follows by the log sum inequality. Applying Equation (9.16) to Equation (9.32), Equation (9.26) follows (where \( H(p) = H(u_x) \) and \( p_i = p(u_x = i) \)). Equation (9.25) follows directly from Equation (9.26).

One can immediately note that the bounds in Equation (9.13) differ by at most \( H(p) \). It is immediately recognized that the upper bound in Equation (9.25) is simply the lower bound in Equation (9.13), plus the discrete entropy of the source mixture component distribution. Taking the difference between the two bounds gives the result.

Since the discrete entropy is maximized by the uniform distribution, we can bound the worst case error based on the model order \( N \):

\[
H(p) \leq \log[N]. \tag{9.33}
\]

This means that the maximum bound difference from the truth is directly tied to our model order, meaning we have incentive to keep our model order low. Of course, under-modeling the mixtures could invalidate the assumptions of our bounds to begin with, rendering them useless. Another way to consider this trade is to observe the
lower bound in Equation (9.13) under the equal variance constraint. If every mixture component has the same variance, then the mutual information terms are equal and may be pulled out of the summation which collapses to unity. This amounts to a fixed variance Gaussian kernel with various means and weightings. As the variance is decreased, the curve fitting approaches a near discrete sampling, and the lower bound loosens considerably, though the distribution is well matched to the truth. As the variances approach the variance of the source, the lower bound tightens to the upper bound, but the low order results in significant distribution mismatch.

Unfortunately, this bound error result does not translate to general noise modeled as a Gaussian mixture, since the bound from Equation (9.26) differs from the lower bound in Equation (9.16) by more than just the addition of $H(p)$.

### 9.4.4 Bounds on Differential Entropy

As noted in Reference [144], a very simple relationship between the mutual information and the differential entropy exist in the I-MMSE integral. Applying this result we can use our mutual information bounds to bound differential entropy. For any Gaussian mixture $x$ as in Equation (9.3):

$$
\sum_i \frac{P_i}{2} \log[2\pi e \Sigma_i G^T] \leq h(x) \leq \frac{1}{2} \log[2\pi e G \Sigma G^T].
$$

(9.34)

This can be seen by taking $n$ as an arbitrary Gaussian mixture, and relating $h(x)$ to $I(x; \Gamma x + n)$ as in Reference [144]. The result in Equation (9.34) follows by applying Equation (9.16) then taking the limit as $\Gamma \to \infty$. Applying the same logic to the isolated upper bound, we also get the following upper bound on differential entropy:

$$
h(x) \leq \sum_i \frac{P_i}{2} \log[2\pi e G \Sigma_i G^T] + H(p).
$$

(9.35)

While this upper bound is novel, the first upper bound is a well known relationship in information theory where the Gaussian distribution maximizes entropy over
continuous, infinite support for an equivalent variance.

9.4.5 Bridging Results to MMSE Bounds

In deriving the relationship between differential entropy and the I-MMSE formula for mutual informations, the authors in Reference [144] noted that the difference between the true MMSE and the MMSE under linear assumptions is quantified by the relative entropy between the true estimation distribution $y$, and the estimation distribution assuming a Gaussian source $y'$:

$$D(y||y') = \frac{1}{2} \int_{0}^{\Gamma} \text{Tr} \left( (G \Sigma G^T \Omega^{-1} + \gamma I)^{-1} \right) - \text{mmse}(H x, \gamma) d\gamma,$$

(9.36)

where $D(x||y)$ is the relative entropy function between distributions $x$ and $y$. This has immediate implications to our bounds, namely that the error from the upper bound in Gaussian noise from Equation (9.13) is exactly $D(y||y')$. That is, the more the estimation problem looks like a Gaussian perturbed by a Gaussian, the tighter the upper bound. Finally, it can be shown [144] that at high SNR, $D(y||y') \to D(x||x')$, where $x'$ is a multivariate Gaussian with the same mean and covariance as $x$. In other words, the bound’s tightness depend on the Gaussianity of the source at high SNR.

We can use Equation (9.36) and the I-MMSE formula to bound the error from the original upper bound on the MMSE and the truth under Gaussian noise assumptions. By taking the derivative of both sides, we get the following expression:

$$2 \frac{dD(y||y')}{d\Gamma} = \text{Tr} \left( (G \Sigma G^T \Omega^{-1} + \Gamma I)^{-1} \right) - \text{mmse}(H x, \Gamma).$$

(9.37)

This has the immediate corollary that $\frac{dD(y||y')}{d\Gamma} \geq 0$. This can be interpreted as follows. The tightness of the true MMSE to the linear MMSE assumption for Gaussian noise is determined by how much a small change in SNR changes how much the problem looks like a Gaussian in a Gaussian estimation problem. Further, since this converges to a
constant as discussed in the previous paragraph, this gap closes as SNR approaches infinity.

9.5 High SNR Asymptotics

In Reference [212], high SNR asymptotic behavior is derived to show convergence of the bounds. These were contributed by the co-authors, and are only summarized in the following table. In Table 9.1, we summarize all of the bounds for MMSE, mutual information, and differential entropy discussed here, as well as any analytical results for tightness and asymptotics covered in Reference [212].

9.6 Examples of I-MMSE Bound Applications

Here we apply the bounds derived in the previous section to a suite of signal processing applications involving mutual information. As stated previously, these applications can range from online, data-driven problems where the mixture models are formed in real time, to low or no-data scenarios where curve fitting to the mixture model is done to bound intractable mathematical results.

9.6.1 Radar Range

For a Gaussian source target tracking kinetic model [182], the range measurement results in a Rician random variable. This is because the source location bivariate Gaussian in Cartesian space is modified by the partial rectangular to polar transformation in a nonlinear way. There is no closed-form solution to Rician entropy [185], and when added to a Gaussian, the measurement distribution becomes even more complex. Using our method, we modeled first a zero-radius Rician, which is equivalent to a Rayleigh. The mixture modeling is shown in Figure 9.7. The mixture model was solved using a differential evolution optimization [190] with respect to the mean.
Figure 9.7: Gaussian mixture model of Rayleigh distribution. The analytic curve is plotted in the dashed purple line, and is the function we wish to model. The light gray lines show all of the individual, weighted Gaussian components that are summed to form the orange line. The orange line is the resulting mixture distribution modeling the Rayleigh distribution.

Square error between the two distributions, but any method of fitting may be used as previously noted. Note that the approximation is not perfect, as the distribution has small, but non-zero support for negative radii. In Figure 9.8, we show the resulting bounds compared to the true mutual information, which was obtained using Monte Carlo simulation, where a large number of trials was used to offset biases associated with the histogram method of estimation [209].

The upper bound from Equation (9.13) and the isolated upper bound from Equation (9.25) are combined in these examples by taking the minimum of the two expressions to form the Joint Upper Bound shown in the black dashed line. For completeness, the dashed gray line shows the maximum of these two bounds to show how they behave in their respective regimes. Also note the truth shown in green is obtained via simulation, where a very large number of trials is used to offset the biases normally
Figure 9.8: I-MMSE bounds for the Rayleigh distribution. The truth is shown in the green, obtained via Monte Carlo integration. The lower I-MMSE bound is given by the dashed blue line, while the joint I-MMSE upper bound is given in the black and gray lines. The black dashed line is the minimum of the Linear upper bound and the isolated upper bound, while the gray dashed line is the maximum of these two bounds.

associated with this method at the expense of greatly increased computational complexity. For many applications, this type of simulation is prohibitively expensive, but we use it here to compare the bounds. As we can see, the upper bound is reasonably tight in this situation, despite the imperfect modeling. If we add a non-zero mean to the underlying Gaussian distributions, or equivalently a non-zero range, then we get a proper Rician distribution. In Figure 9.9, we again model the distribution using a Gaussian mixture model optimized and fitted using differential evolution. Note now that the upper and lower bound nearly converge, as the distribution with these parameters is nearly Gaussian.

In both of these cases, the tight upper bound is due to the Linear upper bound which assumes a Gaussian source with the same variance as the source distributions. Both examples have distributions that are reasonably close to Gaussian. However,
Figure 9.9: Gaussian mixture model of Rician distribution. The analytic curve is plotted in the dashed purple line, and is the function we wish to model. The light gray lines show all of the individual, weighted Gaussian components that are summed to form the orange line. The orange line is the resulting mixture distribution modeling the Rician distribution.

Figure 9.10: I-MMSE bounds for the Rician distribution. The truth is shown in the green, obtained via Monte Carlo integration. The lower I-MMSE bound is given by the dashed blue line, while the joint I-MMSE upper bound is given in the black and gray lines. The black dashed line is the minimum of the Linear upper bound and the isolated upper bound, while the gray dashed line is the maximum of these two bounds.
there are many overlapping mixture components which violates the isolated upper bound assumptions in low and mid-range SNRs. Therefore, the dashed gray upper bound is loose to the true curve.

9.6.2 Radar Range and Bearing

In general, for a Gaussian perturbation model in Cartesian coordinates [182], the radar range is Rician distributed, while the correlated bearing is a complicated statistical distribution (von Mises distribution when conditioned upon the range, as detailed in Appendix A) [219]. We can derive a Gaussian Mixture model for the, in general, correlated bivariate distribution, and subject it to a linear transformation to whiten the resulting measurement covariance obtained via the Crâmer-Rao lower bound (CRLB).

The two-dimensional (2D) distribution can be represented as a contour plot, as shown in Figure 9.11. The purple lines represent the ideal distribution in angle-range space, or the polar transformation of our 2D Cartesian Gaussian function. The orange lines represent the mixture model summation of the individual 2D Gaussian components, which are given in gray.

In Figure 9.12, we show the resulting bounds compared to the true mutual information, which was obtained using Monte Carlo simulation. Again we note the bounds are reasonably spaced, and the simulated line appears very tight to the upper I-MMSE bound. The Linear upper bound continues to dominate the upper bound performance as well, as once again we have many overlapping components.

9.6.3 Radar Association

If two disagreeing measurements arrive in the validation gate of a tracking radar, uncertainty in association produces a bi-modal distribution [189]. This is sometimes
Figure 9.11: Gaussian mixture model of 2D polar distribution as a contour plot. The analytic curve is plotted in the purple contours, and is the function we wish to model. The orange contours are the resulting mixture distribution modeling the true polar distribution after summing the gray contours.

Figure 9.12: I-MMSE bounds for the joint Range-Bearing distribution. The truth is shown in the green, obtained via Monte Carlo integration. The lower I-MMSE bound is given by the dashed blue line, while the joint I-MMSE upper bound is given in the black and gray lines. The black dashed line is the minimum of the Linear upper bound and the isolated upper bound, while the gray dashed line is the maximum of these two bounds.
remedied by making hard decisions, or averaging to return to a single modality [220]. However, information is lost in both cases. These simplifications are often made to make the Kalman formulation tractable, as dealing with mixture hypotheses can be computationally prohibitive. We can easily bound this mixture distribution information using the I-MMSE bounds. In this example, we assume we are tracking in 1-D space, and we obtain two range measurements in the range validation gate region. The one closer to the prediction is weighted with higher confidence, and thus has a higher mixture weight. Since we do not want to average or make a hard decision, we carry both hypotheses forward, updating the posterior using each measurement and weighting and summing to produce a mixture state distribution, shown in Figure 9.13.

In Figure 9.14, we see the resulting I-MMSE bounds. At low SNR, the outlier hypothesis is severely degraded by noise and ignored, making for a near Gaussian
Figure 9.14: I-MMSE bounds for the association distribution. The truth is shown in the green, obtained via Monte Carlo integration. The lower I-MMSE bound is given by the dashed blue line, while the joint I-MMSE upper bound is given in the black and gray lines. The black dashed line is the minimum of the Linear upper bound and the isolated upper bound, while the gray dashed line is the maximum of these two bounds. The lower bound starts to become tight at high SNR where the outlier probability is significant. The upper bound starts out with the Linear bound dominating until 1 dB SNR, where the two sets of mixture distributions stop interacting, making the isolated upper bound more reasonable.

distribution. For high SNR, it becomes significant. Since the distributions are well isolated, their mutual information integrals barely interact. In this case, the lower bound becomes tight, which sums and weights the individual mutual informations of the mixture components. Since the weighting is so uneven, the entropy lost from the genie knowledge of the mixture component is small, but still significant enough to offset from the lower bound. Also noted, we see the upper bound has two regimes. At lower SNRs, the Linear bound dominates since the strong hypothesis by itself makes the source look more unimodal. After about 1 dB of SNR, the two mixture modes separate enough to strengthen the assumptions of the isolated upper bound, and it out performs the Linear upper bound.
As mentioned in Section 9.4, the model order greatly affects the bound error and the isolated support assumption of the isolated upper bound. If we force the number of components in this example to be two, arguably exploiting prior knowledge and perhaps tolerating some under-modeling, we see the results greatly improve in Figure 9.15. Now the upper bound is tight, as the isolated upper bound assumption is satisfied by the reduced model order. Since the variance is unchanged, the Linear upper bound remains unchanged.

### 9.6.4 BPSK Communications

We now explore a simple communications example, where the source distribution is randomly and uniformly drawn from a binary phase-shift keying (PSK) (BPSK) constellation, or $x \in \{-1, 1\}$. The constellation is perturbed by additive white Gaussian noise (AWGN), and so the observation is in fact a GMM. In this case, due to the
Figure 9.16: I-MMSE bounds for the BPSK communications mixture problem. The dashed black line is the joint upper bound, obtained by taking the minimum of the original I-MMSE upper bound and the new isolated upper bound. The maximum of these two bounds is shown in the dashed gray line for reference. The truth is shown in the solid green line, obtained via simulation using a very large number of trials. At low SNRs, the Gaussian noise dominates the Linear upper bound. At high SNR, the discrete approximation takes over and the isolated upper bound converges to the discrete entropy of the underlying BPSK distribution.

For example, at around 4 dB SNR, the Gaussian source approximation starts to fail, and the mutual information is dominated by the source entropy.

9.6.5 Gaussian-Frequency Channel Hopping

We can construct a more complicated communications problem to demonstrate how the assumptions of the various bounds affects performance. In this example, we assume we have 5 adjacent frequency bands we can communicate over. At any given
Figure 9.17: I-MMSE bounds for the a Gaussian-frequency channel hopping communications problem. The dashed black line is the joint upper bound, obtained by taking the minimum of the original I-MMSE upper bound and the new isolated upper bound. The maximum of these two bounds is shown in the dashed gray line for reference. The truth is shown in the solid green line, obtained via simulation using a very large number of trials. The lower bound shown in the dashed blue line is non-trivial since the source is continuous. The lower bound is achieved since the receiver has knowledge of the hopping sequence (or which mixture is drawn from).

instance, we only communicate over one of the bands. When we communicate over a given band, we do so with a Gaussian input distribution over our bandwidth. That is, we select a frequency in that band randomly with a Gaussian distribution. This covert scheme assumes the receiver knows the hopping sequence, and is subsequently listening over the correct band at each instance and measures the random frequency to obtain the source information. The receiver is corrupted by AWGN. The information bounds are shown in Figure 9.17. In this case, we have a non-trivial lower bound since our source is continuous. In fact, the lower bound perfectly captures this scenario. This is because the lower bound is exploiting knowledge of which mixture is being drawn from. Since this is known at the receiver (the hopping sequence), this uncertainty is removed from the mutual information in the actual problem, and the
bound reflects this perfectly. The upper bound is fairly loose, as 5 evenly spaced, evenly weighted Gaussian mixtures presents a very non-Gaussian shape. Similarly, the isolated upper bound is loose due to the proximity and number of mixtures.

9.6.6 Classification Information

As mentioned previously, some applications have naturally arising Gaussian mixtures. Classifiers built on GMMs already perform the model parameter and order estimation from the underlying data as a part of the classification process [213]. Further, mutual information of GMMs in this context can be used to optimize the feature selection space [221, 222]. A large mutual information can mean, in a particular feature space, a large separation or spread to facilitate classification and or that mixtures have small variance around their means (tight clusters). A classification problem in 2D feature space is shown in Figure 9.18 as an example. In this illustration, we see three clusters with equal covariance separated by their means, and with an asymmetrical underlying cluster probability. In fact, the purple class has twice as many members from this data set as orange or yellow class.

This is much like the first communications example employing BPSK, where the source distribution is in fact discrete so the lower bound is not applicable. The bounds for this example are shown in Figure 9.19. This looks strikingly similar to Figure 9.16, but with the SNR transition regions shifted slightly. The I-MMSE bounds in this example are pretty tight in both regimes and during the transition. Since these bounds employ parameters already solved in the GMM clustering algorithm, they are a simple method for optimizing feature selection.

We can also explore how the performance varies in more complicated clustering scenarios. We took the same data from the previous example, and increased the covariance of the orange class, while correlating features in the yellow class, as shown
Figure 9.18: Example classification problem. The purple class is twice as likely as the orange class or the yellow class. Their spread covariances are identical, but each cluster has a unique 2D mean.

in Figure 9.20. The bounds resulting from this complication are shown in Figure 9.21, where the Gaussian-regime upper bound has loosened. We can see the asymmetry has made the Gaussian assumption weak. Further, the different covariance spreads result in the “point-like” entropy assumption occurring at different SNRs for the different clusters. Thus, the ascent to the isolated upper bound happens in a smoothed stair-step profile. Nevertheless, the bounds are still reasonable and would provide utility in feature selection scenarios.

9.6.7 RCS Estimation in Non-Gaussian Noise

Often times, a radar target cross-section, or RCS, is modeled statistically due to amplitude fluctuation from measurement to measurement throughout the tracking
Figure 9.19: I-MMSE bounds for the classification problem. The dashed black line is the minimum of the Gaussian upper bound and the isolated upper bound (maximum shown in gray), while the truth is shown in green. At low SNRs, the Gaussian noise dominates the Linear upper bound. At high SNR, the discrete approximation takes over and the isolated upper bound converges to the discrete entropy of the classifier distribution.

scenario. Because of this randomization, there is information in observing the amplitude. As discussed in Bell’s seminal work, this information can be maximized through waveform design employing water-filling [137]. Often times, radars are dominated by non-Gaussian noise from clutter in nearby range cells [48]. Similarly, this distribution is ultimately a distribution on clutter RCS. In this example, we consider a Rayleigh distribution for our source cross section and a Weibull distribution for our clutter distribution.

We see the I-MMSE bounds in Figure 9.22 for this radar example. Though the Rayleigh is reasonably approximated through a Gaussian [175], the noise mixture assumption loosens the upper bound similar to how the lower bound is loose under typical scenarios (where the mixture knowledge is not readily available). The isolated upper bound is outperformed by the Linear-Genie upper bound since the mixture
**Figure 9.20:** Classification data set with various correlations and covariance matrices. The same data set is used with the orange class covariance being increased, and the yellow class now containing correlated features.

**Figure 9.21:** I-MMSE bounds for the more complicated classification problem. The Gaussian upper bound has loosened due to asymmetry, and the truth now approaches the isolated upper bound in a smooth stair-step fashion.
components overlap significantly in modeling the non-Gaussian distributions.

### 9.6.8 Estimation Rate

In this example, we look at bounding radar estimation rate. In previous chapters, where nonlinear estimation converted prior distributions to non-Gaussian densities, linear approximations like the extended Kalman filter (EKF) were used. To put this approximation to the test, and show how I-MMSE bounds can instead be used to simplify this analysis, we bound estimation rate in a rectangular-to-polar tracking scenario much like Section 9.6.2.

To avoid coordinate-coupled tracking physics, target tracking priors are maintained and physically propagated in the Cartesian-domain [182]. To make simple and tractable filters, priors are assumed to be Gaussian [183]. If we assume a 2D tracking
Figure 9.23: I-MMSE bounds for radar estimation rate. In addition to the truth and bounds shown previously, the linearized mutual information is shown in the yellow line, using a process similar to the extended Kalman filter. This is not a bound, but an approximation.

scenario, then the 2D Cartesian Gaussian becomes a complicated distribution when the measurement is performed, since the radar measures range and bearing [8]. This is effectively a rectangular-to-polar transformation. Therefore, the EKF is used as an approximation. The results of the bounds are shown in Figure 9.23. We see again the Linear-Genie bound out performs the isolated upper bound. The remaining bounds are reasonably tight. The linearized mutual information is shown in the yellow line. Note this is not a bound, but an approximation. In general, it could be larger or smaller than the true measure. Here we see it slightly overestimates the truth.

9.7 Summary

In this chapter, estimation information bounds were found by bridging MMSE bounds for GMMs to the information estimation domain using the I-MMSE connection [144]. The results included arbitrary noise distributions, also modeled using
Gaussian mixtures, as well as an isolated upper bound. In some cases, bound errors were solved for, with one case bridged back to the MMSE domain demonstrating the MMSE bound error as a function of the Gaussianity rate. This new result can be used to measure performance of the bounds given in Reference [211] which were input into the I-MMSE integral to form the mutual information bounds here. Arbitrary entropy models were bounded with a simple tweak of the I-MMSE formula under asymptotically infinite SNR.
<table>
<thead>
<tr>
<th>Bound Name</th>
<th>Formula</th>
<th>Bound Error (e)</th>
<th>Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMSE Genie Bound</td>
<td>$\text{nmse}(x, \Gamma) \geq \sum p_i \sigma_i^2 / (\sigma_i^2 \Gamma + \omega^2)$</td>
<td>Unknown</td>
<td>$\Gamma \to \infty$</td>
</tr>
<tr>
<td>MMSE Linear Bound</td>
<td>$I(x; y) \leq \sum p_i / 2 \log[\sigma_i^2 / \omega^2 \Gamma + 1]$</td>
<td>$e \leq H(p)$</td>
<td>$\Gamma \to \infty$, $e \to 0$</td>
</tr>
<tr>
<td>MI Genie Bound</td>
<td>$I(x; y) \leq 1 / 2 \log[\sigma_i^2 / \omega^2 \Gamma + 1]$</td>
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</tr>
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<td>MI Linear Bound</td>
<td>$I(x; y) \leq \sum p_i / 2 \log[\sigma_i^2 / \omega^2 \Gamma + 1] + H(p)$</td>
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Table 9.1: Summary of estimation and information bounds

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In this work, we have presented background research of the spectral congestion problem, identified RF convergence as the solution to this problem and the required foundation for future systems, identified metrics for joint sensing-communicating systems, bounded these metrics, and demonstrated systems and algorithms achieving various levels of RF convergence on this joint information plane. Future researchers have the challenging but exciting task of pushing systems to the upper-right hand corner of the manifold on the multiple access channel (MAC) information plane. This paradigm shift goes beyond cognitive radio and cognitive radar and far beyond coexistence. Systems of the future must be co-designed to mutual benefit. The result maximizes spectral efficiency, now encompassing radar information, and optimizes both phenomenologies by cherry-picking and sharing the best technological features from both. The work presented here is a framework and foundation for future researchers to build upon, giving them the tools and introductory systems to achieve RF convergence.
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processing phase noise mitigation performance comparison in a coherent dis-

In this appendix, we derive information measures for a typical tracking radar. We start with the source tracking model, and investigate the isolated and joint source entropy. We then work through the measurement model, showing the difficulties of mapping the tracking distribution in the nonlinear measurement domain. From this, measurement noise is added and the Cramér-Rao lower bound (CRLB) for the radar measurements is derived. Finally, to make solutions to radar estimation rate tractable, we linearize the problem using an extended Kalman filter formulation.

A.1 Target Source Entropy Model

For the target motion, we assume a constant velocity, linear two-dimensional (2D) motion model with a Gaussian perturbation acceleration distribution [182]:

\[\mathbf{s}_k = \begin{bmatrix} x_k \\ \dot{x}_k \\ y_k \\ \dot{y}_k \end{bmatrix} = \mathbf{A}(T) \mathbf{s}_{k-1} + \mathbf{w}_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ \dot{x}_{k-1} \\ y_{k-1} \\ \dot{y}_{k-1} \end{bmatrix} + \mathbf{w}_k, \quad (A.1)\]

where the \(\mathbf{w}_k\) is the process noise with covariance defined as

\[\mathbf{Q}_k(T) = \begin{bmatrix} q_{x,k} \frac{T^3}{3} & q_{x,k} \frac{T^2}{2} & 0 & 0 \\ q_{x,k} \frac{T^2}{2} & q_{x,k} T & 0 & 0 \\ 0 & 0 & q_{y,k} \frac{T^3}{3} & q_{y,k} \frac{T^2}{2} \\ 0 & 0 & q_{y,k} \frac{T^2}{2} & q_{y,k} T \end{bmatrix}, \quad (A.2)\]

\(x_k\) is the target position along the \(x\)-axis at discrete time step \(k\), \(\dot{x}_k\) is the target velocity projected on the \(x\)-axis, \(T\) is the revisit time of a particular target (duration between time steps \(k\) and \(k - 1\)), \(\mathbf{s}_k\) is the state vector, and \(q_{x,k}\) is the process model error intensity for the \(x\)-axis (similarly for the \(y\)-axis). We assume the \(x\)-position and velocity have an independent process noise power from the \(y\)-position and velocity.
These powers are estimated to track model mismatch along each dimension. We have parameterized the linear motion model matrix $A(T)$ and the process noise covariance $Q_k(T)$ by the revisit time $T$, as this is our dynamic parameter in some scenarios like in Chapter 8 for the constant information radar (CIR). Note that here we show independent process noise power for $x$ and $y$, but these may be combined to a common parameter in some cases.

Note that although we assume the velocity is constant (and therefore acceleration is zero), in reality the small perturbations of the velocity modeled by the dynamical error covariance matrix allow for small variations in acceleration. This is the stochastic state estimation approach that assumes, by construction, that the motion model contains some error [46]. In this sense, changing velocity is tracked assuming there is uncertainty in the motion model. The recursive Bayesian network therefore includes this probability cloud in likelihood decisions, allowing for a measurement that indicates a change in velocity to drive probability away from the constant velocity track in response to changes in actual velocity. Though this simplifies the model (and subsequently the tracking process), it means that large deviations in velocity may be difficult to handle. Other models can be used if the target exhibits higher order dynamics [182, 183].

The differential entropy of the target is defined by the model process covariance, and for a multivariate Gaussian can be shown to be [172]

$$h(x_{k+1}, x_{k+1}, y_{k+1}, y_{k+1}) = \frac{1}{2} \log_2 [(2\pi e)^4 |Q_k|] = \frac{1}{2} \log_2 \left[ \frac{(2\pi e)^4 q_k^4 T^8}{144} \right], \quad (A.3)$$

where we have dropped the parameterization $T$ of the model covariance matrix and assumed $q_{x,k} = q_{y,k} = q_k$ for simplicity. Though this calculation was relative simple and straightforward, it offers little physical interpretation since the entropy is differential. We track in the Cartesian domain so that the linear motion assumptions
and the straightforward motion model remain true regardless of the target’s location. Tracking in the polar domain results in coordinate-coupled physics [182]. Note also the form given in Equation (A.3) is true in general, but we have simplified by ignoring the Kalman prediction equations as outlined in Chapter 8. Ultimately, including this additional level of detail results in a different matrix in the determinant, but the form remains unchanged, so we ignore for the sake of the argument and analysis here.

A.2 Measurement Model

In all cases in this work, we assume a monostatic radar configuration. The radar may be pulsed or operating with continuous signaling. The state parameters to be estimated are the position in 2D space \([x \ y]\), and the velocity, which has an \(x\) and \(y\) component. The observed parameters, after Doppler processing [46], are the range \(r_k\), range-rate \(\dot{r}_k\), and bearing \(\theta_k\). In some cases, only a subset of these measurements are assumed available. The range-only case, for example given in Chapter 5, is much simpler since the linear Gaussian assumptions remain. Doppler is added in Chapter 6 for the frequency-modulated continuous-wave (FMCW) radar as a necessary construct considering the long integration periods, and complicates analysis as we see. Bearing and Doppler are included in Chapter 8 for the CIR which demonstrates a complete tracking and radar scheduling scenario.

We assume a narrowband environment such that only a frequency shift (Doppler shift) is induced in the returned waveform, not the more general Doppler time scaling. This is possible under the assumption [173]:

\[
T_p B \ll \frac{c}{2\dot{r}},
\]  

(A.4)

where \(c\) is the speed of electromagnetic waves in air, \(T_p\) the signal duration, and \(B\) the signal bandwidth. This means that the range and range-rate which offset the peak of
the narrowband cross-ambiguity function at the measurement processor relate to the position and velocity state as follows [173, 184]:

\[ r = \sqrt{x^2 + y^2}, \text{ and} \]

\[ \dot{r} = \frac{x \dot{x} + y \dot{y}}{\sqrt{x^2 + y^2}}. \]  

(A.5)

(A.6)

Independently, we exploit spatial degrees of freedom [47] to obtain a bearing measurement which relates to the source position as

\[ \theta = \text{atan2}(y, x). \]  

(A.7)

For the range, we can see that it only depends on \( x_{k+1} \) and \( y_{k+1} \), which are seen in our model (ignoring the tracking prior) to be independent with variance \( \sigma_{xy}^2 = q_k T^3/3 \). The source entropy of just the position is therefore given by

\[ h(x_{k+1}, y_{k+1}) = \log_2[2\pi e \sigma_{xy}] = \log_2 \left[ \frac{2\pi e \sqrt{q_k T^3}}{\sqrt{3}} \right]. \]  

(A.8)

If we first assume for simplicity our previous target was at the origin, the range measurement, excluding measurement error for the time being, is a Chi distribution with entropy [172]

\[ h(r_{k+1}) = \ln \left[ \frac{\sigma_{xy} \Gamma(1)}{\sqrt{2}} \right] - \frac{1}{2} \Psi_0(1) + 1 = \log_2 \left[ \frac{e \sqrt{q_k T^3}}{\sqrt{3}} \right] + \left( \frac{\gamma}{2 \ln 2} - \frac{1}{2} \right), \]  

(A.9)

where \( \Gamma \) is the gamma function, \( \Psi_0 \) is the digamma function, and \( \gamma \) is the Euler-Mascheroni constant [172]. We have separated these terms for a specific reason that is evident soon.

Note that this entropy is not equal to the entropy of the source position in Equation (A.8), as we have projected the two-dimensional position uncertainty onto a one-dimensional range axis. That is, multiple perturbations about the true position result in the same range measurement. These aliases are arcs passing through the
two-dimensional Gaussian, and each range along these radial cut-set integrate all of the probability along that Gaussian contour.

For the bearing measurement, probability contours defined by a linear cutset of the two-dimensional Gaussian encapsulate the remaining entropy after the transformation. We can start by recognizing that the ratio of the two position coordinate Gaussians is a Cauchy distribution \([223]\). Since the variance of the two parameters are equal, the result is the standard Cauchy distribution, as they cancel in the division. The standard Cauchy distribution is related to the scaled/shifted uniform distribution as \([223]\)

\[
X \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ then } \\
\tan(X) \sim \text{Cauchy}(0, 1)
\]  

(A.10)

Therefore, the entropy of the simplified bearing measurement (one argument arctangent) is given as \([172]\)

\[
h(\hat{\theta}_{k+1}) = \log_2[\pi] .
\]

(A.11)

However, we have an additional bit of information. Equation (A.11) maps our angle between \((-\frac{\pi}{2}, \frac{\pi}{2})\), or two quadrants of our coordinate system. It is easily seen that the signs of the inputs allow mapping to another 2 quadrants, and so an additional bit of information is introduced. Therefore,

\[
h(\theta_{k+1}) = \log_2[\pi] + 1 = \log_2[2\pi] .
\]

(A.12)

It should be obvious that the range and bearing represent a deterministic, invertible coordinate change, and so the joint entropy of the two is exactly the entropy of the source given in Equation (A.8), with some additive entropy factor from the coordinate stretch factor in the nonlinear transformation \([172]\). We refer to this value as \(\beta\), the Jacobian entropy. Therefore, the following identity must be true because of
the invertibility of the polar transformation

\[ h(r_{k+1}, \theta_{k+1}) = h(x_{k+1}, y_{k+1}) + \beta, \quad (A.13) \]

where \( \beta \) is the Jacobian entropy in mapping from rectangular to polar coordinates. We can now solve for this entropy by equating Equation (A.13) and the sum of the range and bearing entropies, and we easily see

\[ \beta = \frac{\gamma}{2 \ln 2} - \frac{1}{2} \approx -0.0836. \quad (A.14) \]

Now the separation in the range entropy becomes clear, and it is apparent that the coordinate stretching is due to the range mapping alone. This is not entirely unexpected, as the Jacobian in mapping from rectangular to polar coordinates is the range \( r \) [223]. Finally, the range may be mapped to a physical measurement of the returned radar delay [173]

\[ r = \frac{c \tau}{2}, \quad (A.15) \]

for \( \tau \) the measured time delay and \( c \) the speed of the propagating wave in our medium, assumed to be free space. Since this is simply a scalar operation, the entropy is found to be [172]

\[ h(\tau_{k+1}) = \log_2 \left[ \frac{e \sqrt{q_k T^3}}{\sqrt{3}} \right] + \left( \frac{\gamma}{2 \ln 2} - \frac{1}{2} \right) + \log_2 \left[ \frac{2}{c} \right], \quad (A.16) \]

where now the additional scaling contributes additional Jacobian entropy.

It should be clarified that the measurement noise we ultimately introduce adds unwanted entropy to the range and bearing, increasing the overall entropy of the polar coordinate representation of the target (but decreasing the amount of desired information). Note the range and bearing tell us no information about the velocity at any given time step, but only when observed over time via the differential. Here we therefore only consider the range-rate when deriving information about the velocity portion of the state vector. Since the range-rate in Equation (A.6) is dependent on
the entire state vector, we must look at the transformation between this measurement and the entire state vector.

For both range-rate pairs, we can easily see they are correlated Gaussians with correlation coefficient $\rho = \sqrt{3}/2$. We can first diagonalize the range/range-rate and show that the result is Wishart distributed and may be written as [224]

$$x \dot{x} = \frac{3\rho}{T^2} x^2 + \sqrt{1 - \rho^2} T \dot{x} \ z,$$  \hspace{1cm} (A.17)

where $z$ is an independent standard normal distributed random variable. If we do the same for the $y$ coordinate, then we can rewrite the range-rate equation as

$$\dot{r} = \frac{3\rho}{T^2} (x^2 + y^2) + \frac{\sqrt{1 - \rho^2} T (x z_x + y z_y)}{\sqrt{x^2 + y^2}} = \frac{3\rho}{T^2} r + \frac{\sqrt{1 - \rho^2} T (x z_x + y z_y)}{r},$$  \hspace{1cm} (A.18)

It can be shown [224] that the ratio of a Gaussian to an independent Chi distribution results in a variable distributed with Student’s t-distribution. Therefore, we can rewrite our result once again

$$\dot{r} = \frac{3\rho}{T^2} r + \sqrt{1 - \rho^2} T (x \hat{z}_x + y \hat{z}_y),$$  \hspace{1cm} (A.19)

where $\hat{z}_x$ and $\hat{z}_y$ are distributed according to Student’s t-distribution. Solving for this entropy analytically could be difficult or impossible, so we go a different approach. The range-rate, despite what the above formula shows, is actually independent of the range [184]. In fact, it can be written as a projection onto the radial axis as [184]

$$\dot{r} = v_x \cos \theta + v_y \sin \theta.$$  \hspace{1cm} (A.20)

While this distribution may be equally difficult to deal with, we can instead forgo the marginal entropy for now and consider the conditional entropy given the bearing when calculating the joint entropy later, our ultimate goal. Finally, the range-rate may be mapped to a physical measurement of the returned radar Doppler shift [173]

$$\dot{r} = \frac{c \omega_D}{2\omega_c},$$  \hspace{1cm} (A.21)
for $\omega_D$ the measured Doppler shift in the carrier frequency $\omega_c$ and $c$ the speed of the propagating wave in our medium, assumed to be free space. Since this is simply a scalar operation, the entropy is found to be [172]

$$h(\omega_{D_{k+1}}) = h(\dot{r}_{k+1}) + \log_2 \left[ \frac{2\omega_c}{c} \right]. \quad (A.22)$$

We can now solve for the joint entropy of all three measurements for our linear motion model, invoking the chain rule of entropy [172]

$$h(\theta, \tau, \omega) = h(\theta) + h(\tau, \omega_D|\theta) = h(\theta) + h(\tau) + h(\omega_D|\tau, \theta). \quad (A.23)$$

We know $h(\theta)$ from Equation (A.12), and $h(\tau)$ from Equation (A.16), and that the Doppler distribution is independent of the time delay from Equation (A.20). However, we consider the marginal density with both for ease of analysis. If both the time delay and bearing are known, the original position vector is known completely. If the position vector is known, then the velocity components are distributed as

$$v_{x,k+1} \sim \mathcal{N}\left(v_{x,k} + \frac{3}{2T} (x_{k+1} - x_k), \frac{q_k T}{4}\right), \quad (A.24)$$

and similarly for $v_{y,k+1}$. For a known position, the bearing is also constant. Therefore the range-rate is simply a scaled sum of Gaussians. For a given bearing, to find $h(\omega_D|\theta)$, we can solve for the conditional entropy

$$h(\omega_D|\theta) = \int p(\theta) h(\omega_D|\theta = a) \, d\theta, \quad (A.25)$$

where $a$ is a specific value of $\theta$. In other words, we want to sum the entropies of the Doppler measurement for all values of $\theta$ (assuming for each the bearing is deterministic), weighted by the probability of $\theta$ taking on this value. For a specific value of $\theta$, the Doppler measurement is simply

$$\omega_D = v_{x,k+1} \cos(a) \frac{2\omega_c}{c} + v_{y,k+1} \sin(a) \frac{2\omega_c}{c}. \quad (A.26)$$
However, this is just normally distributed as

$$\omega_D \sim \mathcal{N}\left(\mu_{\omega_D}, \sigma^2_{\omega_{e,k+1}} \cos^2(a) \frac{4\omega_c^2}{c^2} + \sigma^2_{\omega_{v,k+1}} \sin^2(a) \frac{4\omega_c^2}{c^2}\right) \equiv \mathcal{N}\left(\mu_{\omega_D}, \frac{q_k T \omega_c^2}{c^2}\right),$$

(A.27)

the entropy of which is given by

$$h(\omega_D|\theta = a) = \frac{1}{2} \log_2 \left[ 2\pi e \frac{q_k T \omega_c^2}{c^2} \right],$$

(A.28)

which is completely independent of the bearing, and position measurement. Therefore

$$h(\omega_D|\theta) = \frac{1}{2} \log_2 \left[ 2\pi e \frac{q_k T \omega_c^2}{c^2} \right],$$

(A.29)

and finally

$$h(\theta, \tau, \omega_D) = \frac{1}{2} \log_2 \left[ \frac{32 e^3 q_k^2 T^4 \pi^3 \omega_c^2}{3c^4} \right] + \beta.$$  

(A.30)

We now consider the more complicated case where our target is not at the origin. It should be immediately apparent that the range is Rice distributed [225], as it may be written as

$$x_{k+1} \sim \mathcal{N}(r_k \cos \theta_k, \sigma^2_{xy})$$  

(A.31)

$$y_{k+1} \sim \mathcal{N}(r_k \sin \theta_k, \sigma^2_{xy})$$  

(A.32)

$$r_{k+1} \sim \text{Rice}(r_k, \sigma^2_{xy})$$  

(A.33)

So we can see that the range distribution depends only on the previous range. It can be shown that the differential entropy for a Rice distribution is given as [185]

$$h(r_{k+1}) = 2 \ln[\sigma_{xy}] + 1 + \frac{r_k^2}{\sigma^2_{xy}} - \frac{1}{\sigma^2_{xy}} \int_0^\infty xe^{-\frac{(x-r_k)^2}{2\sigma^2_{xy}}} I_0\left(\frac{r_k x}{\sigma^2_{xy}}\right) \left(1 + \ln \left[I_0\left(\frac{r_k x}{\sigma^2_{xy}}\right)\right]\right) \, dx,$$

(A.34)

where \(I_0(\cdot)\) is the modified Bessel function of the first kind. Unfortunately, there is no closed-form solution for the integral in the above equation [185]. However, we can compute this entropy using numerical integration or Monte Carlo methods [183].
Though we have looked at the marginal entropy of the range, we do not know for certain, as we did in the previous case, that the range and bearing in the non-central case are independent. We can start by finding the joint distribution of the transformed position state \( [223] \):

\[
p(r, \theta) = p_{x,y}(r \cos \theta, r \sin \theta) r, \tag{A.35}
\]

where the extraneous \( r \) is the Jacobian variable of the rectangular to polar transformation \([223]\). For our multi-variate Gaussian, this becomes

\[
p(r_{k+1}, \theta_{k+1}) = \frac{r_{k+1}}{2\pi \sigma_{xy}^2} \exp \left[ -\frac{(r_{k+1} \cos \theta_{k+1} - x_k)^2 - (r_{k+1} \sin \theta_{k+1} - y_k)^2}{2\sigma_{xy}^2} \right]. \tag{A.36}
\]

Note we have added the time index in this context, as the mean of the bivariate Gaussian is given by the previous position \([x_k y_k]\). We can observe the conditional angle distribution to assert independence. If not independent, then we have a path to compute the joint entropy via the chain rule \([172]\):

\[
p(\theta_{k+1}|r_{k+1} = a) = \frac{p(r_{k+1}, \theta_{k+1})}{p(r_{k+1})} = \frac{r_{k+1}}{2\pi \sigma_{xy}^2} e^{-\frac{(r_{k+1} \cos \theta_{k+1} - x_k)^2 - (r_{k+1} \sin \theta_{k+1} - y_k)^2}{2\sigma_{xy}^2}} \frac{r_{k+1}}{2\sigma_{xy}^2} e^{-\frac{r_{k+1}^2}{2\sigma_{xy}^2}} I_0 \left( \frac{r_{k+1} r_k}{\sigma_{xy}^2} \right)
\]

\[
= e^{-\frac{r_{k+1} r_k}{\sigma_{xy}^2}} I_0 \left( \frac{r_{k+1} r_k}{\sigma_{xy}^2} \right),
\]

where we have exploited the rectangular to polar transformation of the previous state vector, and some trigonometric identities. This distribution is known as the von Mises distribution \([219]\), with \( \kappa = \frac{r_{k+1} r_k}{\sigma_{xy}^2} \) and \( \mu = \theta_k \). This distribution is commonly used in physics \([219]\), however it is unclear if this connection to the phase of a non-central circular Gaussian conditioned upon a specific range has been made in literature previously. In Figure A.1, we plot this distribution for various previous position state bearings. In Figure A.2, we plot this distribution for various previous position radii.
Figure A.1: Probability distribution of non-central circular Gaussian angle with 5 uniformly (over the angular support) chosen previous bearings. The mean is given by the previous bearing.

It should be obvious that this plot also applies to varying the current range. Clearly, this indicates that the bearing and range under a non-central Gaussian source are dependent random variables, despite the bijective mapping. Therefore

$$h(r, \theta) = h(r) + h(\theta | r).$$ (A.37)

While the conditional distribution is given by the von Mises distribution, this is only for a specific value of the range, and does not encompass the conditional entropy for all potential values for the range, weighted by the probability of that particular range value.

We once again can note that the polar transformation under non-zero source mean is still a bijective mapping, and so we can assume independence. In addition, our derivation of the Doppler entropy relied on no assumptions of zero source mean, and so the result still holds

$$h(\theta, \tau, \omega_D) = h(\theta) + h(\tau) + \frac{1}{2} \log_2 \left[ \frac{8\pi e \sigma_{\text{vxy}}^2 \omega^2_c}{c^2} \right],$$ (A.38)
where the first two entropies can be calculated using numerical or Monte Carlo methods.

We are now ready to consider measurement noise. If we assume additive white Gaussian noise (AWGN), the already complicated probability density functions are now convolved with the additive Gaussian distribution function. The delay and Doppler measurements originate from the same signal source, the matched filter receiver, and therefore their noise is, in general, correlated [173]. So we must consider the delay and Doppler entropy jointly. Therefore, instead of our closed-form solution for the Doppler entropy, we can estimate it as a Rician-Gaussian pair corrupted by correlated Gaussian noise:

$$h(\tilde{\theta}, \tilde{\tau}, \tilde{\omega}_D) = h(\tilde{\theta}) + h(\tilde{\tau}, \tilde{\omega}_D | \tilde{\theta}),$$  \hspace{1cm} (A.39)

where $\tilde{x}$ represents a noise corrupted version of the random variable $x$.

The mutual information between the measurement vector and the noisy measure-
moment vector is therefore given by

\[ I(\theta, \tau, \omega_D; \tilde{\theta}, \tilde{\tau}, \tilde{\omega}_D) = h(\tilde{\theta}, \tilde{\tau}, \tilde{\omega}_D) - h(\tilde{\theta}, \tilde{\tau}, \tilde{\omega}_D | \theta, \tau, \omega_D), \tag{A.40} \]

where

\[ h(\tilde{\theta}, \tilde{\tau}, \tilde{\omega}_D | \theta, \tau, \omega_D) = \frac{1}{2} \log_2 ((2\pi e)^3 \sigma^2_{\theta} | J^{-1} \rho \rho^T ). \tag{A.41} \]

The bearing noise is independent of the joint delay-Doppler processor, and \( J^{-1} \) is the inverse Fisher information matrix (FIM) of the joint delay-Doppler processor. Therefore \[ R_{\text{est}} \leq \frac{h(\tilde{\theta}, \tilde{\tau}, \tilde{\omega}_D) - \frac{1}{2} \log_2 ((2\pi e)^3 \sigma^2_{\theta} | J^{-1} \rho \rho^T )}{T}, \tag{A.42} \]

which can be rewritten as

\[ R_{\text{est}} \leq B \log_2 \left[ 1 + \left( \frac{\frac{1}{2} h(\tilde{\theta}, \tilde{\tau}, \tilde{\omega}_D)}{\sqrt{(2\pi e)^3 \sigma^2_{\theta} | J^{-1} \rho \rho^T )}} - 1 \right) \right]. \tag{A.43} \]

Unfortunately, this still requires numerical methods to solve for the noise-corrupted joint measurement entropy.

### A.3 Linearized Model

To facilitate the analysis of these systems, we can take an alternative route to estimate the mutual information using the extended Kalman filter (EKF). The distributions are ultimately Gaussian and driven by the linearization matrix. This is given by computing the Jacobian of the measurement matrix \[ C = \begin{bmatrix} \frac{2 \partial R}{c \partial x} & \frac{2 \partial R}{c \partial y} & \frac{2 \partial R}{c \partial \dot{x}} & \frac{2 \partial R}{c \partial \dot{y}} \\ \frac{2 \partial \omega}{c \partial x} & \frac{2 \partial \omega}{c \partial y} & \frac{2 \partial \omega}{c \partial \dot{x}} & \frac{2 \partial \omega}{c \partial \dot{y}} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial \dot{x}} & \frac{\partial \theta}{\partial \dot{y}} \end{bmatrix}, \tag{A.44} \]
where $R$, $\dot{R}$, and $\Theta$ are the measurement functions for the range, range-rate and bearing respectively, and the partial derivatives are all to be evaluated at the predicted state. For the range, we get the following:

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x} \left[ \sqrt{x^2 + y^2} \right] \bigg|_{\{x,y\} = \{x,y\}_{k|k-1}} = \frac{x_{k|k-1}}{\sqrt{x_{k|k-1}^2 + y_{k|k-1}^2}}$$

$$\frac{\partial R}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left[ \sqrt{x^2 + y^2} \right] \bigg|_{\{x,y\} = \{x,y\}_{k|k-1}} = 0$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y} \left[ \sqrt{x^2 + y^2} \right] \bigg|_{\{x,y\} = \{x,y\}_{k|k-1}} = \frac{y_{k|k-1}}{\sqrt{x_{k|k-1}^2 + y_{k|k-1}^2}}$$

For the range-rate, we get the following:

$$\frac{\partial \dot{R}}{\partial x} = \frac{\partial}{\partial x} \left[ \dot{x} \ddot{x} + y \ddot{y} \right] \bigg|_{x = \dot{x}_{k|k-1}} = \frac{\dot{x}_{k|k-1}}{\sqrt{x_{k|k-1}^2 + y_{k|k-1}^2}} - \frac{x_{k|k-1} \ddot{x}_{k|k-1} + y_{k|k-1} \ddot{y}_{k|k-1} x_{k|k-1}}{\left( \sqrt{x_{k|k-1}^2 + y_{k|k-1}^2} \right)^3}$$

$$\frac{\partial \dot{R}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left[ \dot{x} \ddot{x} + y \ddot{y} \right] \bigg|_{x = \dot{x}_{k|k-1}} = \frac{\ddot{x}_{k|k-1}}{\sqrt{x_{k|k-1}^2 + y_{k|k-1}^2}}$$

$$\frac{\partial \dot{R}}{\partial y} = \frac{\partial}{\partial y} \left[ \dot{x} \ddot{x} + y \ddot{y} \right] \bigg|_{x = \dot{x}_{k|k-1}} = \frac{\ddot{y}_{k|k-1}}{\sqrt{x_{k|k-1}^2 + y_{k|k-1}^2}} - \frac{x_{k|k-1} \ddot{x}_{k|k-1} y_{k|k-1} + y_{k|k-1} \ddot{y}_{k|k-1} y_{k|k-1}}{\left( \sqrt{x_{k|k-1}^2 + y_{k|k-1}^2} \right)^3}$$

$$\frac{\partial \dot{R}}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} \left[ \dot{x} \ddot{x} + y \ddot{y} \right] \bigg|_{x = \dot{x}_{k|k-1}} = \frac{\ddot{y}_{k|k-1}}{\sqrt{x_{k|k-1}^2 + y_{k|k-1}^2}}$$
Finally, for the bearing, we get the following:

\[
\frac{\partial \Theta}{\partial x} = \frac{\partial}{\partial x} \left[ \text{atan2}(y, x) \right] \bigg|_{x, y = x_{k|k-1}, y_{k|k-1}} = \frac{-y_{k|k-1}}{x_{k|k-1}^2 + y_{k|k-1}^2}
\]

\[
\frac{\partial \Theta}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left[ \text{atan2}(y, x) \right] \bigg|_{x, y = x_{k|k-1}, y_{k|k-1}} = 0
\]

\[
\frac{\partial \Theta}{\partial y} = \frac{\partial}{\partial y} \left[ \text{atan2}(y, x) \right] \bigg|_{x, y = x_{k|k-1}, y_{k|k-1}} = \frac{x_{k|k-1}}{x_{k|k-1}^2 + y_{k|k-1}^2}
\]

\[
\frac{\partial \Theta}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} \left[ \text{atan2}(y, x) \right] \bigg|_{x, y = x_{k|k-1}, y_{k|k-1}} = 0
\]

The source uncertainty is our desired information, and is modeled as a multivariate Gaussian with covariance \( P_{k|k-1} \). This covariance is obtained by taking the last target distribution at time step \( k-1 \), and advancing the prediction in time using the linear motion model \( A \):

\[
P_{k|k-1} = A P_{k-1} A^T + Q_k. \quad (A.45)
\]

The covariance of the measured distribution, after linearizing about the predicted state, is given by \( C^T P_{k|k-1} C \), where \( C \) is the Jacobian of the measurement matrix linearized about the predicted state as defined above. The linear transform of the multivariate Gaussian results in another multivariate Gaussian. As shown in Chapter 9 in the final example, this approximation works reasonably well and is straightforward. For highly nonlinear tracking scenarios, full distribution propagation techniques may be used, such as the particle filter.

### A.4 Measurement Noise

We now must find the estimation noise of our measurements. For simplicity, we only consider a single target scenario. We consider the range-Doppler processor independently from the bearing measurement. To start, for a detection-formed array,
the received signal is given by
\[ s_{RX}(t) = \sqrt{2} \Re \left[ \sqrt{E_t} s_{\text{rad}}(t - \tau) e^{j(\omega_c + \omega_D)(t - \tau)} \right] + n(t), \] (A.46)

where \( E_t \) is the transmit energy, \( s_{\text{rad}}(t) \) is our complex baseband waveform, \( \tau \) is the time delay of the received signal in seconds, \( \omega_c \) is the carrier frequency in radians/second, \( \omega_D \) is the Doppler shift in radians/second, and \( n(t) \) is our complex baseband receiver noise with variance \( \sigma_{\text{noise}}^2 \). We are assuming the CRLB is achievable, and sufficiently high signal-to-noise ratio (SNR) that we can define the FIM to be [173]
\[ J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \] (A.47)

where
\[ J_{11} = 2 \frac{\text{ISNR}}{2\pi} \int_{-\infty}^{\infty} \omega^2 |S_{\text{rad}}(j\omega)|^2 d\omega = 2T_p s_{\text{rms}}^2 \text{ISNR} \left( 2\pi B_{\text{rms}} \right)^2 \] (A.48)
\[ J_{22} = 2 \text{ISNR} \int_{-\infty}^{\infty} t^2 |s_{\text{rad}}(t)|^2 dt = 2T_p s_{\text{rms}}^2 \text{ISNR} \left( T_{\text{rms}} \right)^2 \] (A.49)
\[ J_{12} = J_{21} = 2 \Im \left\{ \text{ISNR} \int_{-\infty}^{\infty} t s_{\text{rad}}(t) \frac{\partial s_{\text{rad}}^*(t)}{\partial t} dt \right\} \] (A.50)

for \( S_{\text{rad}}(j\omega) \) the Fourier transform of our baseband radar signal, ISNR the integrated SNR and the waveform finite root mean square (RMS) defined as
\[ \text{ISNR} = \frac{T_p B a^2 P_{\text{rad}}}{\sigma_{\text{noise}}^2}, \quad s_{\text{rms}} = \sqrt{\frac{1}{T_p} \int_{-T_p/2}^{T_p/2} |s_{\text{rad}}(t)|^2 dt} \] (A.51)

respectively, \( B_{\text{rms}} \) the RMS bandwidth, and \( T_{\text{rms}} \) the RMS pulse duration. For the integrated SNR, \( a \) is the combined radar antenna gain, \( P_{\text{rad}} \) is the radar transmit power, and \( \sigma_{\text{noise}}^2 \) is the receiver thermal noise power defined as
\[ \sigma_{\text{noise}}^2 = k_B T_{\text{temp}} B, \] (A.52)
where $k_B$ is the Boltzmann constant in Joules/Kelvin and $T_{\text{temp}}$ is the absolute temperature in Kelvin. The combined radar antenna gain is given by

$$a^2 = \frac{G^2 \text{RCS} c^2}{(4\pi)^3 r^4 f_c^2}, \quad (A.53)$$

where $G$ the radar antenna gain relative to an isotropic antenna, RCS the target radar cross section in square meters, $r$ the actual target range in meters and $f_c$ the carrier frequency in Hz. Note we have assumed the carrier is chosen such that the mean frequency is 0 [173]. For signals that may be written in polar form [189], we can only rewrite the third term as:

$$J_{12,21} = -4\pi \text{ISNR} \int_{-\infty}^{\infty} t f(t)|s_{\text{rad}}(t)|^2 dt, \quad (A.54)$$

where $f(t)$ is the instantaneous frequency function of our waveform. Then the scaled inverse FIM, or CRLB becomes:

$$J^{-1} = \frac{1}{J_{11}J_{22} - J_{12}^2 \begin{bmatrix} J_{22} & -J_{12} \\ -J_{12} & J_{11} \end{bmatrix}}. \quad (A.55)$$

All of these quantities are easily calculated. For example, if we assume a flat spectrum, then $B_{\text{rms}} = B/\sqrt{12}$. For a constant modulus signal, $T_{\text{rms}} = T_p/\sqrt{12}$. If we normalize the energy of our pulse, then $s_{\text{rms}}^2 = 1/T_p$. This fits a wide range of signals of interest for radar systems. Finally, with knowledge of the instantaneous frequency, $J_{12}$ is easily found. For example, for a linear frequency-modulated (FM) chirp:

$$J_{12} = -4\pi \text{ISNR} \int_{-\infty}^{\infty} t (f_0 + k t) |s_{\text{rad}}(t)|^2 dt = -2\pi k J_{22}, \quad (A.56)$$

where $f_0$ is the start frequency of the chirp in Hz, $k$ is the chirp rate in Hz/seconds, and we have made use of the assumption that our signal envelope is symmetric in time about the origin. Since this term drives the correlation coefficient between the variance of the delay and Doppler measurements [173], the chirp rate directly affects

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the measurement correlation. This is the well-known phenomenon for linear FM chirps known as range-Doppler coupling [48].

In Chapter 5, for example, we assume a Gaussian windowed linear FM chirp [173] with chirp rate $B/T_p$, and so:

$$J^{-1} = \frac{1}{\text{SNR}} \begin{bmatrix} T_p^2 & -4\pi T_p B \\ -4\pi T_p B & T_p^{-2} + (4\pi B)^2 \end{bmatrix}. \tag{A.57}$$

Phase noise considerations for pulse-Doppler processing of long-range targets in clutter could easily swamp the thermal estimation noise, reducing Doppler information [226]. While standalone Doppler estimation performance tends to be relatively unaffected [46], derived measurements such as geolocation using frequency difference of arrival (FDOA), coherent sensor networks, and clutter mitigation can be significantly impacted by degrading phase noise. Ignoring effects of target resolution and clutter ridge spreading due to phase noise, we can model the effects on our information as an SNR degradation dependent on the range to the target. Close-in targets are relatively well correlated with the source oscillator, while distant targets provide more time and opportunity for drift [227]. We model the additional degrading source covariance as follows:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{\text{PN}}^2 \end{bmatrix}. \tag{A.58}$$

where $\sigma_{\text{PN}}^2$ is the range-dependent phase noise power. Note that we are treating it here as an additional AWGN term, while in practice it is highly colored close to the oscillator frequency [46]. Therefore, this model is most accurate for ranges that are only degraded by the flat portion of the phase noise density. The shaped spectrum is more important for characterizing target resolution limitations and clutter cancellation [228], but not appreciably detrimental to SNR affecting information. While cross terms may arise in the covariance matrix because of waveform dependent correlation
of the range with the range-rate, in our example, we assume this contribution is negligible. That is, a perturbation of phase is unlikely to significantly impact range. For the example plot in Chapter 5, we set $\sigma_{PN}^2 = 1000 \text{ Hz}^2$. Note this means our effective noise covariance in the example is given by

$$\hat{J}^{-1} = J^{-1} + P.$$  \hspace{1cm} (A.59)

For specific applications, $\sigma_{PN}^2$ would be determined by the amount of phase noise power due to integration time and range to target for a specific local oscillator model. More sophisticated models outside of Gaussian modeling would need to formulate the mutual information with the more complicated distributions. While clutter modeling is outside of the scope of this work, the effect of clutter would contribute an additional, non-Gaussian distribution to the noise degradation in general [48].

Finally, CRLB for the bearing estimate can be considered independently as shown in Chapter 8, and is given by [47]:

$$\sigma_\theta^2 = \frac{1}{\text{ISNR} \, 2k^2 \, N_A \sigma_A^2},$$  \hspace{1cm} (A.60)

where $N_A$ is the number of antennas, ISNR is the integrated SNR, $k$ is the wavenumber, and $\sigma_A^2$ is the RMS element position. For a uniform linear array with a target off the platform boresight, the simplified form given in Equation (8.7) is used [48].
APPENDIX B

I-MMSE BOUND FORMULATION
In this appendix, we form a set of upper and lower bounds on estimation information by formally invoking the I-MMSE relationship on bounds on the minimum mean square error (MMSE) for estimation problems. This proof is presented in Reference [175].

B.1 Univariate Source, Gaussian Noise

For the univariate case, the MMSE bounds presented in Chapter 9 becomes drastically simpler:

\[
\sum_i p_i \left( \sigma_i^2 - \frac{\sigma_i^4}{\sigma_i^2 \Gamma + 1} \right) \leq \text{mmse} (x, \Gamma) \leq \sigma_x^2 - \frac{\sigma_x^4 \Gamma}{\sigma_x^2 \Gamma + 1},
\]

for \( \Gamma \) a scalar signal-to-noise ratio (SNR) term, \( \text{mmse} (\cdot) \) the MMSE function, \( \sigma_x^2 = (\sum_i p_i \sigma_i^2 + p_i \mu_i^2) - (\sum_i p_i \mu_i)^2 \) the variance of the source \( x \) and Gaussian mixture model (GMM) parameters \( \{\mu_i, \sigma_i^2, p_i\}_{i=1...N} \) for the source. Since \( \sigma_x^2 \) is a constant, and \( \sigma_i^2 \) is a constant for a given index on the lower bound, we can solve for the integral and obtain the upper and lower bound, the latter by interchanging the linear summation and integral operations. The integral is given by

\[
\int_0^\Gamma \sigma^2 - \frac{\sigma^4 \gamma}{\sigma^2 \gamma + 1} d\gamma = \Gamma \sigma^2 - \sigma^2 \int_0^\Gamma \frac{\sigma^2 \gamma}{\sigma^2 \gamma + 1} d\gamma.
\]

Noting that the following are equivalent

\[
\frac{\sigma^2 \gamma}{\sigma^2 \gamma + 1} = \frac{\sigma^2 \gamma + 1}{\sigma^2 \gamma + 1} - \frac{1}{\sigma^2 \gamma + 1} = 1 - \frac{1}{\sigma^2 \gamma + 1},
\]

and using the integral table for the natural logarithm, we get the following:

\[
\int_0^\Gamma \sigma^2 - \frac{\sigma^4 \gamma}{\sigma^2 \gamma + 1} d\gamma = \log [\sigma^2 \Gamma + 1].
\]

Finally,

\[
\sum_i \frac{p_i}{2} \log [\sigma_i^2 \Gamma + 1] \leq I(x; y) \leq \frac{1}{2} \log [\sigma_x^2 \Gamma + 1].
\]
The upper bound is immediately recognized as the mutual information between a Gaussian source distribution with variance $\sigma^2 x \Gamma$ perturbed by unit variance Gaussian noise. Though we arrived at this through bounds on the MMSE, it is noted we can arrive at this result by exploiting the single-crossing property [144], which states that the MMSE of an arbitrarily distributed random variable with finite variance is strictly less than the equivalent MMSE for a Gaussian input with the same variance.

The lower bound is the weighted sum of the individual Gaussian mixture component mutual informations, which is derived from the “genie” bound for the MMSE [211]. This bound assumes the estimator has perfect knowledge of which component of the mixture is being drawn from at any given time [211]. This knowledge reduces uncertainty that would have otherwise been reduced through measurement, and so the estimation information is greater than or equal to this amount.

B.2 Multivariate Source, Gaussian Noise

Here we extend the univariate mixture model in Gaussian noise to vector measurements. That is, source distributions are now a vector quantity in general, and consist of correlated multi-dimensional GMMs. The general formula for the I-MMSE is given by Equation (9.12). The MMSE bounds in Reference [211] are shown in their vector form by Equation (9.8). Note one subtly here that must be addressed. The MMSE bounded in Reference [211] is the MMSE for $x$, while the I-MMSE formula given in Equation (9.12) references the MMSE for $H x$. This must be accounted for when converting Equation (9.8). To do so, we assign a new covariance matrix $\Lambda = G \Sigma G^T$. Then the equivalent ‘$H$’ in Equation (9.8) is simply the root SNR which modifies the whitened source distribution, which we denote $\sqrt{\Gamma}$ to maintain convention. Note we can simplify for the argument of the trace without loss of generality. Therefore, our
new problem is given by

\[ \text{Tr}(\cdot) = \text{Tr}\left( \Lambda - \Lambda \sqrt{\Gamma} \left( \sqrt{\Gamma} \Lambda \sqrt{\Gamma} + I \right)^{-1} \sqrt{\Gamma} \Lambda \right). \]  

(B.6)

Invoking the matrix inversion lemma, we get a simplified form:

\[ \text{Tr}(\cdot) = \text{Tr}\left( (G \Sigma G^T + \gamma I)^{-1} \right). \]  

(B.7)

With some simple manipulation [47], we can interchange the trace operator and the integral:

\[
\sum_i \frac{p_i}{2} \text{Tr}\left( \int_0^\Gamma (G \Sigma_i G^T + \gamma I)^{-1} d\gamma \right) \leq I(x; y) \leq \frac{1}{2} \text{Tr}\left( \int_0^\Gamma (G \Sigma G^T + \gamma I)^{-1} d\gamma \right). \quad (B.8)
\]

To solve for the trace, we note that the eigenvectors of \( G \Sigma G^T \) are the same as the argument of the inverse; only the eigenvalues change [229]. If the eigenvalues of \( G \Sigma G^T \) are given by \( \lambda_j \), then the eigenvalues of \( (G \Sigma G^T + \gamma I)^{-1} \) are given by [229]

\[ \lambda_j' = \frac{\lambda_j}{\gamma \lambda_j + 1}. \]  

(B.9)

Similar to the univariate case, the integral of each eigenvalue individually is then easily found:

\[ \lambda_j'' = \log[\gamma \lambda_j + 1]. \]  

(B.10)

Since the argument of the trace is now an eigenvalue decomposition, only the sum of the eigenvalues just defined emerge from the trace operation. Given the sum of logs can be defined as the log of the product, and the equality of the product of eigenvalues and the determinant of a matrix, we can define our bounds as follows:

\[
\sum_i \frac{p_i}{2} \log[|G \Sigma_i G^T \Gamma + I|] \leq I(x; y) \leq \frac{1}{2} \log[|G \Sigma G^T \Gamma + I|]. \quad (B.11)
\]
APPENDIX C

LIST OF ACRONYMS
1D  one-dimensional

2D  two-dimensional

3D  three-dimensional

ADC  analog-to-digital converter

ADS-B  automatic dependent surveillance-broadcast

ASK  amplitude-shift keying

AWGN  additive white Gaussian noise

BER  bit error rate

BPSK  binary phase-shift keying (PSK)

CDMA  code division multiple access

CIR  constant information radar

CPI  coherent processing interval

CRLB  Cramér-Rao lower bound

CSS  chirped spread spectrum

DARPA  the Defense Advanced Research Projects Agency

DE  differential evolution

DSSS  direct-sequence spread spectrum

EKF  extended Kalman filter

ELINT  electronic intelligence
FDOA  frequency difference of arrival

FIM  Fisher information matrix

FM  frequency-modulated

FMCW  frequency-modulated continuous-wave

GMM  Gaussian mixture model

GPS  Global Positioning System

GSM  Global System for Mobile Communications

IBFD  in-band full-duplex

ISB  isolated sub-band

ITS  intelligent transportation systems

$k$-NN  $k$-nearest neighbors

LTE  Long-Term Evolution

MAC  multiple access channel

MIMO  multiple-input multiple-output

MMSE  minimum mean square error

MSE  mean square error

MTI  moving target indicator

MUDR  multiuser detection radar

OFDM  orthogonal frequency-division multiplexing
**PAPR** peak-to-average power ratio

**PRF** pulse repetition frequency

**PRI** pulse repetition interval

**PSK** phase-shift keying

**PSP** principle of stationary phase

**RCS** radar cross-section

**RFID** radio-frequency identification

**RMS** root mean square

**SDR** software-defined radio

**SIC** successive interference cancellation

**SINR** signal-to-interference-plus-noise ratio

**SNR** signal-to-noise ratio

**SSPARC** shared spectrum access for radar and communications

**TBD** track-before-detect

**TDD** time-division duplexing

**TDM** time-division multiplexing

**UMTS** Universal Mobile Telecommunications System

**V2V** vehicle-to-vehicle

**WF** water-filling