Structure and Proper Orthogonal Decomposition in Simulations of Wall-Bounded Turbulent Shear Flows with Canonical Geometries

by

Jon Ronald Baltzer

A Dissertation Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy

Approved April 2012 by the Graduate Supervisory Committee:

Ronald J. Adrian, Chair
Ronald Calhoun
Anne Gelb
Marcus Herrmann
Kyle Squires

ARIZONA STATE UNIVERSITY

May 2012
ABSTRACT

Structural features of canonical wall-bounded turbulent flows are described using several techniques, including proper orthogonal decomposition (POD). The canonical wall-bounded turbulent flows of channels, pipes, and flat-plate boundary layers include physics important to a wide variety of practical fluid flows with a minimum of geometric complications. Recent studies establish the importance of very long motions of streamwise velocity fluctuation, but significant questions remain regarding their forms, how smaller motions may organize with respect to these larger motions, and the characteristics of associated vortical structures. POD extracts highly energetic structures from flow fields and is one tool to profitably analyze the direct numerical simulation fields considered in this study.

Since POD modes require significant interpretation, wall-normal, one-dimensional POD modes for a set of turbulent channel flows are used to establish important features. The modes’ scaling is interpreted in light of flow physics, also leading to a method of synthesizing one-dimensional POD modes. Properties of a pipe flow simulation are then studied via several methods. Statistical quantities, including energy spectra, are used to quantify the very long streamwise motions and compare with similar experiments. Further properties of energy spectra, including their relation to fictitious forces associated with mean Reynolds stress, are considered in depth. Turbulent structures in the pipe flow are then examined in the light of observations from relevant experiments. A variety of methods reveal organization patterns of structures in instantaneous fields and their associated vortical structures. POD modes for boundary layer flows are examined. Finally, very wide modes that occur when computing POD modes in all three canonical flows are compared.

The results demonstrate that POD extracts structures relevant to characterizing wall-bounded turbulent flows. However, significant care is necessary in interpreting POD results, and POD modes can be categorized according to their self-similarity.
Additional analysis techniques reveal the organization of smaller motions in characteristic patterns to compose very long motions in pipe flows. These very-large-scale motions are observed to contribute large fractions of turbulent kinetic energy and Reynolds stress. The associated vortical structures possess characteristics of hairpins, and their organization is considered in light of the hairpin packet model.
ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Ronald Adrian, for his guidance, support, and encouragement throughout my graduate research, as well as for the knowledge and research skills I have gained from him. In the course of my studies, he has provided freedom in pursuing many different aspects of this project while generously providing time to discuss research. He also made many opportunities available for me to attend and present at conferences. I would also like to thank Professor Xiaohua Wu for the opportunity to collaborate in analyzing pipe and boundary layer simulations and for helpful discussions. I am grateful to my dissertation committee for serving on my committee and for the knowledge I have gained from many of their classes.

I have had the privilege to learn from and work with a number of individuals in Professor Adrian’s group at ASU. Specifically, I thank Dr. Kyoungyoun Kim for many helpful discussions, teaching me many aspects of turbulent flow DNS, and his friendship as we have discussed research over the past years. Thanks are also due to Dr. Michael J. Murphy for teaching me much about experimental fluid methods and about how to teach the experimental methods laboratory, as well as for many interesting conversations as we worked together. I have also enjoyed interesting discussions with fellow graduate students and visitors in the lab including Venkatraman Radhakrishnan, Dr. Gerrit Elsinga, Isaac Ziskin, Praveen Kumar Parthasarathy, Goutam Hiranandani, Stefano Discetti, and Qiang Zhong. The discussions, assistance, and friendship from other fellow students both within the department and beyond who are too numerous to list have also been greatly appreciated throughout my studies.

Support of the National Science Foundation with Award CBET-0933848 is gratefully acknowledged. Although additional computational and funding resources are noted in the acknowledgments stated in the relevant chapters, the author
performed many computations were performed using the ASU Advanced Computing Center (A2C2), formerly known as Ira A. Fulton High Performance Computing Initiative, facilities at Arizona State University. The computer time is gratefully acknowledged. The author further wishes to acknowledge the experimental and computational data made available for comparison by a number of authors, who are listed specifically in the relevant sections.

I am grateful for the opportunity to participate in the Stanford University Center for Turbulence Research (Director Prof. Parviz Moin) 2010 Summer research program and the financial support they provided. The discussions with other visiting researchers were appreciated. I would specifically like to thank Dr. Johan Larsson of CTR for the interesting and informative conversations.

I would also like to thank Prof. Don Boyer, Prof. Kyle Squires, and Dr. Marion Vance for providing inspiration to pursue computational fluid dynamics and graduate study, as well as much helpful advice, during the course of my undergraduate studies. I am grateful to Dr. Lynn Cozort for her helpful advisement and looking out for me as I was joining the graduate program and in my graduate studies. Thanks are due to Durella O’Donnell for her administrative help and enjoyable conversations over the years.

Finally, I would like to thank my parents, Ronald and Darlene Baltzer, for their love and support over the years that I could never repay. Their great dedication to my education has been vital to my pursuit of graduate studies. I am thankful for their support throughout my studies and for all that they have taught me in every area of life.

Soli Deo gloria.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>LIST OF FIGURES</th>
<th>xii</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF TABLES</td>
<td>xxx</td>
</tr>
</tbody>
</table>

## CHAPTER

1 **INTRODUCTION**  2
   1.1 Overview of Present Study  9

2 **DIRECT NUMERICAL SIMULATIONS AND POD METHODOLOGY**  14
   2.1 Governing Equations and Numerical Simulation of Canonical Wall-bounded Turbulent Flows  14
   2.2 DNS Database  21
   2.3 Proper Orthogonal Decomposition Theory  24
      2.3.1 One-Dimensional POD  24
      2.3.2 Discrete Approximation  26
      2.3.3 Method of Snapshots  27
      2.3.4 Higher-Dimensional with Homogeneous Directions  28
      2.3.5 POD Domain  30

3 **STRUCTURE, SCALING, AND SYNTHESIS OF ONE-DIMENSIONAL POD MODES OF INHOMOGENEOUS TURBULENCE**  31
   3.1 Introduction  32
   3.2 Background  35
      3.2.1 POD Equations  35
      3.2.2 Data Sets  36
      3.2.3 POD Modes  37
   3.3 Reynolds Number Similarity of the Eigenvalue Spectra  40
      3.3.1 Outer Scaling  40
      3.3.2 Inner Scaling  42
<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3.3</td>
<td>Power Laws</td>
</tr>
<tr>
<td>3.4</td>
<td>Mode Decomposition</td>
</tr>
<tr>
<td>3.4.1</td>
<td>Decomposition Form</td>
</tr>
<tr>
<td>3.4.2</td>
<td>Hilbert Transform and Phase Zero Methods</td>
</tr>
<tr>
<td>3.4.3</td>
<td>Envelope Spline Method</td>
</tr>
<tr>
<td>3.5</td>
<td>Phase and Envelope Functions</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Results</td>
</tr>
<tr>
<td>3.5.2</td>
<td>Scales and Similarity</td>
</tr>
<tr>
<td>3.6</td>
<td>Mode Synthesis</td>
</tr>
<tr>
<td>3.7</td>
<td>Asymptotic POD Modes</td>
</tr>
<tr>
<td>3.8</td>
<td>Discussion</td>
</tr>
<tr>
<td>3.9</td>
<td>Conclusions</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>4.2</td>
<td>Computational Details</td>
</tr>
<tr>
<td>4.3</td>
<td>Validation of DNS Results</td>
</tr>
<tr>
<td>4.4</td>
<td>Energy Spectra</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Definitions</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Streamwise Spectra</td>
</tr>
<tr>
<td>4.4.3</td>
<td>Streamwise Cumulative Spectrum Wavelengths</td>
</tr>
<tr>
<td>4.4.4</td>
<td>Azimuthal Spectra</td>
</tr>
<tr>
<td>4.4.5</td>
<td>Two-Dimensional Spectra</td>
</tr>
<tr>
<td>4.5</td>
<td>Time Evolution</td>
</tr>
<tr>
<td>4.6</td>
<td>Space-Time Correlation</td>
</tr>
</tbody>
</table>

4 DIRECT NUMERICAL SIMULATION OF A 30R LONG TURBULENT PIPE FLOW AT \( R^+ = 685 \): LARGE- AND VERY LARGE-SCALE MOTIONS | 72  |
4.1 Introduction | 73  |
4.2 Computational Details | 78  |
4.3 Validation of DNS Results | 80  |
4.4 Energy Spectra | 83  |
4.4.1 Definitions | 83  |
4.4.2 Streamwise Spectra | 85  |
4.4.3 Streamwise Cumulative Spectrum Wavelengths | 103 |
4.4.4 Azimuthal Spectra | 108 |
4.4.5 Two-Dimensional Spectra | 110 |
4.5 Time Evolution | 112 |
4.6 Space-Time Correlation | 118 |
<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.7 Convection Velocity</td>
<td>124</td>
</tr>
<tr>
<td>4.8 Net Force Spectra</td>
<td>131</td>
</tr>
<tr>
<td>4.9 Conclusions</td>
<td>142</td>
</tr>
<tr>
<td>5 STRUCTURE OF LARGE AND VERY-LARGE SCALES IN TURBULENT PIPE FLOW SIMULATION</td>
<td>147</td>
</tr>
<tr>
<td>5.1 Introduction</td>
<td>148</td>
</tr>
<tr>
<td>5.2 Computational Details</td>
<td>155</td>
</tr>
<tr>
<td>5.3 Long Streaks of Streamwise Velocity Fluctuation</td>
<td>156</td>
</tr>
<tr>
<td>5.4 Radial Extent of Velocity Structures</td>
<td>166</td>
</tr>
<tr>
<td>5.5 Visualization of Structures</td>
<td>170</td>
</tr>
<tr>
<td>5.5.1 $x-r$ Planes</td>
<td>170</td>
</tr>
<tr>
<td>5.5.2 Estimate of Conditional Average for Swirl Event in the $x-r$ Plane</td>
<td>186</td>
</tr>
<tr>
<td>5.6 Vortices and Structures in 3D Volumes</td>
<td>194</td>
</tr>
<tr>
<td>5.6.1 Core Tracing</td>
<td>200</td>
</tr>
<tr>
<td>5.7 Streamwise-Azimuthal Organization of Structures</td>
<td>208</td>
</tr>
<tr>
<td>5.7.1 Conditional Average: General $u'$ Event and Two-point Correlation</td>
<td>209</td>
</tr>
<tr>
<td>5.7.2 Other Studies of Correlation in Wall-bounded Turbulence</td>
<td>209</td>
</tr>
<tr>
<td>5.7.3 Conditional Average: General $u'$ Event and Two-point Correlation Results</td>
<td>210</td>
</tr>
<tr>
<td>5.7.4 Conditional Average on Long Negative $u'$ Regions</td>
<td>214</td>
</tr>
<tr>
<td>5.7.5 Model of Streaks</td>
<td>217</td>
</tr>
<tr>
<td>5.7.6 Interpretation of Organization</td>
<td>219</td>
</tr>
<tr>
<td>5.7.7 Three-Dimensional Correlation</td>
<td>222</td>
</tr>
<tr>
<td>5.8 Azimuthal Structure and Scale Growth</td>
<td>225</td>
</tr>
<tr>
<td>CHAPTER</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>5.9</td>
<td>234</td>
</tr>
<tr>
<td>6</td>
<td>237</td>
</tr>
<tr>
<td>6.1</td>
<td>237</td>
</tr>
<tr>
<td>6.2</td>
<td>240</td>
</tr>
<tr>
<td>6.3</td>
<td>241</td>
</tr>
<tr>
<td>6.4</td>
<td>251</td>
</tr>
<tr>
<td>7</td>
<td>252</td>
</tr>
<tr>
<td>7.1</td>
<td>252</td>
</tr>
<tr>
<td>7.2</td>
<td>254</td>
</tr>
<tr>
<td>7.3</td>
<td>256</td>
</tr>
<tr>
<td>7.4</td>
<td>266</td>
</tr>
<tr>
<td>8</td>
<td>267</td>
</tr>
<tr>
<td>8.1</td>
<td>268</td>
</tr>
<tr>
<td>8.2</td>
<td>270</td>
</tr>
<tr>
<td>8.3</td>
<td>271</td>
</tr>
<tr>
<td>8.4</td>
<td>276</td>
</tr>
<tr>
<td>8.5</td>
<td>282</td>
</tr>
<tr>
<td>9</td>
<td>284</td>
</tr>
<tr>
<td>9.1</td>
<td>284</td>
</tr>
<tr>
<td>9.2</td>
<td>284</td>
</tr>
<tr>
<td>9.3</td>
<td>285</td>
</tr>
<tr>
<td>9.4</td>
<td>287</td>
</tr>
<tr>
<td>9.5</td>
<td>292</td>
</tr>
</tbody>
</table>
# APPENDIX

## A ADDITIONAL ASPECTS OF ONE-DIMENSIONAL POD MODES OF TURBULENT CHANNELS

A.1 Further Details on Phase/Envelope Decomposition Methods .......................... 309
  A.1.1 Hilbert Transform Method .................................................. 309
  A.1.2 Hilbert Transform Testing .................................................. 311
  A.1.3 Phase Zero Method .......................................................... 321
  A.1.4 Envelope Spline Method .................................................... 322
A.2 Recursive Relation for POD Modes .................................................. 327
  A.2.1 Theory ................................................................. 327
    A.2.1.1 Motivation ......................................................... 327
    A.2.1.2 Recursion Relation for Synthetic POD ............................ 328
    A.2.1.3 Generalization .................................................... 329
    A.2.1.4 Orthogonality in Recurrence Relation ............................ 330
  A.2.2 Implementation .......................................................... 331
    A.2.2.1 $y$-Multiplied Results ............................................. 331
    A.2.2.2 Generalized Recursion Formula Results .......................... 335

## B ADDITIONAL PIPE ENERGY SPECTRA COMPARISONS

B.1 Energy Normalization .......................................................... 340
B.2 Energy in the $k_x = 0$-modes ............................................... 342
B.3 Definition of Reynolds Numbers ............................................. 343
B.4 Kim & Adrian (1999) Reynolds Numbers .................................... 343
B.5 Rescaling of KA99 Spectra .................................................. 345
  B.5.1 Outer Scaled Spectra .................................................... 346
<table>
<thead>
<tr>
<th>APPENDIX</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.5.2 Inner Scaled Spectra</td>
<td>349</td>
</tr>
<tr>
<td>B.6 Spectra Comparison of KA99 with Perry &amp; Abell (1975)</td>
<td>350</td>
</tr>
<tr>
<td>B.7 Spectra Comparison of KA99 with Guala et al. (2006)</td>
<td>353</td>
</tr>
<tr>
<td>B.8 Spectra Calculations for Balakumar &amp; Adrian (2007)</td>
<td>354</td>
</tr>
<tr>
<td>B.9 KA99 Scaling Summary</td>
<td>355</td>
</tr>
<tr>
<td>B.10 Guala et al. (2006) Spectra and Cumulative Wavelengths</td>
<td>355</td>
</tr>
<tr>
<td>B.12 Comparison of DNS with Experiments</td>
<td>360</td>
</tr>
<tr>
<td>B.13 Comparison of Pipe DNS with Channels</td>
<td>364</td>
</tr>
<tr>
<td>B.14 Uncertainty of Energy Spectra</td>
<td>369</td>
</tr>
<tr>
<td>C NET FORCE BALANCES FOR PIPE AND CHANNEL</td>
<td>371</td>
</tr>
<tr>
<td>C.1 Channel Mean Axial Momentum Equation</td>
<td>372</td>
</tr>
<tr>
<td>C.2 Pipe Mean Axial Momentum Equation</td>
<td>372</td>
</tr>
<tr>
<td>C.3 Total Shear Stress</td>
<td>373</td>
</tr>
<tr>
<td>C.4 Far-wall Mean Force Balance for Channel</td>
<td>375</td>
</tr>
<tr>
<td>C.5 Far-wall Mean Force Balance for Pipe</td>
<td>376</td>
</tr>
<tr>
<td>C.6 Pipe DNS Results</td>
<td>376</td>
</tr>
<tr>
<td>D EXAMPLE x–r PLANES OF PIPE SIMULATION</td>
<td>379</td>
</tr>
<tr>
<td>D.1 Streak A of $t = 252R/U_{bulk}$ field</td>
<td>383</td>
</tr>
<tr>
<td>D.2 Streak B of $t = 252R/U_{bulk}$ field</td>
<td>389</td>
</tr>
<tr>
<td>D.3 Streak C of $t = 252R/U_{bulk}$ field</td>
<td>392</td>
</tr>
<tr>
<td>D.4 Streak D of $t = 252R/U_{bulk}$ field</td>
<td>396</td>
</tr>
<tr>
<td>D.5 Streak E of $t = 252R/U_{bulk}$ field</td>
<td>398</td>
</tr>
<tr>
<td>D.6 Streak F of $t = 252R/U_{bulk}$ field</td>
<td>405</td>
</tr>
<tr>
<td>D.7 Streak G of $t = 252R/U_{bulk}$ field</td>
<td>408</td>
</tr>
<tr>
<td>D.8 Streak H of $t = 252R/U_{bulk}$ field</td>
<td>415</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------------------</td>
<td>-------</td>
</tr>
<tr>
<td>E.1 Author Permissions for Dissertation Material</td>
<td>422</td>
</tr>
<tr>
<td>E.2 Publisher Permissions for Dissertation Material</td>
<td>423</td>
</tr>
<tr>
<td>E.3 Permissions for Figures Reproduced from Other Publications</td>
<td>426</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Characteristic eddy reproduced from Moin &amp; Moser (1989)</td>
<td>6</td>
</tr>
<tr>
<td>1.2</td>
<td>Reproduced from Zhou <em>et al.</em> (1999): vector plots in cross-sections of a</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>conditional eddy obtained by linear stochastic estimation</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>Reproduced from Adrian <em>et al.</em> (2000): vector plot of streamwise-wall</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>normal plane of a turbulent boundary layer ($Re_\theta = 930$) with constant</td>
<td></td>
</tr>
<tr>
<td></td>
<td>velocity subtracted to illustrate hairpin vortex signature cross-sections.</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>Reproduced from Liu <em>et al.</em> (2001): vector plot of streamwise-wall normal</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>plane for a POD mode obtained from a turbulent channel.</td>
<td></td>
</tr>
<tr>
<td>2.1</td>
<td>Canonical geometries for wall-bounded turbulent flows of (a) channel,</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>(b) pipe, and (c) boundary layer, after Pope (2000).</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>Comparison of $n = 1$–4 POD modes for the streamwise velocity</td>
<td>38</td>
</tr>
<tr>
<td>3.2</td>
<td>Comparison of $n = 20$ POD modes for several Reynolds numbers.</td>
<td>39</td>
</tr>
<tr>
<td>3.3</td>
<td>Contour plot of 1D POD modes for $u$ component of $Re_\tau = 395$ turbulent</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>channel.</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>Comparison of eigenvalue spectra for the channels at all Reynolds numbers</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>considered with outer and inner scaling.</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>Unscaled and scaled POD mode phase of the $Re_\tau = 395$ channel.</td>
<td>47</td>
</tr>
<tr>
<td>3.6</td>
<td>Scaled phase of the $n = 1$ to 5 modes for the $Re_\tau = 395$ channel</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>obtained using the envelope spline and Hilbert transform methods.</td>
<td></td>
</tr>
<tr>
<td>3.7</td>
<td>Envelope functions of the channels comparing the Reynolds number effects.</td>
<td>49</td>
</tr>
<tr>
<td>3.8</td>
<td>Scaled phase of the idiosyncratic POD modes of the channels and thermal</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>convection.</td>
<td></td>
</tr>
<tr>
<td>3.9</td>
<td>Scaled POD mode phase of the channels for $n = 50$ to 100 modes comparing</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>the Reynolds number effects.</td>
<td></td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>3.10</td>
<td>Comparison of POD mode phase functions with different velocity components for the ( Re_\tau = 395 ) channel and convection ( u ) modes.</td>
<td>53</td>
</tr>
<tr>
<td>3.11</td>
<td>Comparison of POD modes and their SPOD counterparts for ( u ) velocity of the ( Re_\tau = 395 ) channel.</td>
<td>56</td>
</tr>
<tr>
<td>3.12</td>
<td>Comparison of the actual zeros and the zeros obtained using POD mode synthesis and the method of Carbone &amp; Aubry (1996).</td>
<td>57</td>
</tr>
<tr>
<td>3.13</td>
<td>Comparison of energy in partial reconstructions for the channels: modal energy contribution and residual energy in reconstruction.</td>
<td>60</td>
</tr>
<tr>
<td>3.14</td>
<td>Comparison of phase for ( u ) velocity modes ( n = 20 ) to 100 of turbulent channels with phase predicted by asymptotic theory of Moser (1994).</td>
<td>63</td>
</tr>
<tr>
<td>3.15</td>
<td>Comparison of ( R_{uu} ) normalized by the centerline variance for ( Re_\tau = 180, 395, ) and 934 turbulent channels and turbulent convection.</td>
<td>68</td>
</tr>
<tr>
<td>4.1</td>
<td>Comparison of wavelengths for spectral peaks in premultiplied streamwise velocity energy spectra for the present DNS and various experiments as a function of wall-normal distance.</td>
<td>75</td>
</tr>
<tr>
<td>4.2</td>
<td>(a) Mean velocity as a function of ( y^+ ). (b) Mean velocity gradients as a function of ( y^+ ) for the present DNS.</td>
<td>81</td>
</tr>
<tr>
<td>4.3</td>
<td>Turbulence intensities and shear stress as a function of ( y/R ).</td>
<td>81</td>
</tr>
<tr>
<td>4.4</td>
<td>Two-point correlation coefficient ( R_{uu} ) as a function of azimuthal separation ( \Delta \theta ).</td>
<td>83</td>
</tr>
<tr>
<td>4.5</td>
<td>Comparison of premultiplied energy spectra at ( y/R = 0.1 ) for the present DNS at ( Re_D = 24 \ 580 ) (black line) with comparable experiments: hot-wire spectra obtained by Hultmark et al. (2010) at ( Re_D = 25 \ 000 ) (filled squares) and hot-film measurements obtained by the authors of KA99 (dashed line).</td>
<td>86</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>Premultiplied one-dimensional spectra as a function of streamwise wavenumber for the present DNS for each velocity component: (a) $\Phi_{uu}$ for $y^+ \leq 60$, (b) $\Phi_{uu}$ for $y^+ \geq 60$, (c) $\Phi_{vv}$, and (d) $\Phi_{ww}$.</td>
<td>91</td>
</tr>
<tr>
<td>4.7</td>
<td>One-dimensional spectra as a function of streamwise wavenumber for the present DNS at $y^+ = 77.5$ ($y/R = 0.113$). Solid: $\Phi_{uu}$; long dashed: $\Phi_{vv}$; dash-dot: $\Phi_{ww}$; short dashed: $-1$ slope; dash-dot-dot: $-5/3$ slope.</td>
<td>92</td>
</tr>
<tr>
<td>4.8</td>
<td>Comparisons of premultiplied $uu$ energy maps with experiments (a,b) and a lower $R^+$ simulation (c). $k_x \Phi_{uu}/u_*^2$ contours are displayed.</td>
<td>94</td>
</tr>
<tr>
<td>4.9</td>
<td>Contour maps of premultiplied spectra $k_x \Phi(\lambda_x)$ for (a) $uu$, (b) $uv$, (c) $vv$, and (d) $ww$.</td>
<td>101</td>
</tr>
<tr>
<td>4.10</td>
<td>Wavelengths corresponding to $uu$ and $uv$ cumulative spectrum values of (a) $\Upsilon = 0.5$ and (b) $\Upsilon = 0.8$. Black solid lines depict $uu$ for the pipe DNS, grey dash-dot-dot lines are $uv$ for the pipe DNS, and symbols represent $uu$ for pipe experiments.</td>
<td>104</td>
</tr>
<tr>
<td>4.11</td>
<td>Contour maps of premultiplied one-dimensional spectra as a function of azimuthal arc length ($s$) wavelength. Contour colours and black contour lines represent the premultiplied spectra for (a) $k_x \Phi_{uu}$, (b) $k_x \Phi_{uv}$, (c) $k_x \Phi_{vv}$, and (d) $k_x \Phi_{ww}$.</td>
<td>111</td>
</tr>
<tr>
<td>4.12</td>
<td>Premultiplied two-dimensional energy spectrum of streamwise velocity at $y^+ \approx 100$ for channel simulations and the present pipe simulation.</td>
<td>113</td>
</tr>
<tr>
<td>4.13</td>
<td>Contours of $u'/U_{bulk}$ shaded from $-0.2$ (black) to $0.2$ (white) in planes at $y^+ = 80$ with $x/R$ and $s/R$ equally scaled.</td>
<td>114</td>
</tr>
<tr>
<td>4.14</td>
<td>Contours of filtered $u'/U_{bulk}$ ($\lambda_x \geq 6R$ and $\lambda_\theta \geq (2/5)\pi$) shaded from $-0.1$ (black) to $0.1$ (white) in planes at $y^+ = 80$ with $x/R$ and $s/R$ equally scaled.</td>
<td>115</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>4.15</td>
<td>Space-time correlation for the present pipe simulation. (a) $R_{uu}(\tau,r_x)/u^2$ contour lines for $y^+ = 50$ (translated upward by 10) and $y^+ = 101$. (b) The unfiltered (solid line) and filtered (dashed line) correlations for $y^+ = 101$ at a sequence of times.</td>
<td>120</td>
</tr>
<tr>
<td>4.16</td>
<td>Comparison of axial convection velocities ($u_c$) calculated for the present pipe DNS.</td>
<td>126</td>
</tr>
<tr>
<td>4.17</td>
<td>Colour contours of the sum of both net Reynolds force spectrum terms for the present DNS. (a) is presented on a linear $y/R$ axis, and (b) uses a logarithmically scaled $y/R$ axis with overlaid $uv$ spectrum contour lines.</td>
<td>134</td>
</tr>
<tr>
<td>4.18</td>
<td>Colour contours of two-dimensional force spectra with contour lines of two-dimensional $uv$ spectra.</td>
<td>141</td>
</tr>
<tr>
<td>5.1</td>
<td>Axial velocity fluctuation $u'/U_{bulk}$ contours of a streamwise-azimuthal cylinder at $y/R = 0.15$ and $y^+ = 101$.</td>
<td>158</td>
</tr>
<tr>
<td>5.2</td>
<td>Black regions of strong negative $u'$ fluctuation in a streamwise-azimuthal cylinder at $y/R = 0.15$ and $y^+ = 101$.</td>
<td>160</td>
</tr>
<tr>
<td>5.3</td>
<td>Histograms of streamwise length of contiguous regions of negative $u'$ fluctuation stronger than a threshold for the streamwise-azimuthal cylinder surface at $y^+ = 101$.</td>
<td>161</td>
</tr>
<tr>
<td>5.4</td>
<td>Histograms of streamwise length of contiguous regions of negative $u'$ fluctuation as in figure 5.3, except $y^+ = 30$.</td>
<td>162</td>
</tr>
<tr>
<td>5.5</td>
<td>Example streamwise velocity fluctuation at $t = 324R/U_{bulk}$ with isosurfaces in three dimensions and grey contours for (b) $y^+ = 20$, (c) $y^+ = 80$, and (d) $y^+ = 270$.</td>
<td>168</td>
</tr>
<tr>
<td>5.6</td>
<td>Three-dimensional isosurfaces of negative $u$ fluctuation ($u'/U_{bulk} = -0.15$) for the pipe section containing low speed streaks A and B.</td>
<td>169</td>
</tr>
<tr>
<td>5.7</td>
<td>Streak A at $t = 252R/U_{bulk}$ with $u - 0.83U_{bulk}$ vectors.</td>
<td>177</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>Uniform momentum zones visualized by contours of $(u-0.88U_{\text{bulk}})/U_{\text{bulk}}$ for (a) streak A and (b) streak B.</td>
<td>181</td>
</tr>
<tr>
<td>5.9</td>
<td>Urushihara et al. (1993) experimental measurement of the $Re_D = 50000$ ($R^+ = 1300$) turbulent pipe flow, reproduced from Adrian et al. (2000a).</td>
<td>185</td>
</tr>
<tr>
<td>5.10</td>
<td>Linear stochastic estimate based on an unsigned 2D swirl event located at $y/R = 0.15$ and $y^+ = 101$.</td>
<td>188</td>
</tr>
<tr>
<td>5.11</td>
<td>(a) Streak A section of figure 5.1 at $y/R = 0.15$ and $y^+ = 101$ with contour lines of $\lambda_{ci}$. $\lambda_{ci}$ isosurfaces surround $u' = -0.21U_{\text{bulk}}$ isosurfaces for (b) streak A and (c) streak B.</td>
<td>196</td>
</tr>
<tr>
<td>5.12</td>
<td>Streak A region displaying white isosurfaces of normalized $\lambda_{ci}/\sigma_{\lambda} = 3$ with a plane at $y^+ = 59$ coloured by streamwise velocity fluctuation and black contour lines of $\lambda_{ci}$ on this plane to indicate the presence of vortex cores.</td>
<td>198</td>
</tr>
<tr>
<td>5.13</td>
<td>Section surrounding negative-velocity fluctuation streak A displaying (a) vortex lines to trace cores compared to scaled $\lambda_{ci}$ isosurfaces and (b) $u'$ isosurfaces with vortex lines.</td>
<td>204</td>
</tr>
<tr>
<td>5.14</td>
<td>Section surrounding streak A displaying $u'$ isosurfaces with vortex lines from a viewing angle to compare the azimuthal positions of the vortex line bundles with those of negative velocity fluctuations.</td>
<td>205</td>
</tr>
<tr>
<td>5.15</td>
<td>Contours of two-point correlation coefficient $R_{uu}(\Delta x, \Delta s)$ in $x-\theta$ cylinder surfaces.</td>
<td>211</td>
</tr>
<tr>
<td>5.16</td>
<td>(a) Conditional average of streamwise velocity fluctuation $u'$ conditioned on contiguous negative $u'$ region events. Also included are a synthetic $u'$ field (a) and associated two-point correlations(c,d).</td>
<td>215</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>5.17</td>
<td>An idealized depiction of concatenated low-velocity streaks with hairpin vortices surrounding the streaks.</td>
<td>221</td>
</tr>
<tr>
<td>5.18</td>
<td>Three-dimensional two-point correlation $R_{uu}$ for reference (event) location of $y^+ = 30$.</td>
<td>223</td>
</tr>
<tr>
<td>5.19</td>
<td>Three-dimensional two-point correlation $R_{uu}$ for reference (event) locations of $y^+ = 101$ (a,b) and $y^+ = 250$ (c,d).</td>
<td>224</td>
</tr>
<tr>
<td>5.20</td>
<td>Example plane at fixed $x$ position for an instantaneous field of the present pipe simulation showing the in-plane velocity with vectors and the plane-normal velocity ($u'$) fluctuations.</td>
<td>227</td>
</tr>
<tr>
<td>5.21</td>
<td>Contours of two-point correlation $R_{uu}(\Delta x = 0, r, r_{ref}, \Delta \theta)$ and LSEs of $-u'$ events.</td>
<td>229</td>
</tr>
<tr>
<td>5.22</td>
<td>Comparison of the azimuthal (arc length) or spanwise distance $l_z$ between the points at which the correlation coefficient $R_{uu}(\delta z, y)$ has decayed from the peak to 0.05.</td>
<td>232</td>
</tr>
<tr>
<td>6.1</td>
<td>Eigenvalue (energy content) spectrum of the $n = 1$ (most energetic) modes for each wavenumber index pair of the present $Re_\tau = 685$ pipe simulation.</td>
<td>242</td>
</tr>
<tr>
<td>6.2</td>
<td>Eigenvalue (energy content) spectrum of the $n = 1$ (most energetic) modes for each wavenumber index pair of a $Re_\tau = 150$ pipe simulation, obtained from data presented in Duggleby et al. (2007).</td>
<td>243</td>
</tr>
<tr>
<td>6.3</td>
<td>Modulus of the eigenmode for the $(1, 5, 1)$ mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$.</td>
<td>245</td>
</tr>
<tr>
<td>6.4</td>
<td>Modulus of the eigenmode for the $(2, 2, 1)$ mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$.</td>
<td>245</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>245</td>
<td></td>
</tr>
<tr>
<td>6.6</td>
<td>246</td>
<td></td>
</tr>
<tr>
<td>6.7</td>
<td>249</td>
<td></td>
</tr>
<tr>
<td>6.8</td>
<td>249</td>
<td></td>
</tr>
<tr>
<td>6.9</td>
<td>250</td>
<td></td>
</tr>
<tr>
<td>7.1</td>
<td>259</td>
<td></td>
</tr>
<tr>
<td>7.2</td>
<td>260</td>
<td></td>
</tr>
<tr>
<td>7.3</td>
<td>260</td>
<td></td>
</tr>
<tr>
<td>7.4</td>
<td>262</td>
<td></td>
</tr>
<tr>
<td>7.4</td>
<td>263</td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td>264</td>
<td></td>
</tr>
<tr>
<td>7.6</td>
<td>264</td>
<td></td>
</tr>
<tr>
<td>7.7</td>
<td>265</td>
<td></td>
</tr>
</tbody>
</table>

- Figure 6.5: Modulus of the eigenmode for the $(1,1,1)$ mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$.  
- Figure 6.6: Modulus of the eigenmode for the $(1,0,1)$ mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$.  
- Figure 6.7: Visualization of the streamwise velocity fluctuation for the $(1,2,1)$ eigenmode of the present pipe flow.  
- Figure 6.8: Visualization of the streamwise velocity fluctuation for the $(2,3,1)$ eigenmode of the present pipe flow.  
- Figure 6.9: Visualization of the streamwise velocity fluctuation for the $(1,5,1)$ eigenmode of the present pipe flow.  
- Figure 7.1: Streamwise velocity isosurfaces of POD modes arranged in sets of constant $k_z$ for selected mode numbers. The modes are viewed obliquely from above the flat plate.  
- Figure 7.2: Streamwise velocity isosurfaces of POD modes for $k_z = 20$ and $n = 1, 5, 10, 20$ (top to bottom).  
- Figure 7.3: Planes of the $n = 1$ (a,b) and $n = 4$ (c,d) $k_z = 0$ modes shaded by streamwise velocity for each streamwise half of the domain. The dotted white line indicates the boundary layer thickness $\delta(x)$.  
- Figure 7.4: Mode velocity of (a) $k_z = 1, n = 1$, (b) $k_z = 5, n = 1$, (c) $k_z = 5, n = 5$, (d) $k_z = 10, n = 5$, (e) $k_z = 10, n = 5$, (f) $k_z = 20, n = 1$.  
- Figure 7.5: $u$ isosurfaces for (a) $k_z = 5, n = 1$ and (b) $k_z = 20, n = 1$ POD modes.  
- Figure 7.6: POD mode eigenvalues.  
- Figure 7.7: Negative $u$ isosurfaces (shaded by $y$) of one DNS field (a,b) and its reconstruction with $k_z = 0-5$ and $n = 1-16$ POD modes (c,d).
8.1 Comparison of $\lambda_{ci}$ isosurfaces in sample fields of the data sets: (a) Ferrante et al. (2004) and (b) Wu & Moin (2010). . . . . . . . . . . . . . . 269

8.2 Comparison of negative $u'$ isosurfaces in sample fields of the data sets: (a) Ferrante et al. (2004) and (b) Wu & Moin (2010). . . . . . . . . . . 270

8.3 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 1$. . . . . . . . . . . . . . . 272

8.4 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 2$. . . . . . . . . . . . . . . 272

8.5 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 3$. . . . . . . . . . . . . . . 273

8.6 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 4$. . . . . . . . . . . . . . . 273

8.7 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 1$. . . . . . . . . . . . . . . 273

8.8 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 2$. . . . . . . . . . . . . . . 274

8.9 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 3$. . . . . . . . . . . . . . . 274

8.10 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 4$. . . . . . . . . . . . . . . 275

8.11 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 5$ and $n = 1$. . . . . . . . . . . . . . . 275

8.12 Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 5$ and $n = 2$. . . . . . . . . . . . . . . 276


<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.15</td>
<td>$k_z = 0$ mode comparison.</td>
</tr>
<tr>
<td>8.16</td>
<td>Lowest $k_z$ modes comparison.</td>
</tr>
<tr>
<td>8.17</td>
<td>$k_z = 5$ mode comparison.</td>
</tr>
<tr>
<td>8.18</td>
<td>Detail of a ramp in the $k_z n = 1$ mode of WM: (a) isosurface visualization and (b) vector plot through its center.</td>
</tr>
<tr>
<td>8.19</td>
<td>Vector plot through a ramp in the $k_z n = 1$ mode of a partly-converged full domain POD computation for WM.</td>
</tr>
<tr>
<td>8.20</td>
<td>$k_z = 20$ mode comparison.</td>
</tr>
<tr>
<td>9.1</td>
<td>$Re_{\tau} = 395$ channel: $u_i(x,y,z_{ref})$ in left and $\tilde{u}_i(x,y)$ in right column.</td>
</tr>
<tr>
<td>9.2</td>
<td>$Re_{\tau} = 395$ channel: $u_i(x,y,z_{ref})$ color contours and $\tilde{u}_i(x,y)$ line contours.</td>
</tr>
<tr>
<td>9.3</td>
<td>$k_z = 0$ or $k_{\theta} = 0$ modal energy contribution fraction for each velocity component.</td>
</tr>
<tr>
<td>9.4</td>
<td>$Re_{\tau} = 395$ channel two-point correlations for $y_{ref} = 0.15h$.</td>
</tr>
<tr>
<td>9.5</td>
<td>$Re_{\tau} = 685$ pipe two-point correlations for $y_{ref} = 0.15R$.</td>
</tr>
<tr>
<td>9.6</td>
<td>$Re_{\tau} = 395$ channel two-point correlations for $y_{ref} = 0.50h$.</td>
</tr>
<tr>
<td>9.7</td>
<td>$Re_{\tau} = 685$ pipe two-point correlations for $y_{ref} = 0.50R$.</td>
</tr>
<tr>
<td>A.1</td>
<td>Hilbert transform test on $n = 20$ sinusoidal function: phase</td>
</tr>
<tr>
<td>A.2</td>
<td>Hilbert transform test on $n = 20$ sinusoidal function: envelope</td>
</tr>
<tr>
<td>A.3</td>
<td>Hilbert transform test on $n = 20$ sinusoidal function: phase error</td>
</tr>
<tr>
<td>A.4</td>
<td>Hilbert transform test on $n = 2$ sinusoidal function: phase</td>
</tr>
<tr>
<td>A.5</td>
<td>Hilbert transform test on $n = 2$ sinusoidal function: envelope</td>
</tr>
<tr>
<td>A.6</td>
<td>Hilbert transform test on $n = 2$ sinusoidal function: phase error</td>
</tr>
<tr>
<td>A.7</td>
<td>Hilbert transform test on $n = 50$ sinusoidal function: phase</td>
</tr>
<tr>
<td>A.8</td>
<td>Hilbert transform test on $n = 50$ sinusoidal function: envelope</td>
</tr>
<tr>
<td>A.9</td>
<td>Hilbert transform test on $n = 50$ sinusoidal function: phase error</td>
</tr>
<tr>
<td>A.10</td>
<td>Hilbert transform test on $n = 20$ synthetic POD function: phase</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>A.11 Hilbert transform test on ( n = 20 ) synthetic POD function: envelope</td>
<td>317</td>
</tr>
<tr>
<td>A.12 Hilbert transform test on ( n = 20 ) synthetic POD function: phase error</td>
<td>318</td>
</tr>
<tr>
<td>A.13 Hilbert transform test on ( n = 2 ) synthetic POD function: phase</td>
<td>318</td>
</tr>
<tr>
<td>A.14 Hilbert transform test on ( n = 2 ) synthetic POD function: envelope</td>
<td>319</td>
</tr>
<tr>
<td>A.15 Hilbert transform test on ( n = 2 ) synthetic POD function: phase error</td>
<td>319</td>
</tr>
<tr>
<td>A.16 Hilbert transform test on ( n = 50 ) synthetic POD function: phase</td>
<td>320</td>
</tr>
<tr>
<td>A.17 Hilbert transform test on ( n = 50 ) synthetic POD function: envelope</td>
<td>320</td>
</tr>
<tr>
<td>A.18 Hilbert transform test on ( n = 50 ) synthetic POD function: phase error</td>
<td>321</td>
</tr>
<tr>
<td>A.19 Example raw phase obtained using the envelope spline method ((Re_\tau = 395) channel for ( n = 5 ) mode) with insets depicting two example scenarios.</td>
<td>325</td>
</tr>
<tr>
<td>A.20 Mode 41</td>
<td>333</td>
</tr>
<tr>
<td>A.21 Mode 41 Error by Recurrence Relation with ( y )-multiplication and Optimal Parameters</td>
<td>334</td>
</tr>
<tr>
<td>A.22 ( \alpha_n ) that provides optimal ( \phi^{(n+1)}(y) ) vs. ( n )</td>
<td>334</td>
</tr>
<tr>
<td>A.23 ( \beta_n ) that provides optimal ( \phi^{(n+1)}(y) ) vs. ( n )</td>
<td>335</td>
</tr>
<tr>
<td>A.24 ( f(y) ) for ( n=40 ) assuming optimal ( \alpha ) and ( \beta ) calculated from ( f(y) = y )</td>
<td>336</td>
</tr>
<tr>
<td>A.25 Recursion relation ( \alpha_n f^{(n)}(y) )</td>
<td>337</td>
</tr>
<tr>
<td>B.1 Fraction of energy in the ( k_x = 0 ) mode for the pipe DNS.</td>
<td>342</td>
</tr>
<tr>
<td>B.2 KA99 premultiplied spectra plot with extracted lines (dashed red) representing lower and upper envelopes of the ( Re_{D,cl} = 33\ 800 ) family of spectrum lines.</td>
<td>347</td>
</tr>
<tr>
<td>B.3 KA99 premultiplied spectra plot with inner scaled wavenumber.</td>
<td>351</td>
</tr>
<tr>
<td>B.4 Premultiplied energy spectra from KA99 pipe experiments with the vertical axis at the original scale.</td>
<td>352</td>
</tr>
<tr>
<td>B.5 Experimental pipe spectrum map of Monty et al. (2009) with lines for digitized PA75 and Guala et al. (2006) experimental spectra.</td>
<td>356</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>B.6</td>
<td>357</td>
</tr>
<tr>
<td>Comparison of streamwise velocity spectra for PA75 pipe experiments at two Reynolds numbers and Hoyas &amp; Jiménez (2006) channel DNS at $Re_\tau = 2003$ for $y/R$ or $y/h$ of 0.2.</td>
<td></td>
</tr>
<tr>
<td>B.7</td>
<td>358</td>
</tr>
<tr>
<td>Comparison of streamwise velocity spectra for PA75 pipe experiments at two Reynolds numbers and Hoyas &amp; Jiménez (2006) channel DNS at $Re_\tau = 2003$ for $y/R$ or $y/h$ of 0.2.</td>
<td></td>
</tr>
<tr>
<td>B.8</td>
<td>359</td>
</tr>
<tr>
<td>Comparison of wavelengths for (a) 0.5 and (b) 0.8 cumulative energy for streamwise velocity in comparable pipe experiments to Guala et al. (2006).</td>
<td></td>
</tr>
<tr>
<td>B.9</td>
<td>361</td>
</tr>
<tr>
<td>Adapted from Figure 5(b) of Hultmark et al. (2010) showing the energy spectra at the inner peak radius, including unfilled circles representing the $Re_D = 25000$ measurements. The spectrum for the pipe DNS at the inner peak location ($y^+ = 14.4$) is superimposed as a red line.</td>
<td></td>
</tr>
<tr>
<td>B.10</td>
<td>361</td>
</tr>
<tr>
<td>Comparison of energy spectra for the present DNS, den Toonder &amp; Nieuwstadt (1997) LDV measurements also at $Re_D = 24580$, and Hultmark et al. (2010) hot wire measurements for $Re_D = 25000$ pipe flow, all for $y^+ \approx 12$.</td>
<td></td>
</tr>
<tr>
<td>B.11</td>
<td>362</td>
</tr>
<tr>
<td>Comparison of energy spectra for the present DNS, den Toonder &amp; Nieuwstadt (1997) LDV measurements also at $Re_D = 24580$, and Hultmark et al. (2010) hot wire measurements for $Re_D = 25000$ pipe flow, all for $y^+ \approx 30$.</td>
<td></td>
</tr>
<tr>
<td>B.12</td>
<td>363</td>
</tr>
<tr>
<td>Premultiplied streamwise velocity spectra are compared for the present DNS, KA99 measurements, and Hultmark et al. (2010) measurements.</td>
<td></td>
</tr>
<tr>
<td>Figure/Label</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>-------------</td>
</tr>
<tr>
<td>B.13</td>
<td>Comparison of energy spectra at $y/R = 0.1$ and $y/h = 0.1$ for the present DNS, channel measurements of Balakumar &amp; Adrian (2007), a channel hot-wire measurement of Monty &amp; Chong (2009), and a spectrum calculated from a single channel field of del Álamo et al. (2004).</td>
</tr>
<tr>
<td>B.14</td>
<td>Comparison of energy spectra at $y/R = 0.3$ and $y/h = 0.3$ for the present DNS, channel measurements of Balakumar &amp; Adrian (2007), and a channel DNS spectrum of del Álamo et al. (2004) presented in Monty &amp; Chong (2009).</td>
</tr>
<tr>
<td>B.15</td>
<td>Comparison of channel DNS (with $Re_\tau$ values indicated in the legend) energy spectra at $y^+ \approx 100$ with inner scaling of wavenumber.</td>
</tr>
<tr>
<td>B.16</td>
<td>Comparison of energy spectra at $y^+ \approx 100$ with inner scaling of wavenumber for the present DNS, a similar pipe flow hot wire measurement of Hultmark et al. (2010), and several channel simulations and measurements.</td>
</tr>
<tr>
<td>B.17</td>
<td>Comparison of energy spectra at $y^+ \approx 300$ with inner scaling of wavenumber for the present DNS, a similar pipe flow hot wire measurement of Hultmark et al. (2010), and several channel simulations and measurements.</td>
</tr>
<tr>
<td>B.18</td>
<td>Pipe DNS energy spectra at several radii with 90% confidence interval bars.</td>
</tr>
<tr>
<td>C.1</td>
<td>Pipe net force balance: View 1</td>
</tr>
<tr>
<td>C.2</td>
<td>Pipe net force balance: View 2</td>
</tr>
<tr>
<td>C.3</td>
<td>Pipe net force balance: View 3</td>
</tr>
<tr>
<td>D.1</td>
<td>Color contour levels for $u$ fluctuation $x-s$ plane plots and $x-r$ plane vector plots; nondimensionalizations are by radius $R$ for length scale and $U_{bulk}$ for velocity scale.</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>D.2</td>
<td>382</td>
</tr>
<tr>
<td>D.3</td>
<td>383</td>
</tr>
<tr>
<td>D.4</td>
<td>384</td>
</tr>
<tr>
<td>D.5</td>
<td>384</td>
</tr>
<tr>
<td>D.6</td>
<td>385</td>
</tr>
<tr>
<td>D.7</td>
<td>385</td>
</tr>
<tr>
<td>D.8</td>
<td>386</td>
</tr>
<tr>
<td>D.9</td>
<td>386</td>
</tr>
<tr>
<td>D.10</td>
<td>386</td>
</tr>
<tr>
<td>D.11</td>
<td>387</td>
</tr>
<tr>
<td>D.12</td>
<td>387</td>
</tr>
<tr>
<td>D.13</td>
<td>389</td>
</tr>
</tbody>
</table>

**Figure**

- D.2: $u$ fluctuation color contours for $t = 252R/U_{\text{bulk}}$ field.
- D.3: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field, low momentum streak A.
- D.4: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.
- D.5: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.
- D.6: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.
- D.7: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.
- D.8: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.
- D.9: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. The field view is the same as figure D.9.
- D.10: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. The field view is the same as figure D.9.
- D.11: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$ and red line contours of 2D $\lambda_{ci}$ signed by $\omega_\theta$.
- D.12: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.
- D.13: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field, low momentum streak B.
<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>D.14 A section of streak B ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>389</td>
</tr>
<tr>
<td>D.15 A section of streak B ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>390</td>
</tr>
<tr>
<td>D.16 A section of streak B ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of $\omega_\theta$. The field view is the same as figure D.15.</td>
<td>390</td>
</tr>
<tr>
<td>D.17 A section of streak B ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>391</td>
</tr>
<tr>
<td>D.18 $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak C</td>
<td>392</td>
</tr>
<tr>
<td>D.19 A section of streak C ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>393</td>
</tr>
<tr>
<td>D.20 A section of streak C ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>393</td>
</tr>
<tr>
<td>D.21 A section of streak C ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>394</td>
</tr>
<tr>
<td>D.22 A section of streak C ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>394</td>
</tr>
<tr>
<td>D.23 A section of streak C ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of $\omega_\theta$. The field view is the same as figure D.22.</td>
<td>395</td>
</tr>
<tr>
<td>D.24 $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak D</td>
<td>396</td>
</tr>
<tr>
<td>D.25 A section of streak D ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
<td>396</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>D.26 A section of streak D ( (t = 252R/U_{\text{bulk}}) ) with vectors at coarse ((6.75^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>397</td>
</tr>
<tr>
<td>D.27 A section of streak D ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>397</td>
</tr>
<tr>
<td>D.28 ( u ) fluctuation color contours of ( t = 252R/U_{\text{bulk}} ) field low momentum streak E</td>
<td>399</td>
</tr>
<tr>
<td>D.29 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>399</td>
</tr>
<tr>
<td>D.30 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of ( \omega_\theta ). The field view is the same as figure D.29.</td>
<td>400</td>
</tr>
<tr>
<td>D.31 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>400</td>
</tr>
<tr>
<td>D.32 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>401</td>
</tr>
<tr>
<td>D.33 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>401</td>
</tr>
<tr>
<td>D.34 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>402</td>
</tr>
<tr>
<td>D.35 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>402</td>
</tr>
<tr>
<td>D.36 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of ( \omega_\theta ). The field view is the same as figure D.35.</td>
<td>403</td>
</tr>
<tr>
<td>D.37 A section of streak E ( (t = 252R/U_{\text{bulk}}) ) with vectors at fine ((4.5^+)) spacing with color contours of 3D ( \lambda_{ci} ) signed by ( \omega_\theta ).</td>
<td>403</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>D.38</td>
<td>u fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak F</td>
</tr>
<tr>
<td>D.39</td>
<td>A section of streak F ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.40</td>
<td>A section of streak F ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.41</td>
<td>A section of streak F ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.42</td>
<td>A section of streak F ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.43</td>
<td>u fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak G</td>
</tr>
<tr>
<td>D.44</td>
<td>A section of streak G ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.45</td>
<td>A section of streak G ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of $\omega_\theta$. The field view is the same as figure D.44.</td>
</tr>
<tr>
<td>D.46</td>
<td>A section of streak G ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.47</td>
<td>A section of streak G ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of $\omega_\theta$. The field view is the same as figure D.46.</td>
</tr>
<tr>
<td>D.48</td>
<td>A section of streak G ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. The field view is the same as figure D.46 but the subtracted velocity is different.</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>D.49</td>
<td>A section of streak G ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. The field view is the same as figure D.46 but the subtracted velocity is different.</td>
</tr>
<tr>
<td>D.50</td>
<td>A section of streak G ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. The field view is the same as figure D.46 but the subtracted velocity is different.</td>
</tr>
<tr>
<td>D.51</td>
<td>A section of streak G ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.52</td>
<td>A section of streak G ($t = 252R/U_{bulk}$) with vectors at fine (4.5$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.53</td>
<td>$u$ fluctuation color contours of $t = 252R/U_{bulk}$ field low momentum streak H</td>
</tr>
<tr>
<td>D.54</td>
<td>A section of streak H ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.55</td>
<td>A section of streak H ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.56</td>
<td>A section of streak H ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of $\omega_\theta$. The field view is the same as figure D.55.</td>
</tr>
<tr>
<td>D.57</td>
<td>A section of streak H ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
<tr>
<td>D.58</td>
<td>A section of streak H ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of $\omega_\theta$. The field view is the same as figure D.57.</td>
</tr>
<tr>
<td>D.59</td>
<td>A section of streak H ($t = 252R/U_{bulk}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.</td>
</tr>
</tbody>
</table>
D.60 A section of streak H \((t = 252R/U_{bulk})\) with vectors at coarse \((6.75^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.59.

D.61 A section of streak H \((t = 252R/U_{bulk})\) with vectors at fine \((4.5^+)\) spacing with color contours of \(3D \lambda_{ci}\) signed by \(\omega_\theta\).

E.1 Copyright permission for figure 1.1.
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Summary of parameters for DNS data sources.</td>
</tr>
<tr>
<td>3.1</td>
<td>Summary of parameters for DNS databases.</td>
</tr>
<tr>
<td>4.1</td>
<td>Symbols for each section of figure 4.6.</td>
</tr>
<tr>
<td>4.2</td>
<td>Symbols for figure 4.10. Each symbol is for $uu$ cumulative energy except for the last line type listed, which is for $uv$.</td>
</tr>
<tr>
<td>4.3</td>
<td>Mean fractions of turbulent kinetic energy and Reynolds shear stress with wavelengths that are within the VLSM, LSM, and shorter ranges for the present pipe simulation. The values are computed from a linear average from the pipe axis to the wall.</td>
</tr>
<tr>
<td>7.1</td>
<td>Eigenvalues ($\lambda/E$).</td>
</tr>
<tr>
<td>8.1</td>
<td>Domain dimensions for turbulent boundary layer comparison.</td>
</tr>
<tr>
<td>E.1</td>
<td>Copyright permission for chapter 3.</td>
</tr>
<tr>
<td>E.2</td>
<td>Copyright permission for chapter 4.</td>
</tr>
<tr>
<td>E.3</td>
<td>Copyright permission for figure 5.9.</td>
</tr>
</tbody>
</table>
Chapters 3, 4, 5, and 7 contain manuscripts published or being prepared for publication in collaboration with other researchers. Permissions from collaborators for including these materials in this dissertation are noted in appendix E. All of these publications have the author of this dissertation as first author, except for Wu et al. (2012) (chapter 4). In this work, Prof. X. Wu performed the turbulent pipe simulation, computed fundamental statistics, created corresponding plots and flow visualizations, and wrote the introductory section of a manuscript. The present dissertation author computed additional statistics, performed analysis, interpreted the results, revised text and figures, and wrote the body of the manuscript. This was performed under the advisement of Prof. R. Adrian, with whom the interpretation of the results was collaboratively performed. All three authors participated in the editing process. In chapter 5, for which the present dissertation author is anticipated to be first author of the published version, work was performed in a similar arrangement, except the present author also commenced the writing of the manuscript.
Chapter 1
INTRODUCTION

Organized motions, known as coherent structures, that persist in time and make significant contributions to statistics are readily observed in wall-bounded turbulent flows (Adrian, 2007). The spatial extents of coherent structures, though varying over a wide range, suggest that much information about the flow can be contained by describing relatively few coherent structures. It is therefore desirable to form compact descriptions of complex turbulent processes by decomposing flows into simpler coherent structures. Advancing knowledge of these simpler elements and their organization would significantly deepen the understanding of turbulent wall flows and this information has the potential to improve control and drag reduction strategies for turbulent flows of critical importance to transportation vehicles, liquid transport, and a wide range of vital applications.

Although these coherent structures are visually striking, they must be described mathematically to be meaningfully analyzed. However, their proper mathematical definition is not self-evident. One method to extract coherent structures is proper orthogonal decomposition, in which modes are generated from a set of field realizations for the quantity or quantities of interest. POD is reviewed in Berkooz et al. (1993) and has the important property that there are no numerical parameters or thresholds that must be adjusted to extract the modes, making the modes less arbitrary than for other techniques. The POD modes, which have the same spatial dimensions as the data fields’ domain, are defined by requiring that each field may be reconstructed by a linear combination of the modes and that the modes are orthonormal under a suitably defined norm (see for example Lumley, 1970; Holmes et al., 1998; Tropea et al., 2007). Furthermore, the POD modes have the property that, in the mean, the reconstruction of a field by partial sums of POD modes con-
verge faster than those of any other set of orthogonal functions (Liu et al., 2001). Another equivalent definition of coherent structure as determined by POD is “the deterministic function which is best correlated on average with the realizations” of the quantity of interest (Lumley, 1967).

Although other quantities may be used, velocity is commonly used for POD of fluid flows. Velocity fluctuations, defined as the field of instantaneous difference from a suitably defined mean for a given flow, are appropriate quantities for POD of turbulent flows. Therefore, realizations of velocity fields are the source data for performing POD analysis, and three-dimensional fields enable the most detailed and clear analysis of the structures. Numerical simulations are suitable because they can provide three-dimensional velocity fields with both high spatial resolution and very large field dimensions.

As canonical wall-bounded flows include important physics that are relevant to important real-world applications but the relatively simple geometries otherwise minimize the complexity, they are appropriate to gain an understanding of important structures. Three important wall-bounded turbulent flow geometries are channels, in which fluid flows between two parallel plates of very large dimensions under the forcing of an applied pressure gradient, pipes, in which which fluid is similarly forced through a circular tube, and flat-plate turbulent boundary layers. In a flat-plate turbulent boundary layer, flow with a fixed free-stream velocity passes over a plate. With sufficient velocity and downstream distance from the beginning of the plate, the flow becomes turbulent. The pressure gradient is arbitrary, and a common configuration for study is a zero-pressure-gradient boundary layer. The turbulent boundary layer could be likened to half of a turbulent channel with only one plate, but the streamwise inhomogeneity due to the developing nature of the boundary as it penetrates higher and higher into the non-turbulent free-stream with downstream distance, as well as lack of influence from an opposing wall and different pressure
gradient, make this flow significantly different, especially at locations significantly above the wall.

The relatively simple geometries of the canonical wall-bounded turbulent shear flows enable simulations to be performed with numerical algorithms that provide highly accurate results with highly efficient computation. In the past, direct numerical simulations of turbulent wall flows have been feasible only at relatively low Reynolds numbers, such as the notable $Re_\tau = 180$ turbulent channel simulation of Kim et al. (1987). Recently, such channel simulations have been performed at increasing Reynolds numbers up to $Re_\tau = 2003$ (Hoyas & Jiménez, 2006), particularly beginning just before the year 2000 (Moser et al., 1999). POD has been previously performed using DNS databases to extract structure, but not at the higher Reynolds numbers that have recently become available. One successful application of POD to data obtained from a numerical simulation is that of Rempfer & Fasel (1994) which studied the three-dimensional structures characterizing transition in a flat-plate boundary layer. They extracted structures that traveled as waves and interpreted other modes as being characteristic of particular stages of transition. It should be noted that disturbances were introduced in particular locations so the structures were essentially ‘phase-locked’ and the flow was not statistically homogeneous in any spatial direction. Rempfer & Fasel (1994) also noted the similarity between dominant structures they extracted and those associated with a bursting event in fully turbulent flow.

A prominent application of POD to a numerical simulation of a fully turbulent wall-bounded flow is that of Moin & Moser (1989), which considered the region from one wall to the centerline of a $Re_\tau = 180$ turbulent channel flow. Interpreting the modes is more complicated than in the case of Rempfer & Fasel (1994), because in a fully turbulent channel the flow is homogeneous in the streamwise and spanwise directions. It can be shown that the eigenfunctions obtained by POD
for a statistically homogeneous function are trigonometric, and therefore the set of POD modes is equivalent to a Fourier expansion (Holmes et al., 1998; Tropea et al., 2007). Thus, the POD modes for a turbulent channel are Fourier modes in the homogeneous directions, but not in the inhomogeneous wall-normal direction. The intuitive concept of a coherent structure is spatially compact, contrary to Fourier modes for which the modes with wavelengths on the order of the domain size contain substantial amounts of energy. In addition, Fourier modes with smaller wavelengths would best correspond to arrangements of smaller structures because all modes span the domain size. To address these issues, Moin & Moser (1989) constructed a ‘characteristic eddy’ that is designed to be consistent with a coherent structure that is randomly scattered in the homogeneous flow directions. This treatment is developed by Lumley (1981) and was based on the shot-noise decomposition. The ‘characteristic eddy’ was constructed with contributions from all of the POD modes. Moin & Moser (1989) noted that the choices of phase for each mode forming the ‘characteristic eddy’ under this framework was not unique, but compactness of the eddy was one possible consideration. Moin & Moser (1989) ultimately developed a characteristic eddy (figure 1.1) that appeared similar to the structure educed by Zhou et al. (1999) using linear stochastic estimation to estimate the conditional structure based on a strong ejection (Q2 event) above the wall (figure 1.2). The present availability of three-dimensional scientific visualization techniques provide new opportunities to gain deeper understanding of the structures.

Recent studies in wall turbulence have focused on motions with long streamwise extent, with significant interest forming after the study of Kim & Adrian (1999). Examining the one-dimensional streamwise energy spectra of turbulent wall flows of various geometries, evidence indicates that such structures contribute significantly to the flow statistics. Guala et al. (2006) consider turbulent pipe flow
Figure 1.1: Reproduced from Moin & Moser (1989): (a) $u$-velocity contours of the characteristic eddy, (b) $v$-velocity contours of the characteristic eddy

Figure 1.2: Reproduced from Zhou et al. (1999): vector plots in cross-sections of a conditional eddy obtained by linear stochastic estimation
and define ‘very-large-scale motions’ to be associated with streamwise wavelengths greater than $\pi R$, where $R$ is the pipe radius. They found that between 50 and 60% of Reynolds shear stress is associated with VLSMs (that is, with Fourier modes with wavelengths in the VLSM range). Balakumar & Adrian (2007) found similar results for channel and boundary layer flows. These results could be explained by large distinct structures and/or organizations of smaller coherent structures, as in the hairpin vortex packet paradigm (see for example Adrian et al., 2000b; Adrian, 2007). Hutchins & Marusic (2007a) note meandering organizations of vortical structures extending into the logarithmic region. Such organizations of smaller coherent structures could also be viewed as coherent structures. This leads to a different concept of what constitutes a coherent structure compared to the prior view in which spatial compactness was deemed necessary, as used to develop the characteristic eddies of Moin & Moser (1989). This suggests that other methods of interpreting POD modes would be appropriate to understand what information they contain with respect to large-scale structures.

Using two-dimensional velocity fields experimentally obtained via particle image velocimetry (in the streamwise-wall normal plane), Liu et al. (2001) performed POD analysis and examined the modes with emphasis on very-large- and large-scale motions but interpreted them in a different manner than previous research not concerned with these motions. Although the modes were trigonometric in the homogeneous streamwise direction, they found that vector fields of some POD modes resembled patterns of the hairpin vortex signature cross-sections considered in Adrian et al. (2000b). Examples of these signatures in an instantaneous vector field of turbulent flow are shown in figure 1.3. A typical POD mode obtained by Liu et al. (2001) showing the arrangement of these signatures is shown in figure 1.4. Liu et al. (2001) proceed by projecting instantaneous fields onto these modes and examining the partial reconstructions, in essence using partial sums of POD modes
as a low-pass filter. The resulting fields indicate which motions are dominant and, because the flow is statistically homogeneous in the streamwise direction, the coefficients obtained by the projections set the phase of the trigonometric (Fourier) functions governing the modes in the streamwise coordinate.

Figure 1.3: Reproduced from Adrian et al. (2000b): vector plot of streamwise-wall normal plane of a turbulent boundary layer ($Re_\theta = 930$) with constant velocity subtracted to illustrate the hairpin vortex signature cross-sections associated with the ‘heads’ of hairpin vortical structures. The heads are identified by circles and associated with the swirling motion that appears as the streamwise velocity switches direction below and above the head when viewed with a constant velocity subtracted.

The results of Rempfer & Fasel (1994), Moin & Moser (1989), and Liu et al. (2001) demonstrate that POD is capable of extracting information in wall turbulence flows that is useful to understanding and explaining the structure of these flows. The present study therefore applies this analysis to the numerical simulations that have recently become available. In particular, the analysis techniques of Liu et al. (2001) are applied to a large simulation of a turbulent boundary layer. This application provides more freedom in the POD modes because the flow is inhomogeneous in both the streamwise and wall-normal coordinates. In addition to being three-dimensional, thereby offering much greater structural information, the DNS
Recent studies have focused on low-order dynamical simulations of turbulent flows using POD modes (see for example Holmes et al., 1998). Examining the temporal evolution of the POD mode coefficients, it has been determined that some modes propagate as waves (Sirovich et al., 1991). This information is useful to understanding the POD modes that will be obtained in the present research and is consistent with hairpin vortex packets considered by Adrian et al. (2000b), in which groups of vortical structures are understood to travel together with low dispersion.

1.1 Overview of Present Study

The objective of this study is to describe structures that are most significant to wall-bounded turbulent flows and their organizations using appropriate analysis tools, including POD, applied to numerical simulation data. While POD is the analysis method generally applied to the flows, other aspects of flow structures are also addressed with specific techniques suited to the topic of interest. For instance, the
turbulent pipe simulation studied herein brought forth new results for a much longer streamwise domain at higher Reynolds number than preceding pipe simulations, so aspects of the energy content at long streamwise wavelengths and the velocity structures in instantaneous fields, including the associated vortical structures, are studied prior to POD analysis.

The present study advances towards full three-dimensional POD modes and the physics they contain by initially concentrating on the inhomogeneous wall-normal direction ($y$) in turbulent channel flows and calculating one-dimensional POD modes, that is, modes of $u(y)$ where $u$ is a velocity component. These are useful in understanding the physical basis for the mode shapes due to the inhomogeneity in the wall-normal direction and is critical to understanding higher dimensional modes. Carbone & Aubry (1996) discuss how symmetries in the flow physics influence the POD mode shapes and apply their analysis to streamwise-velocity POD modes in only the wall-normal direction of $Re_{\tau} = 180$ turbulent channel flow. They were able to synthesize modes of other orders based on one mode and empirical parameters combined with their theory involving mode stretching. They observed excellent agreement with the actual modes in all but the lowest modes, which were related to the largest length scales. Moser (1994) developed WKB analysis to determine the asymptotic properties of POD modes in the limit of high mode number and compared the POD eigenvalues (which indicate the mean energy carried by each mode) predicted by theory and calculated from $Re_{\tau} = 180$ turbulent channel flow. Although these indicate that POD modes are predictable at the highest mode numbers, the lowest modes (those associated with the longest length scales) are less predictable but are most important because they carry the most energy and are most relevant to large scale motions on which the present research focuses. In addition, Carbone & Aubry (1996) bring up questions of how the scaling behaves at higher
Reynolds numbers. Chapter 3 answers these questions using higher-Reynolds number DNS data as noted above.

The availability of other studies for the $Re_\tau = 180$ turbulent channel flow also allows the POD analysis to be verified and provides a baseline for comparisons to other Reynolds numbers and geometries. The turbulent channel mode shapes also have properties that make them straightforward to interpret, such as the no-slip boundary conditions at each wall guaranteeing a zero in every POD mode at the walls and the wall-normal symmetry about the centerline of the channel. Besides explaining the POD modes themselves, the Reynolds number effects on POD modes shed light on the physics of the turbulence. For example, based on similar one-dimensional analysis of experimental PIV data of a turbulent channel at different Reynolds numbers, Liu et al. (1994) postulated that for a certain range of mode numbers, and sufficiently far from the wall layer “both the eigenfunctions and the spectrum of the eigenvalues when scaled by outer layer variables are independent of the Reynolds number for sufficiently large Reynolds number.” By performing exactly the same analysis on a series of channel flow simulations at different Reynolds numbers, trends in the results can be attributed to physical phenomena and not differences in measurement or analysis. In addition, the higher resolution of the numerical data relative to experimental ensures the validity of modes at higher mode numbers.

Related to the physical understanding of the scaling of turbulent channels, Adrian et al. (1995) analyzed the oscillatory behavior of one-dimensional POD modes and developed theory approximating the POD modes based on scaling the phase relation. This theory is further developed in chapter 3. Fernandes (2001) applied this mode synthesis theory to turbulent thermal convection, and, although there are similarities in the mode behavior, the mode shapes are significantly different than for a turbulent channel. The different one-dimensional modes therefore
reflect different physics and this provides evidence that the POD modes are not simply mathematical constructs independent of the physics. To assess the convergence of reconstructions using these synthetic modes and compare with that of full POD and other orthogonal polynomials, energy spectra are calculated.

The very long domain for the pipe simulation discussed in chapters 4 and 5 allows the velocity fields to contain features that are also profitably analyzed using other techniques. Chapter 4 focuses on the statistical aspects of the flow and their relation to experimental measurements. This is necessary to verify the accuracy of the simulation. However, experimental measurements are affected by several considerations, including the effect of differing convection velocities for motions of different scales. These are analyzed in depth with the goal of determining the importance of motions with long streamwise scales in turbulent pipe flows based on simulations.

Having verified the pipe simulation is in accord with experiments and is consistent other relevant simulations, Chapter 4 focuses on extracting structure using several techniques other than POD. Since POD necessarily involves averaging, additional statistical techniques are also appropriate to investigate certain aspects of instantaneous structure. In particular, the variation in the geometries of relatively thin vortical structures associated with velocity structures results in details of vortices generally being smeared out when averaging based on velocity structures in performed. While POD could be performed on vorticity instead of velocity, this study employs other methods suitable to compare with vortical structures that have been commonly observed in other flows.

The study continues with higher-dimensional POD analysis. Since POD modes are more distinctive in inhomogeneous coordinates, the turbulent boundary layer flow offers more freedom with two inhomogeneous directions than the other canonical flows. POD has also been relatively rarely applied to turbulent boundary layers,
so the application of POD to the high resolution simulations that have recently become available yields new structural information.

Finally, the two-point correlation functions that are used to compute very wide POD modes of channel and pipe flows. In simulations for all three canonical flows, the variations are periodic in the spanwise \((z)\) or azimuthal \((\theta)\) coordinates. The widest structures are those that are constant in these directions (those with wavenumbers \(k_z\) or \(k_\theta\) of zero). These modes, however, arise for different reasons: due to the specification of a spanwise periodic domain for this homogeneous coordinate of channel and boundary layer simulations or due to the periodicity that naturally occurs of the azimuthal coordinate in the cylindrical coordinate system for a pipe. Since a typical DNS simulation may include a significant amount of energy in these modes, it is important to characterize them and compare them between the flow geometries to better understand their physical meanings.

This study addresses many questions associated with the structure of turbulent wall flows. The POD analysis technique has little subjectivity in that no numerical parameters need be chosen to extract structures. As extracted POD modes represent statistically dominant structures in the flow, this research establishes a basis for interpreting these modes in terms of physical flow structures. This analysis is analogous to extracting vibration modes in that the physics is understood in terms of entities that make contributions to a complex process. By analyzing the properties of simpler constructs (such as one-dimensional modes) that are related to the complicated three-dimensional POD modes, proper orthogonal decomposition gains a basis for developing better insight into the more complicated modes. The higher-dimensional POD modes and structural information gained from the use of additional techniques also advance the understanding of turbulence structure, which may find further application in turbulence modeling.
Chapter 2

DIRECT NUMERICAL SIMULATIONS AND POD METHODOLOGY

The present research examines a set of direct numerical simulations for wall-bounded turbulent flows. While each section of this study contains summaries of the DNS data set and any aspects of POD pertinent to the analysis being performed, a more general overview including further simulation details and theory of POD is now given for clarity. POD is first explained in a single dimension and then extended to multiple dimensions.

2.1 Governing Equations and Numerical Simulation of Canonical Wall-bounded Turbulent Flows

Incompressible turbulent flows of Newtonian fluids in canonical geometries are considered for the present study. The assumption of incompressibility, i.e. density remaining constant, accurately describes flow behavior at low Mach number (typically below 0.3) and without heat transfer considerations (Panton, 2005; Anderson, 2003). The Mach number $M = V/a$ is defined as the ratio between a characteristic velocity $V$ of the flow and the speed of sound $a$ of the fluid medium. Newtonian fluids are defined as those obeying Newton’s viscosity law in which stress on a fluid parcel is linearly related to rate of strain.

The behavior of such fluids is described by the Navier-Stokes equations describing the momentum balance, which in Cartesian coordinates are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j},$$

(2.1)

using tensor notation with summing assumed on repeated indices. With the numerical subscript $i$ referring to components in each of the Cartesian coordinate directions of vector quantities, $x$ is position and $u$ is velocity. Pressure, a scalar denoted by $p$, is mechanical pressure and not thermodynamic pressure for this incompressible case. Constant scalars are fluid density $\rho$ and kinematic viscosity $\nu = \mu/\rho$, where
μ is dynamic viscosity. This equation is solved in conjunction with an equation enforcing of mass conservation, which for incompressible flows is

\[
\frac{\partial u_i}{\partial x_i} = 0. \tag{2.2}
\]

Important parameters are conveniently expressed by nondimensionalizing (2.1). If \( U_0 \) is a characteristic velocity scale (by which to nondimensionalize \( u \)) and \( L_0 \) is a characteristic length scale (by which to nondimensionalize \( x \)), the key nondimensional parameter governing (2.1) is Reynolds number \( Re = U_0 L_0 / \nu \). Based on the terms the constants are associated with, Reynolds number can be conceptualized as a ratio between inertial forces (related to the nonlinear term associated with advection) and viscous forces. This parameter is very important for turbulent flows, and it has been observed that turbulent motions, such that a flow remains unsteady but statistically stationary when a constant forcing (such as pressure) is applied, only persist when a critical Reynolds number is exceeded (Reynolds, 1883).

While the set of nonlinear partial differential equations (2.1) and (2.2) describing the fluid flow are independent of the flow geometry, the geometry enters the problem through boundary conditions. Where a solid boundary exists, regions of fluid flowing past the boundary have been found to generally obey the no-slip boundary condition, whereby the fluid velocity matches the velocity of the solid boundary at their interface. In the case of canonical wall-bounded flows, the walls are specified as zero velocity, so the boundary conditions for the flows are that the fluid velocity is \( u = 0 \) at the walls.

The specific canonical flow geometries are depicted in figure 2.1. It is common practice to use \( \{x, y, z\} \) to denote Cartesian coordinate directions \( \{x_1, x_2, x_3\} \) and \( \{u, v, w\} \) to denote the corresponding velocity components \( \{u_1, u_2, u_3\} \). In all three cases, the general flow direction is from left to right. For the channel and pipe (figure 2.1(a,b)), the flow is induced by an imposed pressure gradient. For the
boundary layer (figure 2.1(c)), the flow exists due to an imposed free-stream velocity $U_\infty$ that passes by the leading edge where the flat plate begins on the left. The boundary layer is very thin near this location, where the velocity is zero at the wall but is the free-stream velocity immediately above the wall. The boundary layer develops and thickens with increasing downstream distance from the leading edge of the plate, and the local fluid velocity approaches a fixed fraction of the free-stream velocity at greater and greater distances above the wall $y$.

The geometric parameters describing the internal geometries of the channel and pipe are the channel half-height $h$, the distance from one wall to the channel centerline, and the pipe radius $R$, respectively. In the case of boundary layers, there is no geometric separation between two walls, so the boundary layer thickness $\delta$ is commonly used as a length scale, where $\delta$ is typically defined as the height above the wall at which the suitably-defined mean streamwise velocity becomes 99% of the free-stream velocity. Another useful quantity is momentum thickness $\theta$, which is defined based on the deficit of streamwise momentum that occurs relative to the unperturbed free-stream velocity profile. Since the boundary layer thickens with downstream distance, the boundary layer thickness quantities similarly increase with $x$, in contrast to the other geometries for which the geometric quantities $h$ and $R$, as well as the flow thickness properties, remain constant with $x$. In the idealized canonical flows, the streamwise ($x$) extents are infinitely long (although
the boundary layer begins at a fixed location). For the channel and boundary layer geometries, the spanwise (z) extents are also infinitely long. y is defined as the wall-normal distance for the turbulent boundary layer. For the channel, distance above a wall is one possible convention for y, but it is also frequently used as a coordinate oriented normal to the walls but originating at the channel centerline (halfway between the walls). This convention is convenient for expressing the symmetry that naturally occurs in this geometry. The pipe geometry is conveniently described in cylindrical coordinates, with r as a radial distance from the pipe axis and θ defining the angular position. It is also convenient to define y as the distance normal to the pipe wall for comparison to other flows, and y is related to radial position as $y = R - r$.

As turbulent flows fluctuate with time, one common procedure to examine the fluctuating components is Reynolds averaging (Pope, 2000). Each velocity component (for example, u) is decomposed into a mean velocity $U$ (or $\bar{u}$) and a fluctuating component $u'$, such that $u = U + u'$. The mean must be suitably defined. In the case of the canonical pipe flows, the quantity driving the flow (applied pressure or free-stream velocity) is assumed constant with respect to time. The mean velocity quantities, as would conceptually be defined with an ensemble average of the chaotic velocity field realizations obtained under a set of nearly identical but infinitesimally perturbed conditions, would also not vary with time (be statistically stationary), and consequently a time average can be used to calculate the mean quantities. In addition, the canonical flows are statistically homogeneous in several spatial directions. The channel is statistically homogeneous with respect to $x$ and $z$, and the pipe is statistically homogeneous with respect to $x$ and $\theta$. The turbulent boundary layer is only statistically homogeneous with respect to $z$. These homogeneities can be used to maximize the convergence of statistics given a finite set of velocity fields spaced by time intervals.
One relatively robust result for wall-bounded turbulent flows is the logarithmic law describing the mean velocity profile. This relationship can be developed by recognizing two sets of scales describing wall-bounded flows. The outer length scales are related to the geometry of the problem or thickness of the boundary layer. The inner or wall velocity scale is related to the flow in the near-wall region, which is characterized by kinematic viscosity $\nu$ and wall shear stress $\tau_w$ (Pope, 2000). The friction velocity is defined as $u_\tau \equiv \sqrt{\tau_w/\rho}$. A length scale can also be formed from near-wall quantities as $\delta_v = \nu/\tau_w$. With arguments that a region of the flow would exist in which a function is independent of various parameters related to the inner and outer scales, it has been developed that the mean velocity scaled by the friction velocity ($U^+ = U/u_\tau$) would have the form $U^+ = (1/\kappa)\ln y^+ + B$, in which $y^+$ is wall-normal height scaled by the viscous length scale ($y^+ = y/\delta_v$) and $\kappa$ and $B$ are constants (Prandtl, 1925; von Kármán, 1930; Pope, 2000).

While this result is subject to debate as to whether the constants are truly constant and independent of the flow geometry, and how high of Reynolds number is necessary for adherence to this law, the scales discussed are very important in describing these flows. Reynolds numbers are defined based on using various combinations of these scales. Commonly used Reynolds numbers include $Re_\tau = u_\tau \delta/\nu$, in which $\delta$ is used generically as the relevant outer length scale ($h,R,\delta$) for the flow of interest. For turbulent pipe flows, this quantity is also sometimes referred to as the Kármán number $R^+$. $Re_\tau$ can also be understood as the outer length scale nondimensionalized by wall units, as $Re_\tau = \delta/\delta_v$. Other Reynolds numbers are commonly defined similarly but with the appropriate outer velocity scale of $U_{\text{bulk}}$, $U_{cl}$, or $U_\infty$. $U_{\text{bulk}}$ and $U_{cl}$ apply to the channel and pipe flows, in which $U_{\text{bulk}}$ is the mean streamwise velocity averaged over the flow cross-section (the plane normal to the streamwise axis) and $U_{cl}$ is the mean streamwise velocity at the centerline, which is the location at which it is highest. $U_\infty$, the free-stream velocity, it
most analogous for turbulent boundary layers. One such common Reynolds number for turbulent pipe flow is based on bulk velocity and pipe diameter, denoted by $Re_D = 2RU_{\text{bulk}}/\nu$. Common Reynolds numbers for turbulent boundary layer flows include $Re_\theta = U_\infty \theta/\nu$, using the momentum thickness previously described.

While the governing equations are available to describe these flows, analytical solutions for turbulent flows are not available. However, instantaneous fields of flow simulations can be computed by solving numerical approximations of (2.1) and (2.2). These are often performed by spatially discretizing onto a computational grid and using finite-difference methods to approximate spatial derivatives or using spectral methods to represent the flow quantities with an expansion of Fourier modes or other orthogonal polynomials. Since the data sets analyzed in this study were obtained using a wide collection of algorithms and computational codes, details of the specific methods are available in references for each specific data set. For data analysis, the postprocessing codes have been written to be consistent with the numerical methods used for simulation to attain maximum accuracy. When simulating the pipe flow, it is appropriate to approximate the governing equations cast in cylindrical coordinates and therefore naturally conforming to the geometry of the flow domain.

The general simulation technique used for all data sets is known as direct numerical simulation (DNS). DNS directly computes numerical approximations to (2.1) and (2.2) without introducing other additional models. Turbulent flows are known to contain motions with wide ranges of spatial scales. While the canonical flows include infinite dimensions as idealizations, in practice a domain size must be selected. In homogeneous dimensions that are ideally infinite, such as the streamwise coordinate for channels and pipes or the spanwise dimension for channels and boundary layers, a periodic domain is used to model this situation, and this establishes the largest scales of motion that can be simulated. It is known that
the smallest motions in the flow dissipate energy by viscosity, and therefore appropriately small scales of motion must also be resolved to accurately simulate the flow and prevent energy from unphysically accumulating. As the smallest scales requiring resolving become smaller, either a finer grid or higher order polynomials must be used to represent the flow, thereby increasing computational cost for a DNS, or a model must be introduced to represent the smallest scales, thereby introducing error into other types of simulations. As previously noted, Reynolds number \( Re_\tau = \delta / \delta_v \) can be viewed as the ratio between the largest (outer) scales and a length scale related to the viscous wall motions, which are related to the dissipative scales (although not directly). This indicates that the computational cost to perform a DNS increases with increasing Reynolds number. For a turbulent channel flow simulation, channel half-height \( h \) is typically used to nondimensionalize length scales, and the quantities typically used to parameterize the simulation are \( Re_\tau = u_\tau h / \nu \) and the periodic dimensions of the computational domain \( L_x/h \) and \( L_z/h \). \( L_x/h \) and \( L_z/h \) indicate the largest motions the periodic domain can accommodate as nondimensionalized by channel half-height, with unphysical behavior introduced as these dimensions shorten relative to the idealized canonical flow.

Simulating high Reynolds numbers is generally of interest because real applications typically involve high Reynolds numbers and the separation of scales implied by high Reynolds numbers are often necessary for theoretical scaling arguments to be valid. Thus, it is desirable high Reynolds number flows with large domains for accurate physical behavior and high resolution to properly resolve all scales, but all of these goals increase the computational cost and data storage requirements. Another computational issue is the amount of time and number of fields that are retained for performing statistics to obtain the necessary convergence.
2.2 DNS Database

A selection of data sets used in this study are summarized in table 2.1. Further details of the simulations are supplied the chapter for which each flow is studied. Turbulent channels have historically been the flow with which simulations are advanced to higher Reynolds number, as the homogeneity in two directions combined with a simple cubic geometry is suited for high order solutions with efficient spectral numerical methods. At present, the channel simulations by del Álamo, Jiménez, Moser, and co-workers cover a wide range of Reynolds numbers from $Re_\tau = 180$ to $Re_\tau = 2003$ using large domain sizes (del Álamo & Jiménez, 2001; del Álamo & Jiménez, 2003; del Álamo et al., 2004; Hoyas & Jiménez, 2006). Their $Re_\tau = 934$ data set (del Álamo et al., 2004) has been studied by other researchers, and is useful for the present study to provide relatively high Reynolds number data. Simulations at lower Reynolds numbers were performed especially for the present study to provide a high level of statistical convergence using the turbulent channel code used in Kim et al. (2008) (with further numerical details described therein), which simulates the Navier-Stokes equations in primitive variables using a spectral method with Fourier polynomials and periodic boundary conditions in the homogeneous streamwise and spanwise directions. A Chebyshev expansion is employed in the wall-normal direction. The simulations of del Álamo, Jiménez, Moser, and co-workers also use the same polynomial expansions but employ a different method for solving the Navier-Stokes equations that involves solving for the wall-normal vorticity and Laplacian of the wall-normal velocity (Kim et al., 1987).

The other geometries of interest include the pipe, which has been simulated by Wu & Moin (2008) using a finite-difference method on a staggered grid with periodic boundary conditions for the pipe inflow and outflow. The Reynolds number of this simulation is most comparable to the $Re_\tau = 547$ channel. The other canonical
flow geometry is the turbulent boundary layer, which lacks the additional parallel wall that is present in the channel and possesses significantly different structure due to the turbulent flow spatially developing into a undisturbed free-stream region. Ferrante et al. (2004) made available to researchers their turbulent boundary layer data set, in which the boundary layer spatially develops and an auxiliary simulation is used to generate the inflow, instead of the periodic inflow and outflow for channel simulations. The finite-difference zero-pressure-gradient flat-plate turbulent boundary layer simulation of Wu & Moin (2010), based on Wu & Moin (2009a) but extended to higher Reynolds number, allows the flow to evolve from a laminar boundary layer to $Re_\theta \approx 1950$, thereby permitting the structures to develop in a natural manner. These boundary layer flows are periodic and homogeneous in the spanwise direction, but the streamwise spatial development makes the flow inhomogeneous in the streamwise coordinate. POD analysis of this flow therefore is performed using a method to reduce the computational expense with two inhomogeneous directions, known as the method of snapshots.

In table 2.1, the included domain dimensions and numbers of grid points are not necessarily directly comparable, due to the different geometries and numerical methods. For example, the channel simulations are computed with a spectral method using Fourier and Chebyshev polynomials but the pipe and boundary layer simulations use finite difference approximations requiring finer meshes for similar accuracy. For the channel simulations, the number of Fourier/Chebyshev modes, not including the dealiasing modes, are reported for consistency. (In publications such as del Álamo et al. (2006), the number of grid points reported include the extra points associated with the extra Fourier dealiasing modes, so the numbers are $3/2$ times the numbers reported here for only the coordinates using Fourier expansions.)
Table 2.1: Summary of parameters for DNS data sources. Simulations for ‘Spectral DNS Code (Kim *et al.*, 2008)’ entries are performed for the present study. $L_x$, $L_y$, and $L_z$ are the streamwise, wall-normal, and spanwise domain sizes, respectively, for the channel and boundary layer. The channel half-height is denoted by $h$ and pipe radius is denoted by $R$. The length scale listed for each boundary layer is $\delta_0$, the boundary layer thickness at the inlet plane (Ferrante *et al.*, 2004), or $\theta_0$, the momentum thickness at the laminar inlet (Wu & Moin, 2010). $N_x$, $N_y$, and $N_z$ are the number of grid points in the streamwise, wall-normal, and spanwise directions, respectively. For the turbulent pipe simulation, the number of azimuthal grid points is listed in the $N_z$ column.

<table>
<thead>
<tr>
<th>Source</th>
<th>Re $\tau$</th>
<th>$L_x$</th>
<th>$L_y$</th>
<th>$L_z$</th>
<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Channel</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>del Álamo <em>et al.</em> (2004)</td>
<td>$Re_\tau = 934$</td>
<td>$8\pi h$</td>
<td>$2h$</td>
<td>$3\pi h$</td>
<td>2048</td>
<td>385</td>
<td>1536</td>
</tr>
<tr>
<td>Spectral DNS Code <em>(Kim et al., 2008)</em></td>
<td>$Re_\tau = 180$</td>
<td>$4\pi h$</td>
<td>$2h$</td>
<td>$\frac{4}{3}\pi h$</td>
<td>128</td>
<td>129</td>
<td>128</td>
</tr>
<tr>
<td>Spectral DNS Code <em>(Kim et al., 2008)</em></td>
<td>$Re_\tau = 395$</td>
<td>$2\pi h$</td>
<td>$2h$</td>
<td>$\pi h$</td>
<td>256</td>
<td>193</td>
<td>192</td>
</tr>
<tr>
<td><strong>Pipe</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wu <em>et al.</em> (2012) (code for Wu &amp; Moin (2008))</td>
<td>$Re_D = 24580, Re_\tau = 685$</td>
<td></td>
<td></td>
<td></td>
<td>2048</td>
<td>256</td>
<td>1024</td>
</tr>
<tr>
<td><strong>Boundary Layer</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ferrante <em>et al.</em> (2004)</td>
<td>$Re_\theta = 1020-1480$</td>
<td>$20\delta_0$</td>
<td>$3.6\delta_0$</td>
<td>$5\delta_0$</td>
<td>512</td>
<td>96</td>
<td>256</td>
</tr>
<tr>
<td>Wu &amp; Moin (2010)</td>
<td>$Re_\theta = 80-1950$</td>
<td>$12\ 750\theta_0$</td>
<td>$2250\theta_0$</td>
<td>$562.5\theta_0$</td>
<td>8192</td>
<td>500</td>
<td>256</td>
</tr>
</tbody>
</table>
2.3 Proper Orthogonal Decomposition Theory
2.3.1 One-Dimensional POD

Let $u_i(x,t)$ be a velocity field (in practice, the fluctuating velocity is used), where $i$ represents the component of velocity and in general the field is a function of a spatial vector $x$ and time $t$. For simplicity in the following derivation, $u_i$ is assumed to be a function of only one spatial variable, although this later will be generalized so $u_i$ is a function of all three spatial directions. Nonetheless, one-dimensional POD is useful and will be computed; for this reason, the spatial variable in the derivation is $y$, which corresponds to the inhomogeneous direction in a turbulent channel flow (Moin & Moser (1989) presents the equations with this notation). In addition, it is assumed that $u_i(y)$ is complex, which will be necessary at a later point. Summation is implied on repeated indices (subscripts).

The proper orthogonal decomposition equations can be formulated by proposing a basis \{${\phi}^{(n)}(y)$\} such that

$$u_i(y) = \sum_{n=1}^{N} a_n \phi^{(n)}(y) \quad (2.3)$$

(For the purpose of applying POD analysis to the numerical simulations, the velocity fields can be perfectly reconstructed with a finite $N$.)

The proper orthogonal decomposition modes are defined by requiring that the basis be optimal in the sense that the normalized projection of $u$ onto these modes is maximized. This is equivalent to minimizing the error of the partial reconstruction for any number of terms included in the reconstruction. From Holmes et al. (1998), the maximization problem is written as

$$\max_{\phi \in L^2([y_1, y_2])} \frac{\langle (u, \phi) \rangle^2}{\|\phi\|^2} \quad (2.4)$$

where $\langle \cdot \rangle$ denotes a mean.

It is therefore necessary to define a norm. Although different inner products/norms may be used for particular circumstances (Tropea et al., 2007), the
$L^2$-norm is appropriate for the turbulent flows considered because the $L^2$-norm of fluctuating velocity corresponds to turbulent kinetic energy. The inner product is

$$(f,g) = \int_{y_1}^{y_2} f(y)g^*(y)dy$$  \hspace{1cm} (2.5)$$

and the $L^2$-norm is defined as

$$\|f\| = \sqrt{(f,f)}$$  \hspace{1cm} (2.6)$$

This is a calculus of variations problem and, requiring $\|\phi\| = 1$, Holmes et al. (1998) shows that the solution is obtained by solving for the eigenfunctions of the integral equation

$$\int_{y_1}^{y_2} R_{ij}(y,y')\phi_j(y')dy' = \lambda \phi_i(y)$$  \hspace{1cm} (2.7)$$

where the two-point correlation is

$$R_{ij}(y,y') = \langle u_i(y)u_j^*(y') \rangle$$  \hspace{1cm} (2.8)$$

Solving the integral equation (2.7) yields the eigenmodes, which are assigned mode numbers $n$. The modes are typically ordered according to the associated energy (to be defined below) or the sequency, defined as the number of internal zero crossings. The modes $\{\phi^{(n)}(y)\}$ are normalized such that

$$\int_{y_1}^{y_2} \phi^{(n)}(y)\phi^{(m)*}(y)dy = \delta_{nm}$$  \hspace{1cm} (2.9)$$

and it can be shown that the reconstruction coefficients for eigenfunctions of different mode numbers $n$ are uncorrelated:

$$\langle a_n a_m^* \rangle = \begin{cases} \lambda^{(n)} & n = m \\ 0 & n \neq m \end{cases}$$  \hspace{1cm} (2.10)$$

(Moin & Moser, 1989).

Moin & Moser (1989) also shows that the turbulent kinetic energy $E$ is related to the eigenvalues by

$$E = \int_{y_1}^{y_2} \langle u_i(y)u_i(y) \rangle dy = \sum_{n=1}^{N} \lambda^{(n)}$$  \hspace{1cm} (2.11)$$
2.3.2 Discrete Approximation

For the numerical simulations considered in the present research, the velocities (and consequently the two-point correlation tensor) are defined at discrete points so the integrals are numerically approximated in the general form

\[
\int_{y_1}^{y_2} u(y) \, dy \approx \sum_{j=j_1}^{j_2} w_j u(y_j)
\]

(2.12)

where \(w_j\) are the quadrature weights. Following Moin & Moser (1989), these weights are determined by the trapezoidal rule for generality (for the turbulent channels, which use Chebyshev polynomials with Gauss-Lobatto points for the wall-normal direction, using Chebyshev weights would be most accurate but do not permit subdomains to be considered).

Using the two-point spatial correlation defined on discrete points and this numerical integration, the integral equation (2.7) becomes an algebraic eigenvalue problem that can be expressed as

\[
C W \phi^{(n)} \phi^{(n)} = \lambda^{(n)} \phi^{(n)}
\]

(2.13)

where the \(C\) matrix is symmetric and contains the two-point spatial correlation, the \(W\) matrix contains the integration weight vector on the diagonal, and the \(\phi^{(n)}\) vector contains the eigenfunction values at each grid point. When the velocity \(u\) contains multiple components (but \(u\) is a function of only one dimension), all of the components of the eigenfunction are placed in one vector (i.e. for three velocity components, the \(\phi^{(n)}\) vector contains \(\phi_1^{(n)}\), \(\phi_2^{(n)}\), and \(\phi_3^{(n)}\)). Thus, all velocity components are associated with each mode. If \(\phi\) for each component is contiguous in the \(\phi^{(n)}\) vector, the two-point correlations in the \(C\) matrix include the velocity component cross-correlations arranged in blocks, each corresponding to the correlation between the same pair of components.

For computational reasons, it is preferable to solve a symmetric eigenvalue function but the \(W\) matrix in general makes the problem asymmetric. However,
it can be transformed to a symmetric problem by factoring $W$ into two diagonal matrices (denoted by $W'$) so $W = W'^2$ with $W'$ containing the square root of each corresponding entry in the $W$ matrix (Moin & Moser, 1989). It can be shown that solving the eigenproblem

$$W'C W' \phi^{(n)} = \lambda^{(n)} \phi^{(n)}$$

(2.14)

with $\phi^{(n)} = W' \phi^{(n)}$ yields the same eigenvalues and eigenfunctions as (2.13) but has the desired symmetric form.

2.3.3 Method of Snapshots

For large numbers of grid points, the eigenproblem in (2.13) can be very large (this is particularly an issue when $u$ is a function of multiple spatial dimensions). A more computationally efficient method for solving such problems when the number of grid points is larger than the number of instants in time fields (known as snapshots) are recorded at is the method of snapshots (also known as the method of strobes), developed by Sirovich (1987). Although the method can be developed more generally (Sirovich, 1987), for the present purpose the following development suffices to demonstrate the results are essentially equivalent to those from the direct method. (These relations are developed more fully in Tropea et al. (2007).) The two-point spatial correlation matrix $C$ used in (2.13) can be formed from a matrix $A$ containing all of the velocities at every time step as $C = (1/N_t) A A^T$, where $N_t$ is the number of fields at different times (snapshots) used.

It is known that for the singular value decomposition

$$B = U \Sigma V^T$$

(2.15)

the singular values and singular vectors can be computed by solving eigenproblems for $B B^T$ (which form the columns of $U$) and $B^T B$ (which form the columns of $V$).
Rearranging the eigenproblem (2.14) for the direct method and defining \( B \equiv A^T W'/\sqrt{N_t} \):

\[
W'CW'\phi'^{(n)} = \lambda^{(n)}(n) \phi'^{(n)} \quad \rightarrow \quad \frac{W' A^T W'}{\sqrt{N_t}} \phi'^{(n)} = \lambda^{(n)}(n) \phi'^{(n)} \quad \rightarrow \quad BB^T \phi'^{(n)} = \lambda^{(n)}(n) \phi'^{(n)}
\]  

(2.16)

From the singular value decomposition, if the eigenvectors of \( B^T B \) are known (which form the columns of the \( V \) matrix), then the SVD equation can be solved for the \( U \) matrix, whose columns are the eigenvectors of \( BB^T \). Comparing with the direct method eigenproblem, this means that by solving the \( B^T B \) eigenproblem the desired POD eigenfunctions can be obtained from the SVD equation. For problems with many points, this is a smaller eigenproblem and corresponds to forming the time correlation instead of the two-point spatial correlation. The method of snapshots approach forms the time correlation instead of the spatial correlation.

### 2.3.4 Higher-Dimensional with Homogeneous Directions

If the POD integral equation (2.7) is applied to a one-dimensional problem with velocity that is statistically homogeneous, the modes are trigonometric (Holmes et al., 1998). In the flows considered in the present research, there is homogeneity in one or more spatial directions so it is advantageous to use Fourier modes in the homogeneous directions when performing POD of two- or three-dimensional velocity fields. Solving for POD modes when \( x \) and \( z \) are homogeneous directions requires the spectral-density tensor

\[
\Phi_{ij}(k_x,y,y',k_z) = \frac{1}{4\pi^2} \int \int e^{-ik_x r_x - ik_z r_z} R_{ij}(r_x,y,y',r_z) \, dr_x \, dr_z
\]  

(2.17)

which is the two-point spatial correlation tensor

\[
R_{ij}(r_x,y,y',r_z) = \langle u_i(x,y,z,t)u_j(x+r_x,y,y'+r_z,t) \rangle
\]  

(2.18)

Fourier transformed in the homogeneous directions (Moin & Moser, 1989).
It can be shown that the relevant eigenproblem is

\[ \int_{y_1}^{y_2} \Phi_{ij}(k_x, y, y', k_z) \hat{\phi}_j(k_x, y', k_z) \, dy' = \lambda(k_x, k_z) \hat{\phi}_i(k_x, y, k_z) \]  (2.19)

where the Fourier transform of the velocity is reconstructed from

\[ \hat{u}_i(k_x, y, k_z) = \sum_{n=1}^{N} \hat{a}_n(k_x, k_z) \hat{\phi}_i(k_x, y, k_z) \]  (2.20)

(Moin & Moser, 1989).

Several publications list the integral equation (2.19) with complex conjugation applied to \( \hat{\phi}_j(k_x, y', k_z) \) inside the integral on the left hand side, such as Moin & Moser (1989) and Tutkun et al. (2008). However, other publications for this complex case do not include the conjugate on the left hand side but report the same form printed above, including Sirovich (1987), Lumley (1981) equation (7.3), Aubry et al. (1988) equation (7), Webber et al. (1997) equation (7), Podvin & Lumley (1998) equation (2), Liu et al. (2001) equation (7), and Tropea et al. (2007) equation (22.89). It has been verified that the form as written above in (2.19) is consistent with initially specifying the eigenproblem in three dimensions with no assumption of homogeneity and then introducing Fourier expansions of velocity into the equation and two-point correlation definitions for the homogeneous directions. It should also be noted that the final integral equation, after numerically integrating along the inhomogeneous, has conjugation compatible with the form specified by standard eigensolver routines, such as those described in Intel Corporation (2009). Duggleby (2006) also notes applying a standard LAPACK eigensolver routine (zheev) in their POD study of turbulent pipe flow.

In the channel simulations, Fourier modes are used in the homogeneous directions and are defined at discrete wavenumbers. The eigenproblem (2.19) therefore must be solved for each wavenumber but each eigenproblem involves only a \( 3N \times 3N \) matrix, where \( N \) is the number of \( y \) grid points and the factor of three is
due to the three velocity components. The modes satisfy orthonormality and energy
relations analogous to the one-dimensional case.

2.3.5 **POD Domain**

In three dimensions, POD defines the norm for which orthogonality is defined by
integrating over a volume. A choice of domain must therefore be made. In the
case of a turbulent channel, two dimensions are homogeneous, so the full extents
of available data are used in these directions, but the choice of domain for the in-
homogeneous direction bounded by two walls remains. Common choices are the
full domain between the walls or half of the domain, from one channel wall to the
centerline, due to the statistical symmetry that is present about the centerline. In
their study of turbulent channel POD, Moin & Moser (1989) principally used the
half channel domain. To focus on the particular structures dominating the near-
wall regions and the core region away from the wall, they also performed POD on
these domain subsets separately. The present study wishes to describe large struc-
tures and their full extents, so the full computational domains are generally used.
However, to gain insight on the effect of the symmetry about channel centerlines
for POD modes, half-channel modes are also computed for comparison of one-
dimensional POD modes that vary along the wall-normal coordinate (Chapter 3).
Also, for turbulent boundary layers in which the flow thickens along the streamwise
coordinate and is therefore not homogeneous, it is useful to use similar streamwise
subdomains for the purpose of comparing between different simulations (Chapter
8).
Chapter 3

STRUCTURE, SCALING, AND SYNTHESIS OF ONE-DIMENSIONAL POD MODES OF INHOMOGENEOUS TURBULENCE

By examining the proper orthogonal decomposition (POD) of turbulent channel flow and turbulent thermal convection in the inhomogeneous direction of each flow it is found that the POD eigenfunctions separate into two types of modes having different structure and scaling. For these flows, the modes can be represented in the form of amplitude-modulated and phase-modulated oscillations in the wall-normal direction. The phase functions can be extracted by several means, and they contain much of the physics embedded in the two-point spatial correlation function. The phase and envelope functions are analyzed to interpret POD mode behavior. The first several most energetic modes are large scale, and their structure is idiosyncratic relative to the rest of the modes and characteristic of the type of the flow. Outside of this group of idiosyncratic modes, the remaining modes are less energetic, smaller scale, and asymptotically self-similar in the sense that the phases scaled using frequency correlate well and approach a single limiting curve with increasing order (mode number). This asymptotic phase function is characteristic of the type of turbulence, i.e. it differs between channel flow and thermal convection, despite it being small-scale and therefore, presumably insensitive to the form of the boundary conditions.

The work of Moser (1994) provides an approximate relationship for the asymptotic phase function in terms of the distribution of viscous dissipation, $\epsilon(y)$. A more general framework for this relationship is presented in which the idiosyncratic modes determine the energy distribution and the asymptotically self-similar modes determine the distribution of viscous dissipation.
The foregoing results are exploited to develop a complete set of orthonormal basis functions that approximate the POD modes and possess convergence properties that are comparable to the optimal convergence of the exact POD and significantly better than those of conventional orthogonal polynomials. This procedure, called ‘synthetic POD’ uses only knowledge of the distribution of mean dissipation (or equivalently, the asymptotic phase function) without a priori computation of the more complicated two-point correlation functions.

Acknowledgments: Substantial motivation came from unpublished work of Z.-C. Liu and R. J. Adrian. Acknowledgement is due to Prof. R. D. Moser for supplying the $Re_t = 934$ channel DNS data, which was produced in conjunction with Prof. J. Jiménez, Prof. J.-C. del Álamo and Dr. P. S. Zandonade. Computations were performed using the Ira A. Fulton High Performance Computing Initiative facilities at Arizona State University and the computer time is gratefully acknowledged.

3.1 Introduction

Analysis of random data by proper orthogonal decomposition (POD) yields an optimal set of orthonormal basis functions (eigenmodes) and a representation of the process in terms of a linear combination of statistically uncorrelated modes. The POD modal expansion is optimal in the sense that it converges faster than any other modal expansion (Berkooz et al., 1993; Lumley, 1970; Holmes et al., 1998). Physical information about the structure of any particular random process enters the POD through the two-point correlation functions. As originally applied to turbulence, POD was intended to extract the coherent structures of a turbulent flow in the form of the modes (Lumley, 1970). However results in this regard require careful interpretation because, in general, the POD modes of any statistically homogeneous process are trigonometric, and POD yields no information about the form of coherent structures in this class of turbulent flows. (All of the interesting physical infor-
mation is contained in the eigenvalue spectrum of the POD.) This lack of structure lead Lumley (1970) to develop the characteristic eddy formalism for homogeneous flows. In the case of statistically inhomogeneous flow, the eigenmodes do contain physical information about the structure, and a rather limited body of experience indicates that the vector structures of individual modes, especially those that are most energetic, can be related to fundamental coherent structures (Aubry, 1991; Moin & Moser, 1989; Liu et al., 2001). Modal decomposition in space-time also provides a means for identifying propagating and non-propagating modes (Sirovich et al., 1991).

In addition to extracting coherent structures, the rapid convergence of POD has been exploited to develop low-order dynamical models of turbulent flows (Aubry et al., 1988; Podvin & Lumley, 1998; Holmes et al., 1998). When such models are used to reduce the order of the equations for numerical simulation purposes, the application of POD is straight-forward. But, when the low order models are also used to explore the dynamics of turbulence, it becomes necessary to develop a deeper understanding of the manner in which POD modes embody the physics of the flow. The present study examines how the physics implied by the two-point correlation is represented by the modes and how the modes should be interpreted for inhomogeneous flow.

A small body of research has already been conducted along these lines. For example, in regard to energy distribution, Knight & Sirovich (1990) reported that POD reveals an inertial-range spectrum for inhomogeneous flows. Regarding scaling, Liu et al. (1994) examined one-dimensional POD modes of streamwise velocity obtained from the inhomogeneous wall-normal direction of a turbulent channel at two Reynolds numbers, and reported POD modes scaled with outer variables were essentially independent of Reynolds number in the region outside of the wall layer.
Regarding scale similarity, Carbone & Aubry (1996) approached POD modes from the perspective of exploiting symmetries in the Navier-Stokes equations and their consequences for the POD modes. They developed stretching relations that related POD modes of different mode number. Significantly, they developed a link between the scaling of energy spectra and the scaling (by stretching) of POD modes. However, they were unable to address how the stretching might vary with Reynolds number. Carbone & Aubry (1996) observed that the stretching laws did not predict the most energetic mode shapes properly. Higher order modes in the energy spectrum exhibited an exponential decay that was consistent with stretching symmetry, but the lower order modes did not, a fundamental difference.

In the present paper, POD is condensed to the simplest case possible: one velocity component as a function of one inhomogeneous spatial dimension. While simplifying the analysis, the results are important to understanding how more complex POD analyses represent turbulent flows. The oscillatory nature of POD modes suggests that they be decomposed to concentrate the oscillations and that analysis of the decomposed form would better reveal the physics than examining the modes themselves. In addition, the form suggests that synthetic modes that retain essential POD features may be constructed.

POD modes of two canonical flows bounded by plane, parallel walls are compared: turbulent channel flow and turbulent Rayleigh convection (Fernandes, 2001). The noise-free and well-resolved nature of the direct numerical simulation (DNS) of channel flow at three Reynolds numbers allows computation of POD modes to high order. Two-point spatial correlation functions extracted from particle image velocimeter measurements of turbulent Rayleigh-Benard convection (Fernandes, 2001) were used to compare POD mode behavior with the channels. The different physics of this flow help clarify what features are common and where the different underlying physics appears in the POD modes.
3.2 Background

3.2.1 POD Equations

The POD equations for a function of one spatial variable are briefly summarized below with reference to results reported elsewhere (Holmes et al., 1998; Moin & Moser, 1989). The proper orthogonal decomposition generates a basis \{\phi^{(n)}(y)\} such that

\[ u_i(y) = \sum_{n=1}^{N} a_n \phi^{(n)}(y), \]  

(3.1)

where \( n \) is mode number that will be subsequently defined. The POD modes satisfy the integral equation

\[ \int_{y_1}^{y_2} R_{ij}(y, y') \phi_j(y') dy' = \lambda \phi_i(y), \]  

(3.2)

where the two-point correlation is \( R_{ij}(y, y') = \langle u_i(y) u_j(y') \rangle \). The modes are orthogonal, and they are normalized such that

\[ \int_{y_1}^{y_2} \phi^{(n)}(y) \phi^{(m)*}(y) dy = \delta_{nm}. \]  

(3.3)

The turbulent kinetic energy \( E \) is related to the eigenvalues of (3.2) by

\[ E = \int_{y_1}^{y_2} \langle u_i(y) u_i(y) \rangle dy = \sum_{n=1}^{N} \lambda^{(n)}. \]  

(3.4)

(Summation is conventionally assumed on the repeated indices, but for the purpose of one-component POD analysis, summation is omitted.)

Each POD mode is assigned a mode number \( n \). Although the mode number is commonly assigned based on ordering the eigenvalues in descending order, another way is by sequency, defined as the number of internal zero crossings \( \phi^{(n)}(y) = 0 \), which is generally different for each mode. We define mode number as

\[ n \equiv \text{sequency} + 1 \]  

(3.5)

because it is convenient for scaling purposes. Ordering by increasing sequency and by decreasing mean energy contribution are equivalent if the energy contribution
Table 3.1: Summary of parameters for DNS databases. The $L$ components are the computational domain dimensions, and the $N$ components are the numbers of Fourier or Chebyshev modes (not including dealiasing modes). Two measures of the statistical convergence are reported: nondimensional run time $TU_b/L_x$ ($U_b$ is bulk velocity) and number of fields included in the averaging $N_f$.

<table>
<thead>
<tr>
<th>$Re$</th>
<th>$L_x$</th>
<th>$L_z$</th>
<th>$N_x \times N_y \times N_z$</th>
<th>$TU_b/L_x$</th>
<th>$N_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>$4\pi h$</td>
<td>$\frac{2}{3}\pi h$</td>
<td>$128 \times 129 \times 128$</td>
<td>9.3</td>
<td>300</td>
</tr>
<tr>
<td>395</td>
<td>$2\pi h$</td>
<td>$\pi h$</td>
<td>$256 \times 193 \times 192$</td>
<td>26.1</td>
<td>150</td>
</tr>
<tr>
<td>934</td>
<td>$8\pi h$</td>
<td>$3\pi h$</td>
<td>$2048 \times 385 \times 1535$</td>
<td>9.2</td>
<td>21</td>
</tr>
</tbody>
</table>

decays monotonically with increasing sequency. This occurs after the first few modes in the present results. We prefer sequency ordering because it is analogous to the frequency ordering of trigonometric functions that is used in Fourier analysis.

Using the two-point correlation computed at every wall-normal grid point, the integral equation (3.2) is converted to a matrix eigenvalue problem by numerically approximating the integration. The trapezoidal rule is sufficient for this purpose, as shown by Moin & Moser (1989).

### 3.2.2 Data Sets

For the turbulent channels with half-height $h$, the coordinates $x$, $y$, and $z$ refer to the streamwise, wall-normal, and spanwise directions, respectively, and the velocity components in corresponding directions are $(u_1, u_2, u_3) = (u, v, w)$. Table 3.1 summarizes the simulation parameters. The domain sizes and grid spacings for the $Re = 180$ and 395 channels match those of prior simulations (Kim et al., 1987; Moser et al., 1999). These direct numerical simulations of the incompressible Navier-Stokes equations were performed using the Fourier/Chebyshev spectral code used by Kim et al. (2008). The $Re = 934$ channel data was computed by del Álamo et al. (2004). For all channels, data is located on a Chebyshev Gauss-Lobatto grid in $y$. To obtain the fluctuations, the streamwise velocity mean profile was obtained from a long time average in addition to spatial averaging in the homo-
geneous directions, and the other velocity components were assumed to have zero mean.

Particle image velocimetry (PIV) measurements of turbulent Rayleigh-Bénard convection were obtained and used to perform POD calculations in Fernandes (2001), and experimental details are documented therein. The Rayleigh numbers were chosen to place the convection in the fully turbulent regime, and the data shown in the following results was at $Ra = 2 \times 10^8$. The turbulent thermal convection Reynolds number is $Re_* = 445$, based on a velocity scale $w_*$ and the entire layer depth (not half height as for the channels) (Fernandes, 2001). The flow is inhomogeneous in the coordinate representing the location between and normal to two parallel plates of different temperature. For both channel and convection flows, we will use $y$ as the wall-normal coordinate with nondimensionalization so $y$ ranges from $-h$ to $h$. Wall-parallel ($u$) and wall normal velocity components were measured for convection. Fernandes notes the PIV resolution with uniformly spaced data is not sufficient to resolve the inner layer, but the small scales in the central 90% of the flow can be considered resolved for this Rayleigh number. The measured velocities are assumed to be the fluctuations because all mean velocities should be zero from symmetry.

3.2.3 POD Modes

Carbone & Aubry (1996) performed one-dimensional POD analysis of $Re_\tau = 180$ turbulent channel flow using all velocity components but examined only the $u$ velocity components. Comparing the single-component $u$ modes presented here with the $u$ component of the POD modes calculated using all three components for channels reveals most modes are virtually identical for either case. Fernandes (2001) verified that wall-parallel component $u$ modes are statistically independent of other components for turbulent convection.
Figure 3.1: Comparison of (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, and (d) $n = 4$ POD modes for the streamwise velocity of turbulent channels at several Reynolds numbers and POD modes of wall-parallel motion for thermal convection. Sinusoidal modes are also included for reference.
The POD modes based on the streamwise $u$ velocity component for all three Reynolds numbers appear similar to those of figure 2 in Carbone & Aubry (1996). Figure 3.1 includes the first four modes and figure 3.2 contains the $n = 20$ mode, which displays oscillating behavior that is typical of higher-$n$ modes. This figure indicates the oscillations are compressed toward the wall with increasing Reynolds number. All modes exhibit steep linear regions near the walls in which slopes increase with increasing Reynolds number. The difference in low-$n$ mode shapes with higher Reynolds number is consistent with 1-D modes at $Re_\tau = 315$ obtained from the data of Liu et al. (2001). The POD analysis produces the same number of modes as grid points so any signal represented on these points may be perfectly reconstructed. Trapezoidal rule integration causes a few underresolved modes with highest $n$ to contain irregular oscillations, but these fine scale modes contribute very little energy.

To provide an overview of how these modes vary with mode number, figure 3.3 displays contours of the $Re_\tau = 395$ channel $u$-only POD modes, where each
Figure 3.3: Contour plot of 1D POD modes for $u$ component of $Re_\tau = 395$ turbulent channel for all modes with $n$ that is (a) odd and (b) even. Black represents negative and white represents positive $\phi$ values. The arrow in (b) indicates the $n = 20$ mode that is shown in figure 3.2.

cross section of constant $n$ is a single POD mode. Displaying the modes in this manner reveals the structure embedded in the modes, but the structure is obscured by how the signs of the oscillations vary with $n$. This presentation of the mode shapes is related to figure 5 of Carbone & Aubry (1996), in which points are plotted representing the zeros of each mode, which are the gray regions of strong gradient between black and white in figure 3.3. However, Carbone and Aubry display the points on a linear axis in mode number but a logarithmic axis in $y$. They also split into even and odd modes to avoid the alternating peaks and zeros at the centerline, and displaying only zero locations makes the alternating peak sign or derivative at the center not visible.

3.3 Reynolds Number Similarity of the Eigenvalue Spectra

3.3.1 Outer Scaling

To examine how the eigenvalue spectra scale with Reynolds number, the channel spectra are presented with both linear-log and log-log axes in figure 3.4(a). This is the classical presentation, except eigenvalues are plotted as a function of $n$ instead of mode number ordered by decreasing energy. The eigenvalue spectra display similarity for low $n$, as found by Liu et al. (1994). They stated the principle of POD similarity as follows: “In the region outside the wall layer and in the range of
sequences where the wavelength is large compared to the wall layer viscous length scale, both the eigenfunctions and the spectrum of the eigenvalues when scaled by outer layer variables are independent of the Reynolds number for sufficiently large Reynolds number.

The behavior at higher $n$ is qualitatively similar to Fourier wavenumber spectra where lower Reynolds number curves exhibit viscous effects at smaller values of $n$ (or wavenumber) than higher Reynolds number curves. Similar behavior exists in the eigenvalue spectra obtained from a Burgers’ model of turbulence at several Reynolds numbers (Chambers et al., 1988; Sung & Adrian, 1994). (The PIV data of Liu et al. (1994) at $Re_T = 315$ and 1414 did not exhibit this behavior as clearly as the present DNS data, presumably due to experimental attenuation of the smaller scales.) Liu et al. (1994) interpreted $n$ to be the analog of the wavenumber of a conventional trigonometric polynomial by noting that the average wavelength of POD mode oscillations is $\lambda_{\text{mean}} = 2h/(n/2)$, and the corresponding wavenumber is $\pi n/(2h)$. Nondimensionalized in outer scaling by the length scale $h$, the wavenumber is $(\pi/2)n$. Nondimensionalizing the eigenvalues by the mean total...
energy is equivalent to normalizing by the $u$-RMS velocity squared (along with the channel half height), and the behavior appears nearly identical when scaled by $u_2^2$ instead. Fernandes (2001) also observed that the eigenvalue spectra of one-dimensional POD in outer scaling agree at low $n$ for turbulent convection over a range of Rayleigh numbers. Thus, the present data support the principle of outer similarity in eigenvalues as stated in Liu et al. (1994).

3.3.2 Inner Scaling

The analogy to Fourier spectra suggests that inner scaling can also be applied using the appropriate inner length and velocity scales. Inner scaling is obtained by nondimensionalizing $n/h$ by the Kolmogorov length scale $\eta \equiv (\nu^3/\epsilon)^{1/4}$ based on viscosity $\nu$ and dissipation $\epsilon$. Although dissipation may vary by two orders of magnitude (for $Re_\tau = 934$) across the channel $y$ location, the Kolmogorov length scale varies less. The nondimensionalization uses $\eta_{\text{mean}}$, the mean length scale obtained by calculating $\eta(y)$ and averaging across the channel in $y$. The eigenvalues have dimensions of [velocity]$^2 \times$[length], with the length scale appearing due to the integration over the channel wall-normal location $y$ in (3.4). They are therefore nondimensionalized by $\eta_{\text{mean}}$ and the Kolmogorov velocity scale $u_\eta \equiv (\nu \epsilon)^{1/4}$, using $u_{\eta,\text{mean}}$ obtained by similar averaging.

Figure 3.4(b) shows the spectra with this inner scaling, and the curves exhibit similarity between Reynolds numbers for large $n$, similar to Fourier wavenumber spectra with Kolmogorov scaling. (Beyond a certain $n$ the eigenvalues are not meaningful due to grid resolution that is insufficient to resolve all of the zero crossings. Therefore, the eigenvalues for these modes are not included in figure 3.4.)

3.3.3 Power Laws

Lines corresponding to $-1$ and $-5/3$ power laws in $n$ are included in figure 3.4(b) for reference. Although the spectra initially decay approximately at the $-1$ rate,
adherence to this law is not convincing. Likewise, the region of the spectra tangent to the $-5/3$ line does not exist over a wide enough range to support a Kolmogorov $-5/3$ power law.

Moser (1994) used the Kolmogorov similarity law for the inertial range to approximate POD modes, which is discussed in §3.7. The analysis suggested a $-5/3$ law in a parameter that we show to be analogous to $n$. However, Moser observed no clear inertial range in the one-dimensional spectrum for three-dimensional POD modes of $Re_{\tau} = 180$ channel flow; this was attributed to the low Reynolds number.

In contrast, Carbone & Aubry (1996) suggested the spectrum decays exponentially after the first few $n$, a feature that is, on casual examination, also observed in figure 3.4. However, on more careful examination it is clear that the entire curve cannot be represented as a single exponential function. (Although, past the first few $n$, exponential functions could be fitted to various approximately linear segments of the linear-log plot.)

3.4 Mode Decomposition
3.4.1 Decomposition Form

The oscillatory nature of the POD modes (figure 3.2) resembles a phase-modulated and amplitude-modulated trigonometric function and suggests the modes be expressed in the form

$$
\phi_i^{(n)}(y) = A_i^{(n)}(y) \sin(\beta_i^{(n)}(y)),
$$

(3.6)

where the envelope (or amplitude) function $A_i^{(n)}(y)$ and phase function $\beta_i^{(n)}(y)$ can always be chosen to exactly reconstruct the mode. This decomposition allows the complicated mode structures to be reduced into two simpler functions that can be more profitably analyzed. This decomposition is not unique as a method for dividing into the amplitude and phase functions must be specified. We have considered three methods: 1) Hilbert transform; 2) phase zero; 3) envelope spline.
3.4.2 Hilbert Transform and Phase Zero Methods

Modes can be decomposed (Fernandes, 2001) using the Hilbert transform to generate the imaginary part of an analytic signal (also known as a pre-envelope in Dugundji (1958)), from which the phase and envelope are extracted. The Hilbert transform of a real-valued function $\phi(y)$, denoted by $\mathcal{H}\{\phi(y)\}$, is defined by $\mathcal{H}\{\phi(y)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\eta)}{y-\eta} \, d\eta$, where the Cauchy principal value of the integral is used.

The transform shifts the phase of each Fourier component of the signal by $90^\circ$ so $\mathcal{H}\{\phi(y)\}$ contains a peak where the $\phi(y)$ has a zero crossing for each contributing Fourier mode. The envelope function is defined as the modulus of the analytic signal $\phi(y) + \mathcal{H}\{\phi(y)\}i$, so this formulation fills between the peaks of the original function, resulting in a smooth envelope function. The phase of the sinusoidal term (responsible for the small-scale oscillation) is obtained from the arctangent of the complex analytic signal components with $2\pi$ phase increases added for continuity.

The finite domain of the POD modes complicates this method, and different remedies produce significant variation in the results. These issues are not significant for long signals with many oscillations (as encountered in other turbulence applications, such as in Mathis et al. (2009a); Schlatter & Orlu (2010)) but POD modes with few oscillations require a more robust decomposition method.

Carbone & Aubry (1996) noted that zero locations are robust features in their analysis of POD modes, and these features are related to the phase. Since the envelope curve is smooth and positive-valued in the domain, the zeros of the POD mode must be matched by the zeros of the sinusoidal function in (3.6), where a zero occurs when the phase function is an integral multiple of $\pi$. Starting with an initial phase of $\beta^{(n)}(y = -h) = 0$, the $y$ value may be determined at each $\phi^{(n)}(y) = 0$ where the phase function has increased by $\pi$ from the last zero assuming that $\beta$ increases monotonically with $y$. Thus, a $(y, \beta^{(n)}(y))$ pair can be obtained for each zero and this
forms a good representation of $\beta_i^{(n)}(y)$ for modes with numerous zero crossings. Since there is a zero in each mode at $y = h$ due to the wall boundary conditions, the maximum phase of each mode is $\beta^{(n)}(h) = \pi n$. Accurate zero locations are obtained using cubic spline interpolation. The phases obtained from the Hilbert transform method exactly satisfy the zero crossing locations.

### 3.4.3 Envelope Spline Method

Since the goal of the representation in (3.6) is to concentrate the physical scale information into the sinusoidal term, oscillations of the envelope should be minimized. The representation can be made unique by specifying a smooth amplitude function and solving for the phase function:

$$\sin(\beta_i^{(n)}(y)) = \frac{\phi_i^{(n)}(y)}{A_i^{(n)}(y)}.$$  \hspace{1cm} (3.7)

A positive-valued envelope forces the sinusoidal function to represent the oscillations. Therefore $\beta_i^{(n)}(y)$ is an integer multiple of $\pi$ at each mode zero (consistent with the zero phase method). Choosing $A_i^{(n)} = \max |\phi_i^{(n)}(y)|$ and solving (3.7) for $\beta_i^{(n)}(y)$ would guarantee that form (3.6) would reproduce each mode everywhere in the domain, but it would also result in phase function discontinuities at each peak of the mode. If the $\sin(\beta(y))$ term oscillates through $\pm 1$ with $\beta$ increasing smoothly, then the resulting phase function is continuous in this region. This requirement is satisfied if the envelope function is tangent to each of the mode’s peaks. Although the envelope function is not an envelope in the common mathematical sense of being tangent to a family of curves, the terminology is common in relation to the analytic function, and also appropriate because $A_i^{(n)}(y)$ is tangent to each peak of $|\phi_i^{(n)}(y)|$.

Thus, selecting a smooth envelope function with minimal oscillation that is tangent to each peak of the mode results in a representation with a (generally) continuous phase function. There is freedom in specifying envelopes that satisfy this re-
quirement when there are few peaks, but very little when several peaks are present. Our approach will be to concentrate the variations in the mode into the sinusoidal function by generating an envelope function with a low-order polynomial spline fit to the mode peaks to smoothly interpolate. The envelope curve is interpolated using a point at each of the \( n \) peaks, and (3.7) indicates the correct phases to attain a zero at the \( n + 1 \) mode zeros are also obtained by this method. Therefore, this method calculates the phase function using approximately twice as much information (data points) as the phase zero method and guarantees envelope smoothness. The sparsity of data points for the first few (low \( n \)) modes makes the phase function less constrained and more dependent on the interpolation spline. However, splines of different polynomial order produce consistent results (§3.5.1).

After establishing the envelope, solving (3.7) yields the phase function. The phase function is adjusted (in ways that preserve its sine value) such that it is smooth and nondecreasing with \( y \). An exception is discontinuities where local extrema occur in the mode with the concavity directed away from \( \phi = 0 \). Results indicate this provides a phase function that is consistent with the phase zeros and the symmetries implied by a symmetric mode.

3.5 Phase and Envelope Functions

3.5.1 Results

Examining the results for streamwise (\( u \)) fluctuating velocity modes of the \( Re_{\tau} = 395 \) channel reveals important features of inhomogeneous turbulent channel modes. The envelope spline method yields the phase functions in figure 3.5. For this and subsequent phase plots, several of the high \( n \) phase curves are obtained from the phase zero method instead because the mode peaks are underresolved on the computational grid, degrading the envelope spline interpolation. We omit from the phase plots the very highest modes that are underresolved so zeros are missed. The
subscripts specifying velocity component are omitted hereafter because the components are indicated in the text and plots.

We have shown that the maximum phase (at the $y = h$ wall) is $\beta^{(n)}(h) = \pi n$ (§3.4.2) and this suggests that scaling the phase functions by mode number would clarify how their behavior changes with $n$. Figs. 3.5 and 3.6 contain the scaled phase functions

$$\beta_0^{(n)}(y) \equiv \beta^{(n)}(y)/n$$

(3.8) calculated by both the envelope spline and Hilbert transform methods. The Hilbert transform method is calculated in Fourier space with the signal zero padded to twice
Figure 3.6: Scaled phase of the \( n = 1 \) to 5 modes for the \( Re_\tau = 395 \) channel obtained using the envelope spline and Hilbert transform methods. Numbers indicate the mode number and are colored by method as are the lines. Black dots indicate the zero locations, with phase increasing by \( \pi \) between each zero (before scaling).

Regardless of the method used, the scaled phase functions all correlate well with each other except for the first few modes, establishing the robustness of the decomposition and indicating asymptotic self-similarity between modes of sufficiently high order. Modes \( n = 1–5 \) are shown in figure 3.6 and the remaining higher modes are shown in figure 3.5. The phase functions of high \( n \) are consistent between the methods considered. Since the methods satisfy the correct phases at each mode zero, the methods agree closely with sufficient zeros, and thus the phase-envelope decomposition concept is robust.

The phase functions of the lowest order modes (figure 3.6) behave differently from those of the asymptotically self-similar modes. They also exhibit greater dependence on the decomposition method. The \( n = 1, 3, 5 \) modes for the \( u \)-component POD of the channel, at all Reynolds numbers considered, each contain a local ex-
Figure 3.7: Envelope functions of the channels (obtained with the envelope spline method) for $n = 30$ to 50 modes comparing the Reynolds number effects. Insets: Envelope curves of $n = 1$ to 100 POD modes for the $Re_r = 395$ channel obtained using the (a) envelope spline and (b) Hilbert transform methods.

tremum with the concavity directed away from $\phi = 0$ at the centerline. The corresponding phase functions calculated using the envelope spline method include discontinuities at $y = 0$, and the phase is nondecreasing everywhere else. The phase functions obtained with the Hilbert transform method are also nondecreasing for all modes except for these, and these modes contain unwanted oscillations in the envelope that are absent for the envelope spline method.

The envelopes (figure 3.7) obtained from the envelope spline and Hilbert transform methods behave similarly. The main difference is envelopes obtained from the Hilbert transform method rapidly decrease approaching the edges of the domain, whereas those obtained from the envelope spline method smoothly increase based on extrapolation of the spline. For low $n$, the scale separation between the
fast oscillation (sinusoidal term) and the slow variation (envelope function) is not strong, and the decomposition is more ambiguous. The envelope spline method yields a constant-valued envelope for the \( n = 1 \) and 2 modes (when the mode possesses symmetry in \( y \)), and all of the mode variation is concentrated in the phase function. After the first few \( n \), the envelopes appear similar and vary little with \( n \) for a wide range of mode numbers.

The results indicate the envelope spline method generates a reliable decomposition in the form of (3.6) that concentrates the variations of the POD modes into the phase function so the envelope is smooth and has minor variation over a wide range of \( n \). Therefore, all phase functions discussed henceforth are generated using this method. Results are insensitive to choice of spline order and endpoint conditions, and natural cubic splines are used for the results presented.

### 3.5.2 Scales and Similarity

The phase and envelope functions yield insight into the form of the POD modes. They represent the oscillatory behavior of the POD modes in a manner that eliminates the issues related to even and odd \( n \) (discussed in connection with figure 3.3). This unifies the behavior of modes with neighboring \( n \) and allows meaningful comparisons for different flows. The POD modes of channel flow (Figs. 3.1 and 3.2) exhibit boundary-layer behavior near the wall as a steep region that transitions into oscillations of progressively longer wavelength as the centerline is approached. Because of the smaller scales of turbulence in the wall region, the one-dimensional phase curves increase more steeply near the wall than the centerline. The buffer layer appears as a particularly steep near-wall region in the phase function.

These scaled phase plots demonstrate behavior that occurs for channels at all Reynolds numbers: the modes clearly group into two regimes based on the scaled phase function. The phase functions of modes with \( n \) above a certain value correlate
well with each other (for a given flow), and these are designated as the asymptotically self-similar modes. The modes with lowest $n$ involve the largest wall-normal length scales, and the scaled phases vary significantly with $n$. From this and other behavior we observe, these modes are termed the idiosyncratic modes. The relatively sharp distinction between idiosyncratic and asymptotically self-similar behavior suggests that the modes represent two different physical mechanisms.

Figure 3.8 shows the phase functions for the idiosyncratic modes for all of the flows considered. The scaled phases of these channel $u$ velocity modes behave differently for each $n$ with only the odd $n$ modes containing discontinuities. This behavior exists at all Reynolds numbers in the idiosyncratic mode group (low $n$) but not in the asymptotically self-similar modes. The difference between the channel and the thermal convection phase functions for the first few $n$ is striking: the
scaled phases for wall-parallel thermal convection velocity increase in a nearly linear manner with $y$ for even these first few modes and are not significantly different from those of the asymptotically self-similar modes (see also figure 3.1).

Focusing on the higher-$n$ modes for channel flow, the scaled phase functions for $n = 50$ to 100 are compared in figure 3.9, and it is clear that the phase curves asymptote to a slightly different non-linear curve for each Reynolds number. The asymptotes of the scaled phase functions for $Re_{\tau} = 395$ and 934 curves are nearly the same, suggesting that there is a high Reynolds number asymptote for the high sequency asymptotes (not including the boundary layer region). The largest difference in the set of scaled phase functions shown for these two Reynolds numbers exists near the walls where the boundary layers are compressed to the walls so the
Figure 3.10: Comparison of POD mode phase functions over one half of the domain with different velocity components for the $Re_\tau = 395$ channel and convection $u$ modes. The inset focuses on the near-wall region with convection omitted for clarity. Modes $n = 50$ to 100 are shown for the channel and $n = 20$ to 45 for convection.

Mode gradient and phase function are steeper for higher Reynolds number. This is consistent with the results of Liu et al. (1994) and the principle of outer-scale similarity stated therein (described in §3.3.1).

To examine how the POD modes of high $n$ vary for different flow physics, figure 3.10 compares the phase functions of other velocity components of channel flow and the turbulent convection flow. The phase functions of the wall-normal and spanwise velocity components for the channel are similar to each other. Those of the streamwise velocity modes are slightly steeper in the region near the wall and have more curvature than those of other components. Closer examination of
the near wall region (in the inset) shows the phase functions of the wall-normal velocity modes differ qualitatively from those of the \( u \) and \( w \) components, probably because the wall-normal derivative is zero for the wall-normal velocity but not for the other components. The idiosyncratic modes (not shown) are markedly different for the \( u \), \( v \), and \( w \) components of the channel. The phase functions for wall-parallel velocity modes of turbulent convection are nearly linear and possess a very different character from those of both the streamwise and spanwise velocity modes for the channel.

The envelope functions for the channel at each Reynolds number are also similar for high values of \( n \) (figure 3.7). The envelope functions increase near the wall with increasing Reynolds number. The series of Reynolds numbers considered here suggests that the high-\( n \) envelope functions approximately asymptote in shape near the center as Reynolds number increases, but the differences with Reynolds number extend further from the wall than for the phase function. A consequence of the normalization in equation (3.3) is that an envelope with higher peaks near the walls will have lower magnitude near the center.

3.6 Mode Synthesis

The preceding analysis demonstrates upon scaling by \( n \) the phase functions for each type of flow collapse well onto each other for all but a few idiosyncratic modes. This suggests we can exploit the similarity of scaled phase to develop a means of synthesizing modes that will closely approximate POD based on a single phase function. Similar synthesis for POD modes of thermal convection was performed by Fernandes (2001). The single phase function is of the form (3.8) and is denoted by \( \beta_0(y) \) without the \( (n) \) superscript to indicate it is shared by the synthetic POD (SPOD) modes of all \( n \).
Placing $\beta^{(n)}(y) = n\beta_0(y)$ for each $n$ into the mode representation (3.6) and substituting this into the POD mode orthonormality relation (3.3) yields

$$\int_{y_1}^{y_2} A^{(n)}(y)A^{(m)}(y) \sin(n\beta_0(y)) \sin(m\beta_0(y)) \, dy = \delta_{nm}. \quad (3.9)$$

Synthetic modes are obtained by requiring each envelope $A^{(n)}(y)$ to be identical:

$$A^{(n)}(y) = A(y).$$

Changing the integration variable,

$$\int_0^\pi A^2(\beta_0) \sin(n\beta_0) \sin(m\beta_0) \, d\beta_0 = \delta_{nm}. \quad (3.10)$$

Orthogonality is satisfied if we require $A^2(\beta_0)/(d\beta_0/\,dy)$ be independent of $\beta_0$, and the modes are orthonormal when it is equal to $2/\pi$. Thus, the SPOD envelope function is also determined by $\beta_0(y)$. The SPOD modes, which we denote by $\psi^{(n)}(y)$, are

$$\psi^{(n)}(y) = \sqrt{\frac{2}{\pi}} \frac{d\beta_0(y)}{\,dy} \sin(n\beta_0(y)). \quad (3.11)$$

They are orthonormal and have self-similar phase functions that are close to the form of the asymptotically self-similar phase functions of true POD. In the range of $n$ corresponding to the idiosyncratic POD modes with non-similar phase, the phase functions of SPOD are necessarily similar.

Although $A^{(n)}(y)$ and $\beta_0^{(n)}(y)$ slowly change with $n$ for true POD, the SPOD assumptions of mode independence should still capture the essential features of POD, especially the phase profile that is characteristic of each type of flow. We will assess how closely SPOD results match the actual POD in several respects. Figure 3.11 compares the true POD modes of the $u$ velocity from the $Re_\tau = 395$ channel flow with SPOD modes based on the scaled phase function $\beta_0^{(100)}(y)$ for the $n = 100$ mode obtained from the POD. The match between modes would be expected to be best for $n = 100$ because the phase would be precisely correct and the difference would be due to the discrepancy between the envelope function $A^{(100)}(y)$ and the SPOD approximation in (3.11). The half of the $n = 65$ SPOD mode shown in the
Figure 3.11: Comparison of POD modes and their SPOD counterparts for $u$ velocity of the $Re = 395$ channel. The synthetic POD is based on the phase function for the $n = 100$ mode. Modes $n = 1$ and 2 are shown in (a) and the $n = 1$ mode calculated on a half channel domain is also included. The left half of (b) contains $n = 25$, and the right half of (b) contains $n = 65$. Spline interpolation is used to generate the lines in part (b) so the sampling effects do not obscure the mode shapes. The lowest-sequency POD mode for a half channel domain is also shown in (a) after renormalizing the magnitude so its norm over the half channel domain is half of the norm of the other modes over the full channel.

right half of figure 3.11(b) matches closely with the actual POD mode. However, to agree perfectly, the SPOD oscillations should have longer wavelength near the centerline and be more compressed in the region approaching the wall layer. This becomes more pronounced in the $n = 25$ mode. Figure 3.11(a) compares the $n = 1$ and 2 modes with the corresponding SPOD modes, and they are substantially different, as expected, because these are in the idiosyncratic mode regime while SPOD is based on the asymptotically self-similar modes. (Stretching the SPOD modes further toward the wall would improve the match.)

Carbone & Aubry (1996) examined the relationship between POD modes of different $n$ in terms of stretching in $y$ and developed stretching laws to generate a set of modes approximating the 1D POD modes of turbulent channel flow. The generation is based on stretching what they term the ‘mother mode.’ It must be of high sequency because the mode is symmetrically stretched outward (toward the walls) with the outermost zeros stretched out of the domain. This process is
repeated to generate subsequent modes of lower sequency. Since the presence or absence of a zero in the center of the channel (determined by oddness or evenness of the sequency) does not change with stretching, there must be one mother mode for odd sequency and another for even sequency. They calculated the stretching laws using the zero crossing locations because they are the most reliable points of the POD modes to track.

Their procedure has been applied to the $u$ velocity of the $Re = 395$ channel flow to evaluate its ability to relate POD modes of different $n$. Figure 3.12 compares the
zeros from the POD modes to the zeros obtained from the Carbone and Aubry procedure and the synthetic POD modes developed above. Carbone and Aubry produced a similar plot of zeros by using an equation they developed that generates all of the zero locations without a mother mode (Eqn. 20 in their manuscript). Figure 3.12 obtains the zero locations using the zeros locations of a mother mode and calculating the zero locations for lower $n$ by repeated stretching. This is consistent with the zero locations that would exist when following their stretching procedure for estimating modes (as they performed to create figure 2 in their manuscript), and it provides superior agreement with the zero locations from the actual POD modes. The $n = 54$ mode was chosen as the mother mode, and the linear fits of zero locations to compute the required stretching parameters were performed using zeros for modes $n = 6$ to 54. The zeros for the SPOD method are directly generated from the phases that satisfy $\sin (n \beta_0(y)) = 0$, with $\beta_0$ chosen from the phase function obtained from the $n = 54$ POD mode. Thus, the figure begins with the correct zero locations for $n = 54$ and calculates the zero locations for other $n$ according to the synthetic POD form (sharing a common scaled phase) and the stretching procedure of Carbone and Aubry, which produces zero locations only for lower $n$ as the mother mode is stretched out of the domain.

Figure 3.12 shows that the qualitative trends of both methods in calculating zero locations are correct, but the zero locations predicted by the two methods differ perceptibly from the POD mode zero locations. In general, the Carbone and Aubry method yields closer results than the synthetic POD zeros, but the error trend appears similar: the predicted oscillation wavelength is too long near the centerline and too short approaching the wall layer. Figure 3.12 contains the same POD mode zeros represented by the locations of sharpest gradient in figure 3.3(b). For both methods, the zero locations match poorly for the first few modes, and Carbone and Aubry also observed that the stretching behaves differently for the first few modes,
which they termed the “energetic range.” The results indicate that the stretching coefficients determined by a linear fit in the Carbone and Aubry model change with Reynolds number, and this function gains curvature for higher Reynolds numbers. One fundamental issue of relating modes by stretching is the change with Reynolds number observed in the near wall region while the phase function in the remainder of the channel was shown to vary little (§3.5.2), implying that the stretching relationship between the regions varies with Reynolds number.

It should be noted that the SPOD modes were developed to guarantee orthonormality so as to form a useful basis on which to represent the fluctuating velocity signal. The Carbone and Aubry stretching formulation does not guarantee the modes obtained are orthogonal. To generate a set of modes in practice, the Carbone and Aubry method requires both a mother mode and two empirical stretching constants for the odd modes and another set for the even modes. The SPOD method requires a single scaled phase function, which is not guaranteed to exist a priori but has been observed to exist due to the nature of the physics, and it can be estimated using a model (§3.7).

Since POD is notable for providing the expansion that converges fastest in an energy sense, comparing the mean convergence of POD and SPOD will reveal if the essence of the POD method is captured by SPOD. In addition, the convergence will also be compared for sinusoidal modes satisfying the boundary conditions (corresponding to a linear phase function) and Chebyshev polynomials. To assess convergence, velocity fields are projected onto the modes using trapezoidal integration, and the mean of the reconstruction coefficient squared is the mean energy the mode $n$ contributes, denoted $E^{(n)}$. Chebyshev polynomials are an exception because their energy norm weighting is not consistent with the energy norm defined in (3.4). For POD, the mean mode energy is also equal to the eigenvalue, and the sum of the eigenvalues is the mean energy of the flow $E$, (3.4). This measure of convergence is
Figure 3.13: Comparison of energy in partial reconstructions for the channels: (a), (b), and (c) are modal energy contribution and (d), (e), and (f) are residual energy in reconstruction up to mode \( n \), with symbols for each discrete mode number omitted for clarity. The columns of subfigures correspond to \( Re_\tau = 180, 395, \) and 934.

Expressed as the fraction of mean total flow energy contributed by mode \( n \), \( E(n)/E \) and plotted for the \( u \)-component of the channels in figure 3.13(a,b,c). The POD spectra are identical to those in figure 3.4. Figure 3.13 includes another measure of convergence, the energy residual for a partial reconstruction up to and including mode \( n \),

\[
E_T^{(n)} = E - \sum_{n' = 1}^{n} E^{(n')},
\]

plotted as a fraction of \( E \) in parts (d,e,f). The energies do not decay monotonically because we define the mode number \( n \) as \( \text{sequence} + 1 \) rather than ordered by decreasing energy.

The phase \( \beta_0(y) \) used to generate the synthetic POD was set equal to the \( \beta^{(n)}_0(y) \) that provided roughly optimal results. The convergence was not sensitive to the choice of \( n \). The SPOD convergence is better for high \( n \) if a \( \beta^{(n)}_0(y) \) was selected from a high \( n \) mode and likewise for low \( n \). Figure 3.13 shows that the \( n = 1 \) and 2 POD modes each account for approximately 20% of the turbulent kinetic energy.
in the \( u \) fluctuations, and this is similar for all three Reynolds numbers. The next few modes contribute less energy but more for the lower Reynolds numbers than for higher, which indicates less energy for the remaining modes to contribute. This is consistent with the energy residual plot, which displays a lower energy residual at a given \( n \) when Reynolds number is lower. This faster convergence at high \( n \) is also indicated by the value of \( n \) required to retain 99.99% of the energy.

The SPOD energy residual decay is close to that of POD after the first few \( n \) and superior to all of the polynomials shown, indicating that SPOD offers near-optimal representation in the energy norm. This also suggests that convergence rates are not extremely sensitive to the mode shape because SPOD modes vary noticeably from the actual POD modes as shown in figure 3.11. Interestingly, the \( n = 1 \) and 2 SPOD modes contribute significantly less energy than the POD modes, but the next few SPOD modes contribute more. The convergence thereafter is similar. Evidently, energy redistributes relatively easily among the large-scale modes with weak dependence on the mode shape, and tests to remove the idiosyncratic POD mode contributions before projection onto the SPOD modes did not significantly improve convergence for higher \( n \). A smooth \( d\beta_0/dy \) for SPOD was obtained by differentiating a polynomial fit to \( \beta_0(y) \), and this was significant in the convergence of higher \( n \) modes. Although the energy residuals of Chebyshev decay slower than the residuals of POD and SPOD, the differences diminish as Reynolds number increases. The sinusoidal basis, which corresponds to a linear phase function in the SPOD form, produces the slowest convergence in all cases, and becomes worse with increasing Reynolds number.

3.7 Asymptotic POD Modes

Moser (1994) developed a theory for POD modes of inhomogeneous turbulent flow based on a Wentzel-Kramers-Brillouin (WKB) expansion of POD modes suggested
by Sirovich & Knight (1985). Interestingly, the expansion form used for one-
dimensional modes

$$\phi(y) = A(y)e^{ig(y)}$$  \hspace{1cm} (3.13)

is essentially the same as (3.6), with $g(y)$ analogous to $\beta_0^{(n)}(y)$, but normalized dif-
ferently. Expressing the oscillation as a complex exponential permits representing
complex variables. Based on this expansion in the POD integral equation with
the necessary information from the two-point correlation modeled using the Kol-
mogorov similarity hypothesis for the inertial range, Moser obtained approxima-
tions to the eigenvalues and the form of the eigenmodes (hereafter referred to as
“Moser’s analysis”). Moser’s use of (3.13) followed the analysis of Sirovich &
Knight (1985), who found that if the kernel (the correlation $R_{ij}(y, y')$) of the integral
equation (3.2) changes more slowly with $(y + y')/2$ than $y - y'$, then an asymptotic
result for the eigenfunction may be developed in this form. The eigenfunctions of a
homogeneous process (with correlation independent of $(y + y')/2$) are trigonometric
(Holmes et al., 1998). Inhomogeneity alters the sinusoidal form of the modes, and
(3.13) conveniently expresses this with an asymptotic correction.

The results in §3.5 confirm this assumption that the scaled phase approaches
an asymptotic form for high $n$. Although the inertial range approximation used to
model the correlation is a source of inaccuracy in the predicted modes and eigenval-
ues, Moser’s analysis explains how important physics appears in the POD modes.
In particular, this analysis forms a theoretical foundation for how the physics of
the flow captured by the two-point correlation appears in the POD modes’ phase
functions. One consequence is a simple result for phase in terms of dissipation.
For one-dimensional POD of $u$ velocity, each asymptotic eigenfunction is given by
(3.13), and the results indicate its eigenvalue $\lambda$ is proportional to $\alpha^{-5/3}$ (cf. Moser
equation 30). $\alpha$ tends to infinity to obtain the asymptotic behavior. Comparing the
forms indicates $\alpha$ plays a role similar to mode number $n$. The result of Moser’s analysis is that

$$
\frac{dg}{dy} \propto [\epsilon(y)]^{2/5},
$$

where $\epsilon$ is dissipation.

Using dissipation calculated from the channel DNS data for the lower Reynolds numbers and provided by the authors of the $Re_\tau = 934$ channel flow simulation (del Álamo et al., 2004; Hoyas & Jiménez, 2006), $g(y)$ has been obtained by integrating (3.14) and renormalizing to the $\beta_0^{(n)}(y)$ convention by setting its range from 0 to $\pi$. These scaled phase functions are compared with the $n = 20$ to 100
phases for the POD modes in figure 3.14. The predicted phase functions agree well, even though the predicted eigenvalue spectrum differs. As discussed in §3.3, the predicted eigenvalue spectrum has a $-5/3$ law decay, but the actual eigenvalue spectrum shows no such behavior. For the $Re_\tau = 395$ channel, the optimal power law in the form of (3.14) relating the phase function derivative to the dissipation is $dg/dy \propto \epsilon^{0.33}$. Similar results are obtained relating the phase function derivative to the mean square velocity gradient by $dg/dy \propto \langle (du/dy)^2 \rangle^{0.33}$ for this flow.

It should be noted that $A(y)$ is undetermined by Moser’s analysis, so the phase function and eigenvalue results are distinct from the synthetic POD previously developed in which a form of $A(y)$ is chosen. Moser’s analysis for an asymptotic solution to the POD integral equation (3.2) does not specify a form of $A(y)$ except that it is independent of $n$ (which is consistent with SPOD), so this analysis alone is not sufficient to obtain modes. Conversely, the $A(y)$ chosen for synthetic POD forces modes to be orthogonal, but there is no requirement that the SPOD modes satisfy any approximation to the integral equation. Combining Moser’s result to estimate $\beta_0^{(n)}(y)$ with the synthetic POD method results in a set of orthogonal modes possessing nearly optimal convergence obtained without first computing the POD. Thus, the two-point spatial correlation is not needed, and the only empirical input is the distribution of dissipation, $\epsilon(y)$. It is interesting that the phase function predicted by (3.14) does not match the actual asymptotic phase for high $n$ but is more characteristic of an intermediate $n$.

3.8 Discussion

The preceding results established two distinct types of modes: idiosyncratic and asymptotically self-similar. The asymptotically self-similar modes describe the modes of sufficiently large $n$, and they approach an asymptotic phase function that is characteristic of the type of turbulent flow. The asymptotic theory of Moser
(1994) yields a relationship between the asymptotic phase function and the distribution of viscous turbulent dissipation, (3.14) that approximates the observed asymptotic phase relatively well. The energy spectra of these modes are similar between Reynolds numbers using inner scaling (figure 3.4). The idiosyncratic modes of low $n$ possess distinctly different mode and energy (eigenvalue) behavior. For channels, the mean energy contributions appear nearly in pairs (1 & 2, 3 & 4, 5 & 6) for idiosyncratic POD modes in figure 3.13(a,b,c), and then the energies decay monotonically with increasing $n$. Besides indicating the split in mode groups, the eigenvalue pairs for low $n$ suggest the eigenmodes in the pairs may be related. Examining the mode derivatives $d\phi^{(n)}/dy$ at the walls reveals that these derivative values exist roughly in pairs for $n = 1$ to 6 before increasing monotonically with increasing $n$.

The lowest order POD mode is related to the analysis of Poje & Lumley (1995), who considered a channel flow (in three dimensions) and performed energy stability analysis with the goal of determining the finite coherent velocity field that has the maximum energy growth rate. They defined the coherent field to be the fluctuating velocity that is represented by a complex exponential form. They showed that the first stability eigenfunction, defined to be the eigenfunction associated with the maximum energy growth rate, was similar to the most energetic POD mode of streamwise velocity fluctuation obtained by Moin & Moser (1989) for the half-channel domain. The lowest-sequence half channel POD mode is included in figure 3.11, and the $n = 1$ and 2 full channel modes are approximately an even/odd symmetric pair with each half shaped like the half channel mode. (Some variation is evident between the $n = 1$ and 2 full channel modes shapes, and the half channel mode appears to be an average of the full channel mode in that domain.) Poje and Lumley’s results are therefore relevant for the present one-dimensional POD and indicate that the $n = 1$ and 2 full channel even/odd symmetric pair mode
shapes are approximately in accordance with energy stability considerations. It is known that coherent motions of large streamwise and wall-normal extent are significant (Kim & Adrian, 1999; Hoyas & Jiménez, 2006; Hutchins & Marusic, 2007a; Adrian, 2007) in wall-bounded turbulent flows and can appear as fluctuations in time of streamwise velocity over the channel length (particularly when the computational domain length is short). Furthermore, it has been shown that the most energetic motions have sizable vertical extent as well as length. It is therefore likely that these three-dimensional motions contribute strongly to the large-scale (in a wall-normal sense) idiosyncratic modes with high energetic contribution in one-dimensional POD for the channel streamwise velocity.

In summary, the one-dimensional POD modes of channel streamwise velocity consist of the low-$n$ modes representing the long, least stable and most energetic motions, and the asymptotically self-similar higher-$n$ modes representing the smaller scale motions. Modes with asymptotically high $n$ have closely self-similar phases so the modes satisfy orthogonality in a simple way similar to the SPOD orthogonality and are profitably analyzed using the scaled phase functions. The modes with scaled phases approaching the asymptotic form are also classified as self-similar modes in an approximate sense. Within the idiosyncratic modes, the modes following the first pair must be orthogonal to this pair and to the self-similar modes but must have few oscillations. As a result, their shapes are most complex. Among these modes, the $n = 3$ and $5$ streamwise velocity channel modes at the centerline are notable with the concavity directed away from $\phi = 0$. This becomes weaker as $n$ increases until the concavity of the center oscillation is directed toward $\phi = 0$ for odd modes above $n = 5$. This transition between the highest-$n$ idiosyncratic modes and the lowest-$n$ asymptotically self-similar modes occurs smoothly, making the division between the two groups somewhat inexact. However, the switch to monotonic decay of eigenvalues at $n \approx 5–6$ also supports this division. The idiosyn-
ocratic channel modes with the concavity directed away from $\phi = 0$ are physically interpreted as representing large motions of increased or decreased momentum on both sides of the channel centerline.

The phase functions of idiosyncratic modes for the channel behave differently than the corresponding functions for turbulent convection. For turbulent convection, the idiosyncratic phase functions are less distinct from those of the asymptotically self-similar modes, in contrast to the sharp change observed in channel flow. This reflects different wall-normal large-scale behavior for the flows. The eigenvalues for convection exhibit regular monotonic decay after the third mode, but the $n = 2$ mode is associated with more energy than the $n = 1$ mode. The $n = 2$ mode is likely associated with the roll cells that span the plate separation in convection (Fernandes, 2001). The $n = 1$ eigenmode closely matches a sinusoidal curve (figure 3.1), with scaled phase similar to an asymptotically self-similar mode. The $n = 2$–$4$ modes display the steep near-wall region boundary layer behavior that the channel flow modes possess, but the variation of oscillation wavelengths in the remainder of the domain is reduced, corresponding to the more linear phase functions relative to the turbulent channel.

The POD modes encapsulate flow physics embedded in the two-point correlation. Comparing the two-point correlations for channel flow and convection in figure 3.15, the channel correlations have much taller peaks along the diagonal (i.e. $R_{uu}(y,y)$) and the minimum along the diagonal for $y$ near the centerline is much deeper for the channel than for convection. The two-point correlation of a homogeneous process depends only on the displacement $y - y'$, and $R_{uu}(y,y')$ is constant along the diagonal. The mean streamwise flow that is present in the channels but not in the convection flow contributes to these strong peaks. The greater variation along the diagonal for the channel indicates greater inhomogeneity, which corresponds to greater deviation in the phase function from linear behavior (§3.7). The strongest
Figure 3.15: Comparison of $R_{uu}$ normalized by the centerline variance $\langle u(0)^2 \rangle = R_{uu}(0,0)$ for (a) $Re_\tau = 180$, (b) 395, and (c) 934 turbulent channels and (d) turbulent convection. The turbulent convection correlation is obtained from Fernandes (2001) and centerline symmetry is not forced on the correlation, whereas for the channels it is. Note the $R_{uu}/\langle u(0)^2 \rangle$ axis for convection is stretched by a factor of 3 relative to the channel axes.

inhomogeneity in the convection two-point correlation is at the boundaries, where the no-slip boundary conditions dictate $R_{uu}(y,y') = 0$. The correlation rapidly increases moving from the wall due to viscous/conductive scaling. Consequently, the phase slope increases most rapidly near the wall and is linear elsewhere.

Synthetic POD captures important physics of the POD modes and produces comparable convergence to true POD modes, where the sinusoidal modes, corresponding to a linear phase function, converge much more slowly (figure 3.13). This indicates the phase curvature relative to linear is very important and the phase functions for channels are not well approximated by linear phase. Sung & Adrian
(1994) examined the proper orthogonal decomposition of a one-dimensional Burgers’ model of turbulence that is inhomogeneous due to boundary-layer behavior at the domain boundaries, similar to the channel, and concluded that high $n$ modes assume a trigonometric form with little difference from a Fourier expansion. However, the present evidence indicates that the modulation of the trigonometric form is important and is not confined to a small region (such as near the wall). The similar convergence behavior of Chebyshev modes is a result of the similarity between their phase functions and the optimal (POD) phase functions for these channel flows, as Chebyshev modes are adapted to boundary layer behavior with more rapid oscillations near the wall. Results indicate that the overall convergence properties are closely linked with phase functions, but relatively insensitive to the precise form of the large scales. Comparing the channels at all three Reynolds numbers, figure 3.9 shows that the phase curves for higher Reynolds numbers depart more from linear behavior, but the difference in the highest Reynolds numbers is concentrated near the wall. This is consistent with the two-point correlations’ near-wall peaks becoming sharper and closer to the wall (relative to $h$) with increasing Reynolds number.

3.9 Conclusions

In conclusion, for the inhomogeneous, wall-bounded turbulent flows considered here, the POD modes in the inhomogeneous direction are of two types: asymptotically self-similar modes and idiosyncratic modes. The former are a set of higher order modes that are characterized by nearly self-similar phase functions when scaled by the sequency according to equation (3.8). The similarity becomes asymptotically correct as the sequency becomes large. The asymptotic limit is a phase function that is characteristic to each type of turbulence. The latter, idiosyncratic modes are the low order modes whose phase functions are not self-similar. They contain much of
the energy, and their scales are of the order of the dimension of the flow domain, making them sensitive to the outer length scale. But, since they must span the entire domain, they also contain a thin near wall boundary layer corresponding to the buffer layer region in which the viscous length scale is important. Thus, the division into idiosyncratic modes and asymptotically self-similar modes is not simply the same as division into inner and outer scaling. The idiosyncratic modes constitute the bulk of the modes that usually comprise a low order model. They contain most of the turbulent kinetic energy, ranging from 79% in modes 1–6 at $Re_\tau = 180$ to 66% in modes 1–6 at $Re_\tau = 934$.

The asymptotically self-similar modes contain most of the viscous turbulent dissipation, ranging from 65% at $Re_\tau = 180$ to 91% at $Re_\tau = 934$. This division of energy and dissipation suggests two analogies, one between the idiosyncratic modes and the large-scale to very large-scale motions in the homogeneous direction of wall turbulence and a second analogy between the classical inertial sub-range plus viscous dissipation range and the asymptotically self-similar modes.

The nearly universal form of the asymptotically self-similar modes suggests a method of synthesizing a completely self-similar set of orthonormal modes that approximate the asymptotically self-similar part of the orthogonal decomposition. The synthetic POD can start with the limiting phase function, or with an approximation to it, such as given by Moser’s phase-dissipation relation. The synthesis can be extended down to the lowest sequency, thereby covering the entire range of scales, including those of the idiosyncratic modes.

Combining the SPOD method of synthesizing approximate POD modes with (3.14), the phase gradient-dissipation relation of Moser (1994), makes it possible to construct a nearly-optimal orthogonal basis without calculating the POD a priori but with only knowledge of the distribution of dissipation. If the dissipation is independent of $y$ (homogeneous), the phase is a linear function of $y$, and the de-
composition is Fourier trigonometric. The more inhomogeneous the dissipation, the more the phase function departs from linearity, and distinctively characterizes the turbulence. These concepts relating simple one-dimensional POD to the physics of a flow may be developed to interpret more complicated POD with multiple velocity components and higher dimensions. The insights and synthetic POD modes may also be combined with further information relating mode coefficients to model the higher modes so computation of turbulent flows can be simplified, as suggested by Carbone & Aubry (1996).
Chapter 4

DIRECT NUMERICAL SIMULATION OF A 30R LONG TURBULENT PIPE FLOW AT $R^+ = 685$: LARGE- AND VERY LARGE-SCALE MOTIONS

Fully developed incompressible turbulent pipe flow at Reynolds number $Re_D = 24,580$ (based on bulk velocity) and Kármán number $R^+ = 684.8$ is simulated in a periodic domain with a length of 30 pipe radii $R$. While single-point statistics match closely with experimental measurements, questions have been raised of whether streamwise energy spectra calculated from spatial data agree with the well-known bimodal spectrum shape in premultiplied spectra produced by experiments using Taylor’s hypothesis. The simulation supports the importance of large- and very large-scale motions (VLSMs, with streamwise wavelengths exceeding $3R$). Wavenumber spectral analysis shows evidence of a weak peak or flat region associated with VLSMs, independent of Taylor’s hypothesis, and comparisons with experimental spectra are consistent with recent findings (del Álamo & Jiménez, 2009) that the long-wavelength streamwise velocity energy peak is overestimated when Taylor’s hypothesis is used. Yet, the spectrum behaviour retains otherwise similar properties to those documented based on experiment. The spectra also reveal the importance of motions of long streamwise length to the $uu$ energy and $uv$ Reynolds stress and support the general conclusions regarding these quantities formed using experimental measurements. Space-time correlations demonstrate that low-level correlations involving very large scales persist over $40R/U_{\text{bulk}}$ in time and indicate that these motions convect at approximately the bulk velocity, including within the region approaching the wall. These very large streamwise motions are also observed to accelerate the flow near the wall based on force spectra, whereas smaller scales tend to decelerate the mean streamwise flow profile, in accordance with the behaviour observed in net force spectra of prior experiments. Net force spectra are
resolved for the first time in the buffer layer and reveal an unexpectedly complex structure.

Acknowledgements: The computer program used in this study was developed by the late Dr. Charles D. Pierce of the Center for Turbulence Research at Stanford. XW was supported by the NSERC Discovery Grant and the Canada Research Chair Program (CRC) in Aeronautical Fluid Mechanics. The calculations were performed at the High Performance Computing Virtual Laboratory (HPCVL). Additional computations were performed using the Arizona State University Advanced Computing Center facilities. RJA and JRB gratefully acknowledge the support of the National Science Foundation with Award CBET-0933848. We gratefully acknowledge the experimental spectra provided by M. Hultmark and A. Smits, and those provided by H. Ng and J. Monty. We also wish to acknowledge J.-C. del Álamo, J. Jiménez, P. S. Zandonade, and R. D. Moser for making two-dimensional energy spectra for channel DNS available, and acknowledge C. Chin for pipe DNS spectra.

4.1 Introduction

The importance of motions with very long streamwise extent in incompressible, fully-developed, turbulent pipe flow was brought into focus more than a decade ago by Kim & Adrian (1999), hereafter referred to as KA99. Using premultiplied, one-dimensional spectra of the streamwise velocity fluctuation, they found that modes of streamwise wavelength significantly longer than one pipe radius make substantial contributions to streamwise turbulent kinetic energy at moderate Reynolds numbers. KA99 observed that premultiplied streamwise velocity spectra exhibit two maxima, suggestive of a bimodal form. One maximum exists at the low-wavenumber end of the inertial subrange, which KA99 associated with large-scale motions (LSMs) with length on the order of a few pipe radii $R$, and the other exists at longer wavelength. The latter scales were called ‘very large-scale
motions’ (VLSMs). LSMs were originally identified as turbulent bulges in turbulent boundary layers, with the average dimensions of bulges accepted to be 2–3 $\delta$ in streamwise length, 1–1.5 $\delta$ in width, and $\delta$ in height (reviewed in Guala et al. (2006)). It has become conventional to define LSMs with these nominal dimensions for the canonical wall shear flows, where $\delta$ is the appropriate length scale of channel half-height $h$, pipe radius $R$, or boundary layer thickness $\delta$. The short-wavelength boundary of LSMs depends on Reynolds number, but at typical laboratory Reynolds numbers is about the thickness of the logarithmic layer or slightly longer (0.2–0.3 $\delta$). It is convenient to define LSMs as wavelengths up to $3\delta$ and define VLSMs as longer wavelengths, but this is nominal and may be subject to change in the light of future findings.

In pipe flow experiments at significantly higher Reynolds numbers than the present simulation, Guala et al. (2006) have calculated that VLSMs with streamwise length greater than $3R$ contain over 65% of the streamwise turbulent kinetic energy at the radii they measured. The bimodal form has also been observed in turbulent channels and boundary layers (for example, Balakumar & Adrian, 2007; Hites, 1997). The wavelengths at which peaks occur as a function of wall-normal position from various experiments are summarized in figure 4.1. The wavelengths of long-wavelength flattened regions or peaks for the present simulation are also included in the plot along with peak locations from a pipe experiment with similar Reynolds number (Hultmark et al., 2010), and these are discussed in §4.4.2.

The preceding spectrum results and others (such as Monty et al., 2007; Bailey et al., 2008) are based on experimental investigations with hot wires or films, which require Taylor’s hypothesis (Taylor, 1938) to estimate the spatial behaviour based on single-point temporal measurements. More recently, doubts have been raised regarding the presence of the bimodal behaviour in spectra obtained from Fourier transforms of true spatial fields without the use of Taylor’s hypothesis. del Álamo
Figure 4.1: Comparison of wavelengths for spectral peaks in premultiplied streamwise velocity energy spectra for the present DNS and various experiments as a function of wall-normal distance. Data from the present $R^+ = 685$ simulation is included (filled diamonds), and all other data is obtained from experiment. The small circles with shaded centres are obtained from the hot-wire spectra of Hultmark et al. (2010) for $R^+ = 690$ pipe flow. Other data is adapted from Balakumar & Adrian (2007): ‘Hash symbol, $Re_\tau = 531$ (channel); plus symbol, $Re_\tau = 960$ (channel); asterisk symbol, $Re_\tau = 1584$ (channel); less than symbol, $Re_\tau = 1476$ (ZPGBL); and greater than symbol, $Re_\tau = 2395$ (ZPGBL).’ The remaining symbols correspond to pipes, mainly at considerably higher Reynolds number than the present simulation, including Guala et al. (2006) at $\circ R^+ = 3815$, $\Box R^+ = 5884$, $\triangle R^+ = 7959$, and Bullock et al. (1978) at $R^+ = 2630$ represented by large black $\bullet$. The other symbols are defined in figure 4 of Guala et al. (2006).

& Jiménez (2009) (see also Moin, 2009) discussed the convection velocities of motions of various streamwise wavelengths in turbulent channel flow, and found that long-wavelength motions moving faster than the local mean lead to errors when Taylor’s hypothesis is invoked, particularly in the buffer and inner logarithmic layers. They showed that the long-wavelength peak for a turbulent channel is clearly present when Taylor’s hypothesis is used, but suggest that the energy spectrum based on purely spatial information could behave more similar to $k_x^{-1}$, where $k_x$ is streamwise wavenumber, making the peak at least partially an artifact due to Taylor’s hypothesis. Their study followed a comparison between DNS and hot-wire
measurements of a turbulent channel at $Re_\tau \approx 1000$ by Monty & Chong (2009), who noted the absence of a DNS longer-wavelength spectrum peak, finding the shape more similar to a shoulder. del Álamo & Jiménez (2009) also compared $Re_\tau = 2000$ channel DNS spectra modified to simulate the effects of using Taylor’s hypothesis with experimental measurements of a pipe. They also indicated the peak induced by Taylor’s hypothesis would become stronger with increasing Reynolds number. Similar studies directly comparing experimental and DNS pipe flows at sufficiently high Reynolds numbers to observe this effect have not been performed.

While DNS of turbulent channels have been performed in long streamwise domains at increasingly high Reynolds numbers (del Álamo et al., 2004; Hoyas & Jiménez, 2006), there has been a distinct absence of DNS to study very long structures in pipe flow. In this paper we present the first DNS evidence of very long coherent structures in fully-developed turbulent pipe flow. Previous pipe flow DNS studies mostly dealt with the mean and second-order turbulent statistics, e.g., Eggels et al. (1994), Wu & Moin (2008). The axial dimensions of the computational domains used in those simulations were no more than $15R$. Chin et al. (2010) examined the effect of DNS pipe simulation length on statistics and also considered energy spectra, but the maximum Reynolds number was $R^+ = 500$, and they concluded that no outer peak was discernible in premultiplied spectrum maps, which present the premultiplied spectra over a range of $y$ (wall-normal position) values. The present simulation adopts an axial dimension of $30R$. It is the first DNS study focusing on the very long structures in fully-developed turbulent pipe flow at this high of Reynolds number. In comparing pipe DNS statistics for varying streamwise lengths, Chin et al. (2010) found that the one-dimensional energy spectrum was the statistic that required the greatest length, and lengths of $8\pi R$ were adequate for the Reynolds numbers they considered. The longest channel simulations of del Álamo et al. (2004) and Hoyas & Jiménez (2006) also used computational domains with
lengths of $8\pi$ times the channel half-height, so our domain accommodates similar motions and is of sufficient length for correct statistics based on the observations of Chin et al. (2010).

Guala et al. (2006) also experimentally observed the importance of VLSM contributions in turbulent pipe flows, with VLSMs contributing 50–60% of the Reynolds shear stress at the radii they measured. The importance of VLSMs in carrying $uv$ Reynolds stress was unexpected based on the idea originally inferred by Townsend (1976) that the very long motions would be inactive, that is, would carry substantial $uu$ energy but little Reynolds stress. del Álamo & Jiménez (2001); del Álamo & Jiménez (2003) had also previously observed the sizable contributions of very long motions to $uv$ Reynolds stress in turbulent channel simulations, and del Álamo et al. (2004) found that these contributions occur mainly in the outer region and that this is how the impermeability of the wall limits the wall-normal motions in relation to Townsend’s idea. Through experimental particle image velocimetry measurements in channels, Liu et al. (2001) also showed that relatively long streamwise scales carried substantial $uv$ without requiring Taylor’s hypothesis, but the maximum streamwise spatial length that they could observe was limited. Guala et al. (2006) also used $uv$ spectra to compute net force spectra that decompose a Reynolds stress term that appears in the mean momentum equation into contributions by streamwise wavenumber. They found that VLSMs played a special role because the net force contribution of these wavelengths changed sign to accelerate the streamwise flow as the wall was approached, but measurements were made no closer than $0.15R$ from the wall. The $uv$ spectral contributions will therefore be quantified for the present pipe simulation, and the DNS will permit computing force spectra without error introduced from the use of Taylor’s hypothesis and with positions nearer to the pipe wall available.
The organization of the present study is as follows: after verifying that single-point statistics are in agreement with experiments and documenting additional statistics characterizing the simulation, we compare energy spectra with experimental measurements and observe the significant amount of energy in very long scales for the simulation (§4.4). We proceed by discussing the axial wavelengths at which particular cumulative energy fractions occur in the light of prior experimental observations (§4.4.3). Having established the significance of these motions in the present simulation and observed similarities and differences with experiments, we visualize contours of axial velocity fluctuation and then filter them to concentrate on the VLSMs (§4.5). Observations from the time evolution of these fields are then related to space-time correlations (§4.6) and the convection velocity of very long streamwise modes (§4.7). Having characterized these properties of very long motions in this pipe simulation, we conclude by examining the net force spectra (§4.8) and summarizing the results.

4.2 Computational Details

In the present study, the unit length scale is the pipe radius $R$, and the unit velocity scale is $U_{\text{bulk}}$, which is defined as the ratio of mean volume flow rate and pipe cross-sectional area. The unit time scale is therefore $R/U_{\text{bulk}}$. The Reynolds number based on pipe diameter $D$ and $U_{\text{bulk}}$, $Re_D = 24580$, is the same as that in the experiments of den Toonder & Nieuwstadt (1997). Superscript + refers to quantities normalized by friction velocity $u_\tau$ for velocity and by viscous wall unit $\nu/u_\tau$ for distance. The Kármán number $R^+$ is equivalent to the friction Reynolds number $Re_\tau$, but here the former is reserved for pipe flows while the latter is generally used for other wall-bounded shear flows.

$r$ is the radial coordinate measured from pipe axis, $x$ is the flow axial direction, and $\theta$ is the azimuthal coordinate. For the purpose of analogy with the spanwise
coordinate of a channel, we introduce the arc length \( s = r\theta \). (The use of arc length in turbulent pipe flows is discussed by Monty et al. (2007).) By analogy with the wall-normal coordinate of a channel, it is also convenient to define \( y = R - r \) for the pipe (also used by Guala et al., 2006). It is also helpful to introduce the analogous velocity components \( u = u_x \), \( v = -u_r \), and \( w = u_\theta \). The subscripts of correlation functions herein use \( u, v, w \) to indicate the velocity components.

The finite-difference grid size used in the current computation is \( 256 \times 1024 \times 2048 \) along the \( r, \theta, \) and \( x \) directions, respectively. Resolution along the axial direction is \( \Delta x^+ = 10.03 \) or \( \Delta x = 0.00732R \). The computed pipe radius and friction velocity based Kármán number \( R^+ \) is 684.8. Along the azimuthal direction maximum grid spacing is achieved at the wall \( (r = R) \) yielding \( \Delta(R\theta)^+ = 4.2 \). The minimum and maximum wall-normal grid spacings are \( 3.578 \times 10^{-4}R \) and \( 9.892 \times 10^{-3}R \), respectively. In wall units, these correspond to 0.144 and 11.3. The maximum wall-normal grid spacing is located at \( r = 0.406R \) rather than at the centreline. The first layer of grid points in the staggered mesh system is located 0.205 wall units from the pipe surface. There are 108 grid points located near the wall between \( 0 < y/R < 0.1 \), and 44 grid points near the centreline between \( 0.9 < y/R < 1.0 \). At the pipe centreline the wall-normal grid spacing is \( 1.435 \times 10^{-3}R \). Below \( y^+ = 50 \), the wall-normal grid spacing \( \Delta r^+ \) is 1.2 or less, and below \( y^+ = 30 \), \( \Delta r^+ < 0.8 \).

The computer program, numerical method and boundary conditions are the same as those described in Wu & Moin (2008). The simulation was performed using 160 threads on a Sun SPARC Enterprise M9000 Server. Each restart data file has a size of 21.7GB. Except for an initial start-up period, the computational time step was fixed at \( \Delta t = 0.009R/U_{\text{bulk}} \). The initial condition was plug flow superimposed with a random number field. The simulation was first advanced for \( 300R/U_{\text{bulk}} \). Statistics were subsequently collected for another \( 260R/U_{\text{bulk}} \). An
overbar represents averaging over time as well as over the two homogeneous directions.

4.3 Validation of DNS Results

To establish the validity of the simulation, one- and two-point statistics are compared to experimental data. Figure 4.2(a) compares the DNS profile of $\overline{u}^+$ as a function of $y^+$ with the experimental data of den Toonder & Nieuwstadt (1997). To document additional aspects of the mean velocity for the present DNS, profiles of the mean velocity gradient, the logarithmic-law indicator function, and the power-law indicator function are plotted in figure 4.2(b). The indicator functions for a Wu & Moin (2008) pipe simulation at $Re_D = 44,000$ (but shorter domain length) are included for comparison, as the higher Reynolds number flow would be expected to possess a wider region that could obey such a law. Interpretation of the indicator functions and their relation to scaling laws are discussed therein. Wu & Moin (2008) noted that at low Reynolds numbers (including their pipe DNS) the narrow logarithmic slope region was not the same as what was implied by the classical complete similarity arguments. $d\overline{u}/dy$ experiences a precipitous drop within the buffer region and its curvature changes from convex (viscous sublayer) to concave and back to convex (core region). $d\overline{u}^+/d\ln(y^+)$ is approximately constant over the region of $40 < y^+ < 100$, indicating an approximately logarithmic dependence of $\overline{u}^+$ on $y^+$. This region is narrower than that of the higher Reynolds number simulation, but the overall behaviour is very similar. The comparison between the power-law indicator functions $d\ln(\overline{u}^+)/d\ln(y^+)$ for the present pipe simulation and the $Re_D = 44,000$ simulation also indicates a narrower flat region of adherence to power-type behavior, where Wu & Moin (2008) found this region was already limited to $70 < y^+ < 120$ for the higher Reynolds number.

The second-order turbulence statistics are compared with those of den Toonder
Figure 4.2: (a) Mean velocity as a function of $y^+$. Symbols: den Toonder & Nieuwstadt (1997); solid line: present DNS; dash-double dotted line: $\overline{u}^+ = y^+$. (b) Mean velocity gradients as a function of $y^+$ for the present DNS. Dashed line: $0.5(R/U_{bulk})d\overline{u}/dy$; dotted line: $2d\overline{u}^+/dln(y^+)$; dash-dotted line: $20dln(\overline{u}^+)/dln(y^+)$. The light grey lines are for the $Re_D = 44 000$ pipe DNS of Wu & Moin (2008).

Figure 4.3: Turbulence intensities and shear stress as a function of $y/R$. Open symbols: den Toonder & Nieuwstadt (1997); filled dots: Hultmark et al. (2010) at $Re_D = 24100$ with hot-wire length $l_w = 4.3$ (dark) and $Re_D = 23800$ with $l_w = 19.1$ (light grey); lines: present DNS. Solid line: $u_{x,rms}';$ dashed line: $u_{\theta,rms}';$ dotted line: $u_{r,rms}';$ dash-dotted line: $\overline{u'_xu'_r}$. 

81
& Nieuwstadt (1997) in figure 4.3. The comparison of $u'_{x,rms}$ also includes hot-wire data obtained by Hultmark et al. (2010). The near-wall peaks are very similar in magnitude and location for the experiments and the DNS. In the DNS data, the $u'_{x,rms}$, $u'_{r,rms}$, and $u'_{\theta,rms}$ peaks occur at $y^+ = 14, 86, \text{ and } 41$, respectively. The $u'_{x}u'_r$ peak occurs at $y^+ = 47$, which agrees closely with the value of $y^+ = 52$ from the correlation of the location of maximum Reynolds shear stress for pipes and channels $y_{RS \text{ max}} \approx 2(Re_\tau)^{1/2}$ (Sreenivasan, 1987; Sreenivasan & Sahay, 1997; Marusic et al., 2010). The values of $u'_{x,rms}$ in the region above the peak to the pipe centreline are appreciably larger in den Toonder & Nieuwstadt (1997) than those of Hultmark et al. (2010) and the DNS. Experimental data of the Reynolds shear stress are noisy and measurements of the azimuthal turbulent intensity are not available. The maximum deviation of the DNS total shear stress $\left(-\nu d\bar{u}_x/dr + u'_xu'_r\right)^+$ relative to the local theoretical linear value $r/R$ is 2.27%.

Figure 4.3 shows that for $0.9 < y/R < 1.0$ near the pipe axis the azimuthal and radial turbulent intensity profiles collapse. This Reynolds-number-independent feature is dictated by the Reynolds-averaged mean radial momentum transport equation

$$-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial r} - \frac{u'_r^2 - u'_\theta^2}{r} - \frac{du'_r^2}{dr} = 0,$$

and serves as an additional check of the simulation quality.

The two-point correlation coefficient $R_{uu}$ as a function of azimuthal separation $\Delta \theta$ is compared with the experimental data of Bailey et al. (2008) and Monty et al. (2007) at two radial locations in figure 4.4. Here $R_{uu}$ represents correlation of the axial velocity component at two locations. The Reynolds number of Bailey et al. (2008) shown in the figure is more than three times the present value. However, Bailey et al. (2008) showed that $R_{uu}(\Delta \theta)$ remains nearly frozen over a two-order magnitude range in $Re_D$. Monty et al. (2007) also showed that $R_{uu}(\Delta \theta)$ is insensitive
to the change of Reynolds number. Overall, evaluations of the two-point correlation coefficient, mean and second-order statistics validate the statistical features of the present DNS.

4.4 Energy Spectra

4.4.1 Definitions

For clarity, the definitions and normalization conventions of energy are summarized below; these are generally consistent with other literature, such as Guala et al. (2006), in which more detail is given. Following common usage, the plots and discussion use the velocity components instead of their number indices as subscripts of correlation $R$ and energy spectrum $\Phi$; however, numerical index subscripts are more convenient in these equations. Given the two-point streamwise correlation $R_{ij}(r_x; y) = \langle u'_i(x, y)u'_j(x + r_x, y) \rangle$ (with averaging over $\theta$ and time implicitly assumed), the co-spectrum is defined as

$$ S_{ij}(k_x; y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}k_x r_x} R_{ij}(r_x; y) dr_x, \quad (4.2) $$

Figure 4.4: Two-point correlation coefficient $R_{uu}$ as a function of azimuthal separation $\Delta \theta$. Filled circles: Bailey et al. (2008) at $y/R = 0.2$ for $Re_D = 76,000$; open diamonds: Monty et al. (2007) at $y/R = 0.15$ for $Re_D = 21,400$ ($R^+ = 615$); solid line: present DNS at $y/R = 0.2$; dashed line: present DNS at $y/R = 0.15$. 
which is the Fourier transform of the correlation. Then the one-sided wavenumber
cospectrum is given by
\[ \Phi_{ij}(k_x; y) = S_{ij}(k_x; y) + S_{ij}(-k_x; y) = 2\text{Re}\left\{S_{ij}(k_x; y)\right\}. \quad (4.3) \]
The spectra are related to the mean-square fluctuation by
\[ \overline{u_i'u_j'} = \int_{-\infty}^{\infty} S_{ij}(k_x) dk_x = \int_{0}^{\infty} \Phi_{ij}(k_x) dk_x. \quad (4.4) \]
The computed spectra (via discrete Fourier transforms) are normalized to discretely
approximate (4.4) such that
\[ \overline{u_i'u_j'} = \frac{N_x}{2} \sum_{n_x=-N_x/2+1}^{N_x/2} S_{ij}(n_x) \Delta k_x = \sum_{n_x=0}^{N_x/2} \Phi_{ij}(n_x) \Delta k_x, \quad (4.5) \]
where \( \Delta k_x \) is the spacing between wavenumbers, which is equal to \( L_x/(2\pi) \), \( L_x \) is the
30\( R \) pipe domain length, and \( N_x \) is the number of grid points in \( x \). The discrete energy spectra are plotted as a function of the associated wavenumber \( k_x = L_x/(2\pi)n_x \) or wavelength \( \lambda_x \). The energy spectra presented include both true wavenumber spectra based on instantaneous spatial fields of these DNS data and, for comparison, spectra from experimental hot-wire or hot-film measurements and Taylor’s hypothesis to infer spatial wavenumbers based on temporal fluctuations.

A related function is the cumulative energy spectrum \( \Upsilon_{ij}(k_x = 2\pi/\Lambda_x; y) \), which
is discussed in greater detail in Guala et al. (2006) and Balakumar & Adrian (2007). For a given \( y \), this is the fractional contribution to \( \overline{u_i'u_j'} \) by all wavelengths from 0 to \( \Lambda_x \) or, equivalently, all wavenumbers from \( k_x = 2\pi/\Lambda_x \) to infinity:
\[ \Upsilon_{ij}(k_x = \frac{2\pi}{\Lambda_x}; y) = 1 - \frac{\int_{0}^{k_x} \Phi_{ij}(\tilde{k}_x) d\tilde{k}_x}{\int_{0}^{\infty} \Phi_{ij}(\tilde{k}_x) d\tilde{k}_x}. \quad (4.6) \]
For the discrete computed spectra,
\[ \Upsilon_{ij}(k_x = \frac{2\pi}{\Lambda_x} = \frac{L_x}{2\pi n_x}; y) = 1 - \frac{\sum_{\tilde{n}_x=0}^{n_x-1} \Phi_{ij}(\tilde{n}_x)}{\sum_{\tilde{n}_x=0}^{N_x/2} \Phi_{ij}(\tilde{n}_x)}. \quad (4.7) \]
This definition is slightly modified from that of Guala et al. (2006) and Balakumar & Adrian (2007) so that in the present definition the cumulative spectrum for a
wavelength $\Lambda_x$ includes the contribution of the mode with that wavelength (and all shorter wavelengths).

4.4.2 Streamwise Spectra

While the simulation’s first and second order statistics agree closely with experimental measurements, the agreement of the DNS energy spectra with experimental spectra must be carefully examined due to possible error induced by the use of Taylor’s hypothesis, as discussed in the introduction. Therefore, it is important to compare with experimental spectra obtained from a pipe at similar Reynolds number. The recent study of Hultmark et al. (2010) includes spectra for a pipe flow at $Re_D = 25\,000$. Otherwise, most recently obtained pipe spectra have been at higher Reynolds number than our computation. The Hultmark et al. (2010) premultiplied streamwise velocity spectra are compared with the present DNS in figure 4.5 at $y/R = 0.1$, which is near where a logarithmic layer would begin to form.

The spectra are generally in good agreement, with the DNS rolling off at slightly higher wavenumbers. The experimental spectra in the lower-wavenumber region are generally larger than those of the simulation, but the presence of noise that would diminish with additional statistical convergence leads to uncertainty in quantifying the difference. For this reason, an experimental pipe spectrum measured by the authors of KA99 at $y/R = 0.1$ is also included in figure 4.5. This spectrum (not published in KA99) was obtained with essentially the same flow parameters as the lowest Reynolds number in KA99. We have consistently defined $Re_D$ herein to be based on the bulk velocity, but for purposes of comparing to the KA99 spectra we note that the Reynolds number of the simulation based on centreline velocity is 30\,940, and that this is within 8\% of the KA99 Reynolds number based on centreline velocity, 33\,430. We judge the two Reynolds numbers to be close enough to allow comparison of the trends. The Kármán number for the experiment is $R^+ = 825$.
Figure 4.5: Comparison of premultiplied energy spectra at $y/R = 0.1$ for the present DNS at $Re_D = 24 580$ (black line) with comparable experiments: hot-wire spectra obtained by Hultmark et al. (2010) at $Re_D = 25 000$ (filled squares) and hot-film measurements obtained by the authors of KA99 (dashed line). The KA99 pipe flow at $Re_{D,cl} = 33 430$ based on centreline velocity is at slightly higher Reynolds number than the present DNS with $Re_{D,cl} = 30 940$. Dotted lines depict the long-wavelength modes if the DNS and KA99 spectra were decomposed into bimodal forms, as discussed by KA99.

Based on $u_r$ calculated from pressure measurements. This value is lower than the value reported in KA99 because table 1 in KA99 used an incorrect value of the viscosity.

The experimental spectra obtained by KA99 clearly show the bimodal shape characteristic of measurements in the logarithmic layer. The lower-wavenumber peak magnitude becomes less pronounced relative to the higher-wavenumber peak with decreasing Reynolds number. In addition, the higher-wavenumber peak magnitudes of the spectra decrease slightly with decreasing Reynolds number. Since the wavenumber is scaled with $R$ (in outer scaling), the spectral drop-off at lower $k_x R$ is expected for the lower-Reynolds-number DNS. At high wavenumber, the DNS energy spectrum does not suffer from attenuation that potentially could af-
fect hot-wire or hot-film spectra due to spatial averaging issues and low-pass filters typically applied. The Hultmark et al. (2010) spectrum rolls off at slightly lower wavenumber than the present DNS spectrum for the radius shown in figure 4.5, but the experiment uses a high sampling rate and small probe, so the DNS could also be responsible for the slight difference.

The behaviour of the Hultmark et al. (2010) spectrum at long wavelength is similar to that of the KA99 spectrum. There is evidence that a low-\(k_x\) peak also appears in Hultmark et al. (2010) because a distinct notch separating the peaks appears between \(k_x R = 1\) and 2. The presence of noise makes the peaks’ wavelengths more difficult to determine, and this is responsible for the scatter in the wavelength data points in figure 4.1. These points were determined by also comparing with nearby radial locations to eliminate spurious peaks.

The long-wavelength peak for the DNS appears attenuated relative to the experimental spectra, most clearly in the KA99 spectrum of figure 4.5 that is well converged at long wavelengths. This difference could potentially be attributed to errors affecting the experiment and the simulation relative to the true spatial wavenumber spectrum. The experiments shown for comparison are generally consistent with each other and other relevant experiments, so these are assumed to be correct representations of single-point thermal anemometer measurements. The most obvious source of the discrepancy is the use of Taylor’s hypothesis to infer spatial wavenumber spectra from the measured temporal fluctuations. del Álamo & Jiménez (2009) have convincingly demonstrated the effects of this by applying a function to simulate errors introduced by the use of Taylor’s hypothesis to turbulent channel DNS spectra, and the resulting spectra contain a more pronounced longer-wavelength peaks that are more comparable to those of an experiment at similar Reynolds number. This will be discussed in greater depth below. Another possible source of the discrepancy is error introduced by the simulation on a finite domain length.
the $\langle u^2 \rangle$ statistics match very closely between simulation and experiment (as indicated by the $u'_{x,rms}$ comparison in figure 4.3), and this quantity is the sum of the energies at each wavelength (as shown in equation (4.5)), this supports the correctness of the spectrum with the possible issue of whether the energy could be incorrectly transferred to other modes as a result of the limited periodic domain length while the total energy remains correct. Focusing on the longest modes that are relevant to the comparison of the longer-wavelength peak and for which this could potentially be a concern, the study of Chin et al. (2010) for pipe simulations with a range of periodic-domain axial lengths (but at lower Reynolds number than the present simulation) indicates that the long scales generally converge even when the length is restricted. Examining the spectrum map for the long wavelengths ($\lambda_x > 2R$) of the $R^+ = 500$ pipe (their figure 7), the premultiplied energy contour lines for the lowest magnitude agree very closely with each other for the $8\pi R$, $12\pi R$, and $20\pi R$ domain lengths. Their $4\pi R$ domain length contour line agrees fairly well with the others for this lowest contour level, but deviates significantly from the longer domain lengths for the next higher-magnitude contour line shown in that significantly greater energy exists at wavelengths greater than about $3R$. Based on their spectrum map, this curve indicates an excess in energy at around $6R$ wavelength that extends for wall-normal distances from below $0.1R$ to above $0.4R$ relative to the simulations on longer periodic domains and potentially could appear to be a peak at these wavelengths. The $30R$ domain length of the present simulation is between the $8\pi R$ and $12\pi R$ lengths that show good agreement for Chin et al. (2010), and the Reynolds number of the present simulation is sufficiently close that it is not expected to dramatically alter the domain length requirements, so the evidence indicates that the computational periodic domain length of the present simulation is sufficient for reliable energy spectra at long wavelengths.
Having considered the other possibilities, the difference at long wavelength between the present DNS and the experiments appears consistent with the del Álamo & Jiménez (2009) conclusions with respect to the use of Taylor’s hypothesis. However, there is evidence in the DNS that a peak may be beginning to form. Close examination reveals that the decay in the DNS spectrum with decreasing wavenumber flattens at approximately the same wavelength as where peaks are observed in experimental spectra. In the case of figure 4.5, this levelling occurs at \( k_x R \approx 1.2 \). Spectra at additional radii and spectrum maps demonstrate that this feature is consistent over a range of \( y \). A more convincing long-wavelength peak occurs at higher \( y \) values for this Reynolds number, as observed in later figures, such as figure 4.6(b).

This long wavelength peak in the pipe DNS spectrum is similar to what occurs in the \( Re_\tau = 934 \) channel DNS spectrum of del Álamo et al. (2004), included as grey squares in figure 2 of Monty & Chong (2009). In that figure, a dip occurs at \( \lambda_x \approx 8h \), and a small peak exists at longer wavelength. However, Monty & Chong (2009) also compute similar spectra by zero-padding the DNS data to increase spectral resolution, and then smoothing the spectra to eliminate oscillations that are induced. This procedure appears to obscure the slight dip and longer-wavelength peak shown in the raw spectra, and they become no longer apparent. Careful examination of the \( Re_\tau = 2003 \) channel energy spectra in figure 10 of del Álamo & Jiménez (2009) (particularly at \( y^+ = 200 \)) also reveals the presence of the dip and the longer-wavelength peak. The consistency of these features between simulations creates a more convincing case that they are not just noise but are characteristic of DNS spectra at relatively high Reynolds number in wall-bounded shear flows.

Returning to figure 4.5, the wavelength at which a short flat region or weak long-wavelength peak occurs in the DNS spectrum matches the wavelength where distinct peaks occur in the KA99 hot-film data. This supports the possibility that a small peak may be beginning to form at the correct wavelength, but the use of Tay-
lor’s hypothesis overemphasizes the energy in this region. The peak wavelengths from the DNS are included in figure 4.1, and they agree with peak wavelengths from the various experiments using Taylor’s hypothesis. The location of the dip in the centre of the bimodal shape in the KA99 measurements also agrees well with the dip or levelling off in the DNS. The experiments indicate that the dips exist at longer wavelength for lower Reynolds number, although this variation may be restricted to relatively low Reynolds numbers. The limited domain length of the pipe (or the channel just discussed) results in Fourier modes being widely spaced for long wavelengths, since the modes occur at integer fractions of the simulation periodic wavelength (i.e., $30R, 15R, 10R, \ldots$). This wide spacing of wavelengths is apparent in the discrete nature of the spectral peak wavelengths of the pipe DNS reported in figure 4.1. However, the agreement between the spectrum maps in Chin et al. (2010) for the three longest domain length simulations suggests that the spectra at long wavelengths are relatively consistent despite the discrete wavelength issue. The peak location $\lambda_\alpha \approx 8h$ at $y/h = 0.3$ estimated from the $Re_\tau = 934$ channel DNS is also consistent with the data reported in figure 4.1.

Figure 4.6 presents the full set of premultiplied streamwise energy spectra from the present simulation for each velocity component. The premultiplied streamwise velocity spectra in figure 4.6(b) include lines representing $y^+ = 60$ and $y^+ = 101$, and both possess the levelling off or slight dip identified in the $y/R = 0.1$ ($y^+ = 69$) spectrum of figure 4.5. At lower $y^+$ values in figure 4.6(a), the overall magnitudes are larger (until very near the wall), and the behaviour is dominated by the rise to the peak at $k_\alpha R \approx 6$, which will be identified with the inner energy site in the discussion of figure 4.9. At higher $y^+$ values in figure 4.6(b), the region of flattened decay transitions into a lower-wavenumber peak, with two examples indicated by arrows. Above where a logarithmic layer could be expected to occur, the lower-wavenumber peak is more comparable in magnitude to the peak at $k_\alpha R \approx 6$ for these
radii. This indicates that these motions of such long wavelength contribute large fractions of the total energy. These wavelengths are classified as very large-scale motions, and will be discussed in further depth in connection with the spectrum maps.
In figure 4.7, the log-log spectra (non-premultiplied) at $y^+ = 77.5$ for all three velocity components indicate the large amount of energy in long wavelengths of $u$ relative to the radial and azimuthal velocity components, which have spectra that are approximately constant for low wavenumber and over an order of magnitude smaller at the lowest wavenumber (which corresponds to the wavelength of the $30R$ periodic domain length). The viscous cutoffs at high wavenumber are similar between components. To compare the $uu$ energy spectra with common scalings, $-1$ and $-5/3$ slope lines are also included. The $uu$ spectrum appears only tangent to the $-5/3$ line, and no clear power law is observed in the inertial subrange, probably because of the low Reynolds number. $\Phi_{uu}$ is approximately parallel to the $-1$ line for $k_xR \approx 4–7$, but, at lower wavenumbers, contains less energy than indicated by the $-1$ line. A slight dip relative to the linear decay trend is also visible in $\Phi_{uu}$ at $k_xR = 1.26$ that corresponds to the dip observed in the premultiplied spectra of figure 4.6(b).

The premultiplied $vv$ spectra in figure 4.6(c) and $ww$ spectra in figure 4.6(d) clearly show a paucity of energy at low wavenumber relative to $uu$. The peaks occur at greater wavenumber for these velocity components than for any peak in $uu$. It is also clear that the peak reaches its greatest magnitude at a greater wall-
normal distance for \( \overline{ww} \) than \( \overline{uu} \), and at an even greater wall-normal distance for \( \overline{vv} \). These trends are more clearly shown in spectrum maps, which are now considered.

Spectrum maps depict contours of premultiplied energy as a function of wavelength and wall-normal position (del Álamo et al., 2004; Hutchins & Marusic, 2007a). In experiments on wall-bounded turbulent shear flows at high Reynolds numbers, the maps clearly reveal a bimodal distribution of \( \overline{uu} \) energy in which the maxima of the two modes visible in an energy spectrum for a particular wall-normal position correspond to two peaks clearly visible in the maps. In turbulent boundary layers, Hutchins & Marusic (2007a) identified an ‘outer energy site’ that corresponds to a long-wavelength peak in the bimodal distribution that exists for a range of wall-normal locations approximately in the logarithmic layer vicinity. They identified the outer peak with superstructures, which correspond to the very large-scale motion terminology used by KA99 and others. Hutchins & Marusic (2007a) also identify an ‘inner energy site’ located at the wall-normal location with maximum turbulence production and associated with the near-wall cycle (Jiménez & Pinelli, 1999; Schoppa & Hussain, 2002). Monty et al. (2009) compared spectrum maps between pipe, channel, and boundary layer flows and found these features are clearly present for each (at sufficient Reynolds number), but are most similar between pipe and channel.

The pipe spectrum map from the hot-wire study of Ng et al. (2011) reproduced in figure 4.8(a) contains the features identified by Hutchins & Marusic (2007a) and Monty et al. (2009). This map is overlaid with premultiplied \( \Phi_{\overline{uu}} \) contours for the present DNS. Since the experimental Kármán number is 52% greater than that of the DNS (and \( Re_D \) is estimated to be 39 500), the figure contains the spectra scaled in both inner (left) and outer (right) units. In the inner-scaled plot, the inner-site peak location found by Hutchins & Marusic (2007a) is marked by a black cross at \( \lambda_+ = 1000 \) and \( y^+ = 15 \) (and is assumed to be constant for these wall-bounded
Figure 4.8: Comparisons of premultiplied $uu$ energy maps with experiments (a,b) and a lower-$R^+$ simulation (c). $k_x \Phi_{uu}/u_c^2$ contours are displayed. (a) Black contour lines for the present DNS ($R^+ = 685$) are superimposed on colour contours with red dashes for hot-wire pipe spectra of Ng et al. (2011) at $R^+ = 1040$. Contour lines and colour levels correspond to values of 0.1 to 0.7 in increments of 0.1 (thinner lines) and 0.9 to 1.9 in increments of 0.2 (thicker lines). (b) Present DNS (solid black lines) and pipe-flow hot-wire measurements of Hultmark et al. (2010) at the same Reynolds number (dashed red lines). The thicker lines are contour levels of 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.6, 2.0, and the thinner lines are 0.3 and 0.5. (c) The present DNS (solid black lines) and $R^+ = 500$ pipe-flow DNS of Chin et al. (2010) (dashed red lines). The dashed red lines and thick black lines are levels of 0.3, 0.7, 1.1, 1.5, 1.9, and the thin black lines are 0.35–0.65 in increments of 0.05.
shear flows). The DNS contour lines suggest the peak is located slightly nearer the wall and at slightly shorter wavelength ($\lambda^+_x \approx 700$). This peak location in the DNS matches closely with the peak at $\lambda^+_x \approx 600$ in the $R^+ = 500$ pipe DNS of Chin et al. (2010). The slight difference between the location in the present DNS and the $\lambda^+_x = 1000$ location approximated from experiments may be due to Reynolds number differences. The DNS contours appear similar to those of Ng et al. (2011) for the near-wall peak region. Along the short-wavelength extreme of the spectra, contour lines for the DNS persist to slightly smaller wavelength than for the experiment, and besides spatial averaging issues with hot-wire measurements (studied in depth by Hutchins et al. (2009) and Chin et al. (2009)), this also might be attributed to numerical reasons in the simulation. Above $y^+ \approx 50$, the DNS uu spectra and the experimental spectra begin to behave differently due to the different Reynolds numbers, and outer scaling is necessary for comparing outer peak features.

The outer-scaled spectrum map of figure 4.8(a) includes the location of the outer energy site for the experiment (black cross) and the DNS (red cross). These locations are calculated from the relationship of Mathis et al. (2009a) and Mathis et al. (2009b), who suggest the outer site occurs at $y/R \simeq 3.9Re^{-1/2}$ and $\lambda_x = 6R$. Near the centre of the pipe ($y/R = 1$), the contours match closely for both flows, except at the longest wavelengths. In the experiment, no contour island representing the local energy maximum of an outer site appears on the contour map, but the site is relatively weak for this Reynolds number (compared to the $R^+ = 3005$ Monty et al. (2009) experiment in which a contour island is clearly present). Although a very small contour island may appear if the contour level is selected carefully, it is not necessary for a contour island to exist for the premultiplied spectra at individual $y$ values to have the bimodal form. Instead, weaker longer-wavelength peaks can appear as protrusions in the contour lines to higher $y$ values (since the spectrum in general decays with increasing $y$), and this occurs for the experiment with wave-
lengths between 6 and 20\(R\) when \(y\) is greater than 0.06\(R\). The wavelength that this feature appears to be centred about is greater than the 6\(R\) suggested for the outer site, possibly as a result of the lower Reynolds number.

The DNS spectrum map in figure 4.8(a) is dominated by approximately concentric circular contour lines representing the decay from the peak of the inner energy site. While this characterizes the spectrum at low \(y\), there is a notable change from sloped lines to approximately vertical contour lines as \(y\) increases for \(\lambda_x/R \approx 10\). At \(y/R = 0.1\), the approximately level region identified in the one-dimensional spectra is visible at \(\lambda_x/R = 5\) (which is the previously noted \(k_xR = 1.26\)) in the line for the DNS 0.5 contour level. This location is similar to that of the protrusion noted in the Ng et al. (2011) spectrum. At higher \(y\) values near 0.3\(R\), the 0.4 contour line for the DNS includes a more significant protrusion that is also located near the protrusion in the experiment contour lines (to within the limited spectral resolution of the DNS spectrum).

The DNS is next compared with hot-wire spectra obtained by Hultmark et al. (2010) for pipe flow at the same Reynolds number as the present simulation in figure 4.8(b). The experimental spectra contain some noise that would reduce with further statistical convergence. The agreement between the simulation and experiment is good to within the noise. The experimental spectrum contains evidence of additional energy near the expected outer site location that is weaker in the DNS. The apparent long-wavelength peak in the 0.4 contour line for the DNS matches closely with a similar feature in the hot-wire spectrum. The decay at high wavenumber was observed to occur at lower wavenumber for the Hultmark et al. (2010) spectra than the present simulation in figure 4.5. This is manifested in the shift of contour lines to longer wavelength for the experiment relative to the contour lines for the DNS in the region of the spectrum map where the energy decays to low magnitude as wavelength decreases.
Very near the wall (i.e., \( y^+ < 10 \)), the lowest contour levels for the experiments (figure 4.8a,b) are approximately horizontal at short wavelength, suggesting that the viscous roll-off occurs at the same wavelength in this region. In contrast, these lines for the DNS are sloped, indicating a shift to viscous decay at longer wavelength as the wall is approached. This behaviour is consistent with another pipe DNS that is compared next in figure 4.8(c). This difference between hot-wire experiment and DNS also occurs in the turbulent channel flow study of Monty & Chong (2009).

Effects of the hot wire within very small distances from the wall could contribute to this discrepancy. It is also possible that the wavelengths could be affected by Taylor’s hypothesis, as del Álamo & Jiménez (2009) found that even ‘small’ modes with \( \lambda_x < 2h \) and \( \lambda_z < 0.4h \) in \( Re_\tau = 934 \) channel simulations travel much faster than the local mean very near the wall, until \( y^+ \) approaches 16 (in their figure 4). As the very long wavelengths that travel faster than the local mean are represented at shorter wavelength to overemphasize the long-wavelength peaks when \( y \) is near the logarithmic layer region and Taylor’s hypothesis is used with the assumption that all motions convect at the local mean velocity, so shorter motions could be shifted to even shorter wavelength when very near the wall under the same assumptions. However, further information would be needed on the convection velocities of the very smallest wavelengths very near the wall to confirm this possibility. In figure 4.8(c), the contour lines at short wavelength become roughly horizontal for both the present simulation and another pipe simulation at a \( y^+ \) value similar to that observed in the \( Re_\tau = 934 \) channel, and the behaviour at very low \( y^+ \) appears consistent among simulations.

The following comparison with a pipe DNS simulation at lower Reynolds number highlights the changes with Reynolds number that may indicate an outer peak forming. figure 4.8(c) contains a \( uu \) spectrum map for the \( R^+ = 500 \) pipe simulation of Chin et al. (2010), who compared simulations at different pipe domain
lengths, and the data digitized from the dotted line of their figure 7(b), which is typical of the long domains presented \((L_x > 8\pi R)\). Chin et al. (2010) found no discernible outer-site peak. Figure 4.8(c) also contains the spectrum map for the present \(R^+ = 684.8\) DNS at the same levels. Additional levels are added to better represent the behaviour where an outer site would be expected. The contour lines for the lower Reynolds number simulation appear more circular than for the present simulation. Particularly when comparing the lowest contour level \((0.3)\) lines, the vertical (flat with respect to wavelength) contour lines at \(\lambda_x/R \approx 10\) (below the arrow) previously discussed are absent at \(R^+ = 500\). The additional contour lines reveal much more of the behaviour. The second-to-lowest contour level line of the \(R^+ = 500\) simulation in figure 4.8(c) reveals some deviation from a circular shape in the high-y/long-wavelength quarter of the contour line, but not as much as for the present \(R^+ = 684.8\) simulation. The contour lines for the \(R^+ = 170\) pipe simulation of Chin et al. (2010) are even more circular. Thus, while no contour line island representing an energy peak at the outer site is visible for the present DNS, there is evidence suggesting that an outer energy site peak may be forming.

The spectrum maps for \(uu\), \(uv\), \(vv\), and \(ww\) of the present simulation are presented in figure 4.9. The spectrum map in figure 4.9(a) displays the same data as that superimposed on figure 4.8. An important feature of the \(uu\) spectrum is very large-scale motions, which are defined as motions with streamwise wavelengths greater than \(3R\) based on \(uu\) energy spectrum behaviour (Balakumar & Adrian, 2007). In all plots, this wavelength is indicated with a dashed line. While the vertical contour lines marking the development toward an outer energy site and protrusions from concentric ellipses appear in the \(uu\) spectra, no such features exist to any extent in the \(vv\) and \(ww\) spectra. The \(uv\) spectrum map has a slight bulge at long wavelength in the contour shapes, and this may be explained by the \(u\) influence. These spectrum contour lines and the positions where they cross the VLSM
line indicate that VLSMs can make large contributions to $uu$, significant contributions to $uv$, and very minor contributions to $vv$ and $ww$. Guala et al. (2006) suggests that large-scale structures of streamwise velocity grow in streamwise length linearly with $y$ in the logarithmic layer. Lines with slope 1 (i.e., $\lambda_x \propto y$) are included in the spectra for reference, but no convincing adherence is apparent. This is not unexpected because the Reynolds number of the present DNS does not create a sufficiently large logarithmic region. The other components also do not exhibit any notable behaviour along this line. An additional line is included on the $vv$ map, which is unique in having an inclined principal axis of the concentric ellipses representing the peak. While the other spectra have approximately vertical or horizontal principal axes, this inclined axis indicates that dominant streamwise wavelengths of wall-normal/radial motions increase with pipe wall-normal distance in a power-law fashion that is slower than linear. When averaged, $uu$ motions would also grow in streamwise length with increasing $y$ because VLSMs become progressively stronger with increasing $y$, whereas the short-wavelength peak (inner site) dominates the behaviour near the wall. These observations are also apparent in the conventional premultiplied spectrum plots (figure 4.6).

The $uu$ spectrum map has been interpreted by Hutchins & Marusic (2007a) in terms of the ‘inner energy site’ being associated with strong $u$ fluctuations and the near-wall cycle (Jiménez & Pinelli, 1999; Schoppa & Hussain, 2002), and the outer site being associated with VLSMs and superstructures. While the inner peak of the $uu$ spectrum is concentrated near the wall and decays quickly with increasing $y$, the $uv$ peak decays more slowly with $y$ and peaks at a higher $y$. This is due to the attenuation of $v$ fluctuation near the wall, which is seen in the $vv$ spectrum, so the regions of $Q2 (-u, +v)$ fluctuation (which contribute strongly to $uv$) are stronger above the $uu$ peak location. The absence of long-wavelength peaks in the $vv$ and $ww$ spectra indicates that these motions possess less long streamwise organization.
than the $u$ motions. The $ww$ spectrum peaks at a height intermediate between $uu$ and $vv$, but also extends to near the wall. The near-wall quasi-streamwise vortices contribute strongly to $w$ (azimuthal velocity) fluctuations (Kim et al., 1987). The peaks of the $vv$ and $ww$ spectrum maps are centred at shorter wavelengths than the short-wavelength peak of the $uu$ map. The $uv$ peak is at a similar wavelength to that of the $uu$ short-wavelength peak, implying that the coherence between $u$ and $v$ is similar to the organization leading to this peak in $uu$. However, energy at the outer site location is weaker in the $uv$ spectrum map, so there is less energy at long wavelengths relative to $uu$. Consequently, the length scales obtained by averaging over motions of all scales are shorter for $uv$ than $uu$. 
Figure 4.9: Contour maps of premultiplied spectra $k_x \Phi(\lambda_x)$ for (a) $uu$, (b) $uv$, (c) $vv$, and (d) $ww$. Solid white lines are cumulative spectra contours at the indicated levels. Solid black lines indicate slopes of unity ($\lambda_x \propto y$), with also a line along the contour major axis in (c). To the right of each map are cumulative spectra: dotted green: $y^+ = 30, y/R = 0.044$; solid black: 101, 0.15; short-dashed blue: 204, 0.30; long-dashed red: 478, 0.70.
Contour lines of cumulative spectra are overlaid onto the spectrum maps in figure 4.9. To the right of each map, cumulative spectra for selected radii are also presented in a line plot, as in Guala et al. (2006) and Balakumar & Adrian (2007). The cumulative spectra quantify the previous observations of the importance of VLSM contributions, indicating that approximately 35–50% of $\langle uu \rangle$ is contributed by motions with wavelengths in the VLSM range. The contributions reach their maximum percentage near $y/R \approx 0.3–0.4$ but increase most rapidly with respect to $y$ near $y/R = 0.1$, indicating the growing relative importance of VLSMs near the region where a logarithmic layer would be expected to occur. VLSMs remain important to $\langle uv \rangle$, contributing 25–40% of the Reynolds shear stress, but make much lower contributions to the other velocity components. The contribution to $\langle vv \rangle$ is typically less than 10%, and the contribution to $\langle wv \rangle$ ranges from 10% to 20%. Interestingly, VLSMs contribute the largest fraction to $\langle wv \rangle$ near the wall, at $y^+ \approx 14$. This may be linked with the organization of quasi-streamwise vortices.

It should be noted that the definition of cumulative spectra (4.7) used to calculate the data for these plots is discrete. As discussed, the long wavelengths are coarsely spaced, so the contour lines presented are based on interpolation. Thus, at particularly long wavelengths, there is some uncertainty in cumulative energy versus wavelength. It should also be noted that the total energy in the definition includes the zero-wavenumber mode, but this mode is not included in the sum of energies of shorter wavelength in the numerator of (4.7) to calculate the cumulative energy. This places the energy at longer wavelength than the periodic domain length and therefore the zero-wavenumber mode is included with the VLSMs. This is appropriate because the zero-wavenumber mode accounts for temporal fluctuations of the spatial mean energy, and this is expected to contain energy from motions that are too long to fit in the domain length. Such motions make contributions up to 5–10% of the energy for the $uu$ spectra, and smaller amounts for other velocity
components. This mean $k_x = 0$ energy fraction is given as a function of radius $r$ by

\[
\frac{\langle u'^2 \rangle_{k_x=0}}{\langle u'^2 \rangle} = \frac{\left(\langle (u_{x,\theta} - \langle u \rangle)^2 \rangle\right)}{\langle (u - \langle u \rangle)^2 \rangle},
\]

(4.8)

where $u(x,r,\theta,t)$ is velocity in an instantaneous field, $\langle \cdot \rangle_{x,\theta}(r,t)$ is the mean in only the homogeneous spatial directions for each radius of an instantaneous field, and $\langle \cdot \rangle(r)$ is the mean in $x$, $\theta$, and time (i.e., over all fields). The cumulative $uv$ does not strictly increase monotonically with increasing $\Lambda$ because the $uv$ spectrum itself is not entirely negative, but the positive contributions are very weak and at very short wavelength, so only the expected monotonic behaviour of cumulative $uv$ is visible.

4.4.3 Streamwise Cumulative Spectrum Wavelengths

The cumulative spectra in figure 4.9 suggest plotting, as a function of radial or wall-normal position, the streamwise wavelengths $\Lambda$ at which the cumulative spectra $\Upsilon(\Lambda)$ are equal to chosen energy fractions. Guala et al. (2006) and Balakumar & Adrian (2007) plot these data for their hot-film experiments. Wavelengths for the present simulation and various pipe experiments with roughly comparable Reynolds numbers are compared in figure 4.10.

Several considerations involving the experimental measurements and DNS are important to interpreting the comparison. Using Taylor’s hypothesis for experiments assuming all motions convect at the mean velocity has been shown to modify the energy spectrum at long wavelengths (del Álamo & Jiménez, 2009), so this is expected to also affect the cumulative spectra. Monty & Chong (2009) found the issue of the weaker long-wavelength peak in DNS relative to hot-wire experiments to be of concern due to the large energies in long wavelengths noted by Guala et al. (2006). Since applying Taylor’s hypothesis with the assumption that all motions convect at the local mean velocity is observed to shift energy in very long wavelengths to shorter wavelengths while in the proximity of the logarithmic layer (del
Figure 4.10: Wavelengths corresponding to $uu$ and $uv$ cumulative spectrum values of (a) $\Upsilon = 0.5$ and (b) $\Upsilon = 0.8$. Black solid lines depict $uu$ for the pipe DNS, grey dash-dot-dot lines are $uv$ for the pipe DNS, and symbols represent $uu$ for pipe experiments. The experiment most comparable to the present $R^+ = 685$ ($Re_D = 24\ 580$) DNS is the hot-wire pipe data of Hultmark et al. (2010) at $R^+ = 690$ ($Re_D = 25\ 000$), depicted as filled squares. The symbols for other experiments are listed in table 4.2. (Figure A8 in the digital appendix of Balakumar & Adrian (2007) presents a similar figure for channel and boundary layer experiments.)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Pipe Flow</th>
<th>$R^+$</th>
<th>$Re_D$</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>- - - - - -</td>
<td>DNS</td>
<td>685</td>
<td>24 580</td>
<td>Present Simulation</td>
</tr>
<tr>
<td>■</td>
<td>Hot Wire</td>
<td>690</td>
<td>25 000</td>
<td>Hultmark et al. (2010)</td>
</tr>
<tr>
<td>▼</td>
<td>Hot Wire</td>
<td>1606</td>
<td></td>
<td>Perry &amp; Abell (1975)</td>
</tr>
<tr>
<td>⊥ ⊤</td>
<td>Hot Wire</td>
<td>49 645</td>
<td></td>
<td>Lekakis (1988)</td>
</tr>
<tr>
<td>+</td>
<td>Hot Wire</td>
<td>55 000</td>
<td></td>
<td>McKeon &amp; Morrison (2007)</td>
</tr>
<tr>
<td>•</td>
<td>Hot Wire</td>
<td>1133</td>
<td>42 700</td>
<td>Vallikivi et al. (2011)</td>
</tr>
<tr>
<td>♦</td>
<td>Hot Wire</td>
<td>1000</td>
<td></td>
<td>Ng et al. (2011)</td>
</tr>
<tr>
<td>- - - - - -</td>
<td>DNS ($uv$)</td>
<td>685</td>
<td>24 580</td>
<td>Present Simulation</td>
</tr>
</tbody>
</table>

Table 4.2: Symbols for figure 4.10. All $Re_D$ values listed conform to the definition used throughout this paper based on bulk velocity. Each symbol is for $uu$ cumulative energy except for the last line type listed, which is for $uv$. The figure numbers in each publication that these data correspond to are: McKeon & Morrison (2007) (Figure 5a), Vallikivi et al. (2011) (Figure 10), and Ng et al. (2011) (Figure 9d).
Álamo & Jiménez, 2009), it is not immediately obvious how the cumulative spectra would be affected besides the presence of less energy at the very longest wavelengths. Guala et al. (2006) also commented that experimental results are subject to the limited frequency response of the data acquisition, the spatial averaging effects of the hot film, and the use of a low-pass filter that would attenuate the $u'u'$ spectrum at high wavenumber. This would tend to overemphasize the fractions of energy at longer wavelength. While the Taylor’s hypothesis and high-wavenumber attenuation effects do not affect the DNS, the finite domain length could affect the cumulative energy due to the coarse spacing of long wavelengths and the periodic domain possibly improperly transferring energy to various modes. However, the domain was chosen to be long relative to other simulations in order to minimize these effects.

For the present DNS (black lines in figure 4.10), the most relevant comparison is with the experimental pipe hot-wire measurements of Hultmark et al. (2010) at the same Reynolds number (filled squares). These agree well for $y/R > 0.1$ based on the $\Upsilon_{uu} = 0.5$ and 0.8 lines shown. While the trends match very closely, the wavelengths are somewhat larger for the hot-wire experiment relative to the present simulation. Averaging over $y/R = 0.1$ to 1, the $\Upsilon_{uu} = 0.5$ wavelengths are on average 6% larger and the $\Upsilon_{uu} = 0.8$ wavelengths are 10% larger than the DNS. The experimental wavelength is greatest relative to the DNS at locations immediately above the near-wall region. Specifically, the $\Upsilon_{uu} = 0.5$ wavelengths have the largest difference of 36% at $y/R = 0.051$, and the $\Upsilon_{uu} = 0.8$ wavelengths have the largest difference of 31% at $y/R = 0.163$. The cumulative wavelengths for the experiment have several points that do not vary smoothly with $y$, but this probably would be eliminated with additional statistical convergence of the spectra. Consequently, the difference from the DNS also contains spikes, suggesting the averages provide a better indication of the difference. Near the wall, particularly below $y/R = 0.03$
\( y^+ = 20 \), the wavelengths for the experiment are significantly shorter than for the simulation. This is consistent with the differences observed and discussed in connection with the energy spectrum map of figure 4.8(b) and approximately matches the \( y^+ \) location at which contour map lines at short wavelength for both simulation and experiment become horizontal.

The energetic importance of long-wavelength motions has been highlighted by the cumulative spectra of Guala et al. (2006) and Balakumar & Adrian (2007). Guala et al. (2006) obtained spectra experimentally for pipes with significantly higher Reynolds numbers than the present simulation, and their results indicate that \( \zeta_{uu} = 0.5 \) and 0.8 occurs at substantially longer wavelengths than for the present results. For reasons we do not fully understand, these also fall significantly above additional cumulative wavelengths that have been computed for other pipe experiments at comparable Reynolds numbers. A comparison of premultiplied spectra with those of comparable experiments (such as Perry & Abell, 1975) indicates that the Guala et al. (2006) spectra roll off on the high wavenumber end at lower wavenumbers. Guala et al. (2006) commented that the spectra and cumulative distributions ‘under-represent the smallest scales of motion’ due to low-pass filtering and spatial filtering effects. However, this faster roll-off relative to other spectra begins at wavelengths significantly longer than those where Guala et al. (2006) suggests these effects are significant. It should be noted that plots of the wavelength of the peak VLSM energy in Guala et al. (2006) (also included here in figure 4.1) agree reasonably well with the data of Perry & Abell (1975) and Bullock et al. (1978). The cumulative energy plots depend on spectra over a wide range of wavenumbers besides those near the peak at long wavelength, and the differences observed could be due to distortion in the Guala et al. (2006) data. For this reason, their cumulative wavelengths are omitted from figure 4.10. We also omit a comparison with the Balakumar & Adrian (2007) cumulative wavelengths for channel experiments be-
cause of possible differences between pipe and channel flows. In general, their $Re_\tau$ Reynolds numbers are comparable to the present simulation, and the cumulative wavelengths are of comparable magnitudes when scaled by channel half-height.

To further compare the behaviour of cumulative energy wavelengths for pipe flows, values were computed from digitized spectra for additional experiments found in the literature. These provide further support of the large amounts of energy existing at long scales, as observed by Guala et al. (2006). The pipe flows included for comparison are generally restricted to those with $Re_D < 50\,000$, and the symbols are described in table 4.2. One experiment with $Re_D = 66\,000$ is also included because it includes a wide range of $y$. At these relatively low Reynolds numbers, the general trend is to longer wavelengths for a given cumulative energy value as Reynolds number increases, but the behaviour for a wider range of Reynolds numbers will be discussed in future work.

Various experiments include different ranges of wavenumbers. While they consistently include sufficiently small scales that the energy decays to a very low level at high wavenumbers, they vary in the degree to which the energy decays to a low level for the longest scales measured. Therefore, extrapolation was used on several data sets that did not decay as much as others, generally by extrapolating a constant non-premultiplied energy for wavenumbers below the lowest included in the data set. Other data sets did not require extrapolation, including Hultmark et al. (2010) and Vallikivi et al. (2011). The latter was obtained using a new nano-scale thermal anemometry probe and is expected to also be very accurate with the smallest scales. The effect of extrapolation is illustrated by the cumulative wavelengths calculated from the data of Lekakis (1988). The cumulative spectrum wavelengths are represented as bars in figure 4.10, with the upper and lower ends of each bar representing the wavelengths with and without extrapolation. For these data, the change
with extrapolation is relatively strong compared to other data sets. The effect is clearly larger for $\gamma_{uu} = 0.8$ than 0.5, but even then the wavelengths are within $3R$.

Regardless of whether experiment or simulation spectra are used, all of the included data support the presence of significant fractions of energy in motions with wavelengths in the VLSM region, and the fractions would be expected to increase for higher Reynolds numbers. In the pipe flows included, the wavelengths corresponding to 50% of the $uu$ energy occur in motions with wavelengths approaching $3R$ for the lowest Reynolds numbers and exceeding $3R$ for a significant fraction of the pipe cross-section for higher Reynolds numbers. It is likely that $\gamma_{uu} = 0.5$ wavelengths would exceed $3R$ if pipe simulations were performed at higher Reynolds numbers. For $\gamma_{uu} = 0.8$, the wavelengths for all of the pipe flows included reside within the VLSM range over essentially the entire $y$ range. Thus, although there is some evidence that the Guala et al. (2006) calculated wavelengths were longer than accepted values due to a combination of experimental errors and the wavelengths calculated here for the simulation and other experiments are somewhat shorter, the results strongly support the conclusion that VLSM motions are energetically important in these pipe flows for both experiments and simulations at lower Reynolds numbers. Figure 4.10 also shows similar trends for $uv$ cumulative energy as for $uu$, except that the wavelengths are shorter. The difference between $uu$ and $uv$ is also consistent with experiments (Guala et al., 2006; Balakumar & Adrian, 2007).

4.4.4 Azimuthal Spectra

Spectrum maps based on decomposition into azimuthal Fourier modes are presented in figure 4.11. The modes are associated with arc length wavelengths and wavenumbers, which are useful for analogy with the spanwise counterparts for channels. The arc length wavelength is defined as $\lambda_s = 2\pi r/n_\theta$, where $n_\theta$ is the azimuthal mode number. The spectrum map domains are therefore curved as $r$
reduces with increasing $y$. Compared to the streamwise maps (figure 4.9), the azimuthal maps attain their maxima at similar $y$ locations, but differ in indicating pronounced scale growth with $y$. While the strongest peak contours for the $uu$ spectrum do not show a clear major axis, the peaks for the other components have clearly inclined major axes. The $uu$ peak occurs at $y^+ \approx 15$ and $\lambda_5^+ \approx 100$, which matches the accepted near-wall streak wavelength and supports the identification of this peak with the motions associated with quasi-streamwise vortices. This feature was also observed in azimuthal correlations from the Chin et al. (2010) pipe DNS. While the lower contour levels of $uu$ are inclined in the map, there is also an apparent peak at $y/R \approx 0.2$ and $\lambda_z/R \approx 1$. A similar peak is observed in the $Re_\tau = 934$ channel DNS spanwise spectrum map of Chin et al. (2009) (their figure 3b), in which it occurs at $y/h \approx 0.3$ and spanwise wavelength $\lambda_z/h \approx 1$. The azimuthal scale growth with $y$ for the present pipe DNS obeys a trend similar to that observed by Tomkins & Adrian (2003), who report linear spanwise scale growth in boundary layers based on the scales of features obtained by conditional averages or their linear stochastic estimates, which are based on two-point correlations. Monty et al. (2007) also observe linear scale growth for the canonical flows above a certain height based on two-point correlations. While we are focused here only on the energy spectrum, these observations appear consistent with the spanwise spectrum map behaviour, and therefore lines corresponding to linear growth are superimposed on the maps. A line connecting the $uu$ peaks has slope less than 1, consistent with the slowing growth that leads Monty et al. (2007) to divide the scale growth into two linear regions. Based on the major axes of the contour ellipses, the $vv$ and $ww$ spectra also indicate slower than linear scale growth, although the highest contour levels are inclined at approximately the correct slope for linear growth.
4.4.5 Two-Dimensional Spectra

Using streamwise and azimuthal wavelengths, the two-dimensional spectrum of $u u$ for the present DNS at $y^+ = 101$ is presented in figure 4.12. Spectra from DNS simulations of channel flow at similar $y^+$ values (matched in inner scaling) are also included, and the presentation is similar to that of del Álamo et al. (2004) in that the wavelengths are nondimensionalized by $y$. This nondimensionalization provides a good match between the contours, and it is also advantageous for the present comparison because it avoids assuming a relationship between $R$ and $h$ characterizing the different geometries. While the pipe spectrum would be smoother with better statistical convergence, the contours clearly indicate that the streamwise energy distributions by scale are very similar in the logarithmic layer region for these pipe and channel flows.

The agreement provides evidence that the effects of pipe curvature do not radically change important large scale motion statistics at wall-normal distances up to $\sim 0.15R$, the nominal top of the logarithmic layer. At this $y^+$ location, the two-dimensional spectrum includes peaks at moderate streamwise wavelength ($\lambda_x/y \approx 15$, which for the pipe is $\lambda_x/R \approx 2$) and also a peak at much longer wavelength. For the pipe, this is at $\lambda_x/R \approx 16$, and this is probably closely related to the outer energy site in the streamwise one-dimensional spectrum maps. At this radius, there appears to be a slight shift of the shortest wavelengths to yet shorter wavelengths in both $\lambda_x$ and $\lambda_s$ for the pipe relative to the channel. Figure 4.12 and the previous spectrum comparisons also indicate that the features observed are consistent irrespective of the flow simulation numerical method used, as the channel simulations employ spectral methods while the present pipe simulation uses finite difference.
Figure 4.11: Contour maps of premultiplied one-dimensional spectra as a function of azimuthal arc length ($s$) wavelength. Contour colours and black contour lines represent the premultiplied spectra for (a) $k_s \Phi_{uu}$, (b) $k_s \Phi_{uv}$, (c) $k_s \Phi_{vv}$, and (d) $k_s \Phi_{ww}$. The white lines have a slope of unity, corresponding to linear growth with $y$. 
4.5 Time Evolution

The preceding results indicate strong similarities between experiment and numerical simulation of pipe flows. The issues brought up relating to Taylor’s hypothesis are closely related to how the spatial fields evolve with time and the convection velocities for modes of various wavelengths. Recalling that streamwise scales of interest for VLSMs can range over $15R$, it is instructive to examine time sequences of streamwise velocity fluctuation contours for the full axial and azimuthal extents of the fields.

Instantaneous streamwise velocity contours for the entire periodic pipe length at $y^+ = 80$ are visualized in figure 4.13 at times spaced by $7.2R/U_{\text{bulk}}$ or $\Delta t^+ = 275$. The axes are scaled such that the arc length has the same scaling as axial length, although azimuthal angle $\theta$ is displayed. Structures traveling at the bulk velocity would convect $7.2R$ downstream between frames, or they would convect by $6.4R$ if they travel at the local mean velocity, as is commonly assumed in conjunction with Taylor’s hypothesis to infer spatial information from temporal measurements. These are relatively large time spacings, considering the findings of Dennis & Nickels (2008) when comparing instantaneous fields obtained using particle image velocimetry (PIV) with spatial fields inferred by Taylor’s hypothesis from initial PIV measurements. For a turbulent boundary layer, they found that the correlation coefficient between the two decayed to 0.2 for the maximum convection distance considered of $6.3\delta$. However, when they calculated the correlation coefficient with filtered versions of the fields to remove the small-scale motions, it remained at 0.37 for the same convection distance, suggesting that the larger scales remain correlated over considerably longer times. Since the present study focuses on the very long scales of motion, the time spacing is appropriate to study their evolution and lifetime. While only one $y^+$ location is presented, the consistent flat
Figure 4.12: Premultiplied two-dimensional energy spectrum of streamwise velocity at $y^+ \approx 100$. Each spectrum is plotted at $\{20\%, 40\%, 60\%, 80\%, 100\%\}$ of its value at the moderate wavelength peak. Solid black lines: $Re_\tau = 547$ channel (del Álamo et al., 2004; del Álamo & Jiménez, 2003); dashed black lines: $Re_\tau = 934$ channel (del Álamo et al., 2004); shaded contours: present pipe (with the peak determined by applying smoothing to the spectrum, but the unsmoothed spectrum is plotted).

region or dip/long-wavelength peak in the one-dimensional streamwise spectrum that persists in a range wider than $y^+ = 60–101$ suggests that conclusions from this position should apply to this entire range.

While a description of the spatial structures is beyond the scope of this study, it is useful to identify several distinctive features and track their evolution in time. For the structures to convect downstream by the full periodic domain length of $30R$, a delay of between 4 and 5 frames is necessary, depending on whether the structures convect at the local mean or a higher velocity closer to the bulk velocity. Strong fluctuations are the most obvious features to visually track over long times. Exam-
Figure 4.13: Contours of $u'/U_{\text{bulk}}$ shaded from $-0.2$ (black) to $0.2$ (white) in planes at $y^* = 80$ with $x/R$ and $s/R$ equally scaled. Evenly-spaced times in terms of $R/U_{\text{bulk}}$ are (a) 302.4; (b) 309.6; (c) 316.8; (d) 324.0; (e) 331.2; (f) 338.4; (g) 345.6; (h) 352.8.
Figure 4.14: Contours of filtered $u'/U_{\text{bulk}}$ ($\lambda_x \geq 6R$ and $\lambda_\theta \geq (2/5)\pi$) shaded from −0.1 (black) to 0.1 (white) in planes at $y^+ = 80$ with $x/R$ and $s/R$ equally scaled. Times in terms of $R/U_{\text{bulk}}$ are (a) 302.4; (b) 309.6; (c) 316.8; (d) 324.0; (e) 331.2; (f) 338.4; (g) 345.6; (h) 352.8.
amples of features are identified by circles, ellipses, and rectangles in figure 4.13. The circles are centred on a relatively compact but strong positive fluctuation of $u$ velocity. The ellipses are drawn for a longer streak of positive velocity fluctuation that is initially stronger and more compact but lengthens significantly with time. Lee & Sung (2011) visualized streamwise–spanwise planes of a turbulent boundary layer simulation at a sequence of times, and they discuss the streamwise lengthening of low-speed regions that frequently occurs. They find these occurrences are associated with streamwise merging of streaks, and various scenarios are discussed by Tomkins & Adrian (2003). It has been recognized that long meandering streaks of streamwise velocity fluctuation organize such that streaks are flanked by streaks of opposite sign in the spanwise direction (Hutchins & Marusic, 2007a; Tomkins & Adrian, 2003), and therefore observations of the lengthening of negative $u'$ streaks may be relevant to positive $u'$ streaks also.

Another interesting feature that persists over a long time is the diamond-shaped region of low-speed streaks surrounding a high-speed streak marked by a rectangle in figures 4.13(c) to 4.13(f). The downstream extents of the rectangles containing this feature are connected by a dashed black line, following Lee & Sung (2011). The slope of this line suggests a convection distance of approximately $6.3R$ between each frame, closely matching the mean velocity. However, the other shapes drawn for the other features indicate some variability in convection distance between each frame (convection velocity will be discussed in §4.7). While the feature identified with rectangles can be visually tracked over four subsequent frames (or five frames including its embryonic stage in figure 4.13(b)), the feature undergoes significant changes in this duration of $28.8R/U_{bulk}$. If the feature from any given frame were to purely convect with no other change, the signature obtained from a single stationary point probe would be significantly different depending on which frame was chosen. Even relatively large scales sometimes change significantly when the flow
has convected by only $6–7R$, although in most cases the patterns in very long scales remain clearly recognizable between frames, and this becomes clearer when spatial filtering is applied.

Significant change as the flow progresses several radii downstream is consistent with the loss of correlation between measurements with and without Taylor’s hypothesis noted by Dennis & Nickels (2008), which they attributed to rapidly changing small scales. In their boundary layer at $y/\delta = 0.16$, they concluded visually there was strong similarity between the fields with and without Taylor’s hypothesis, but noted there were also significant differences even to large scales. The scales visually tracked in the pipe DNS are also relatively large, but the spectral analysis indicates that even longer scales that could be less visually obvious are energetically important. Several structures that were identified appear to convect at approximately the local mean velocity between neighbouring frames, but it is difficult to precisely define their centres (or other equivalent points) between frames due to the changes in structure.

To examine the very large-scale motions of interest, spatial filtering is applied to the velocity fields. Lee & Sung (2011) visualized large scales in a boundary layer field by applying a Gaussian filter. For the present pipe flow, applying Fourier filtering in the two periodic coordinates ($x$ and $\theta$) is appropriate because the structures clearly correspond to wavelengths of interest identified in the energy spectra of figure 4.6(b). The black arrow near the $y^+ = 60$ premultiplied spectrum indicates the $\lambda_\chi = 6R$ wavelength at which a longer-wavelength peak of the bimodal spectrum shape may begin to form. The sharp-cutoff Fourier filtering will retain this and longer-wavelength modes (the first five positive wavenumbers). Applying only streamwise filtering yields a confusing pattern, so azimuthal filtering is also performed to retain wavelengths of $\lambda_\theta = 0.4\pi$ (72 degrees or 760 wall units at $y^+ = 80$) and greater (also the first five positive wavenumbers in $\theta$). This az-
imuthal wavelength is relatively wide, and the portion of energetic motions retained by these wavelengths is described by the two-dimensional spectrum in figure 4.12. Zero-wavenumber modes are also retained in the filtering. The motions these wavelengths extract are clearly VLSMs because the streamwise wavelength is well within the VLSM range ($\lambda_x = 3R$ is the nominal cutoff). This filtering retains only significantly larger scales than that of Lee & Sung (2011).

The filtered fields in figure 4.14 include the same circles, ellipses, and rectangles shown in the unfiltered fields. Since much of the energy is removed when the filtering is applied, the contour levels are reduced to half of those for the unfiltered fields. The circles each have a high-speed streak in the filtered field passing through their center, but the streamwise filtering removes the small details necessary to identify the original feature. The ellipses appear to remain centered around the strong high-speed streaks in the filtered field, although merging is evident at the last two times shown. The rectangle regions still capture the diamond-like low-speed streaks surrounding the high-speed streak, although these would probably be clearer if additional azimuthal modes were retained. The significantly differing filter lengths in $x$ and azimuthal arc length $s$ complicate the result. In the filtered fields, the dashed line consistently follows the upstream sides of the low-speed streaks that are located immediately downstream of the high-speed streaks that were identified in the unfiltered fields. The filtered fields indicate that many large-scale streak features identifiable in the unfiltered fields correspond closely with long-wavelength Fourier modes, and the patterns remain visibly correlated while they convect considerable distances (though the lengths of streaks may change somewhat).

4.6 Space-Time Correlation

As a quantitative means of assessing the amount of change as the field evolves, we have computed the space-time correlation. Dennis & Nickels (2008) experimentally
computed the space-time correlation of their turbulent boundary layer flow for spatial distances up to 6δ, but DNS allows quantification of the amount of correlation over very long convection distances. For their turbulent boundary layer simulation, Lee & Sung (2011) have calculated space-time correlations and used their inclinations to determine convection velocities. Chung & McKeon (2010) calculated spatio-temporal correlations and spatio-temporal spectra (Kx–ω spectra) for large-eddy simulations (LES) of turbulent channels with emphasis on VLSMs. From the space-time correlations, they examine the degree to which the region of strong correlation is symmetric with respect to time delay and streamwise displacement, as a symmetric correlation indicates a linear relationship and consequently Taylor’s hypothesis is valid. Since the entire periodic spatial frame of the present pipe DNS evolves in time, it is appropriate to correlate the entire x-θ plane of streamwise velocity fluctuation. The space-time correlation is

$$R_{uu}(r_x, y, \tau) = \langle u(x, \theta, y, t)u(x + r_x, \theta, y, t + \tau) \rangle,$$  \hspace{1cm} (4.9)

where the averaging \(\langle \cdot \rangle\) includes averaging over \(x, \theta,\) and time \(t.\)

We examine the space-time correlation at two wall-normal locations: \(y^+ = 50\) and 101. The contours in figure 4.15(a) focus on space-time correlation values that correspond to \(R_{uu}(r_x, y, \tau)/\left\langle u^2(x, \theta, y, t)\right\rangle\) (coefficient) values of 0.25 and less. These low fractions of the correlation maxima at \((r_x = 0, \tau = 0)\) persist for long times. In the present correlation for unfiltered \(u\) at \(y^+ = 101,\) the lowest two contour levels correspond to correlation coefficient values of 0.10 and 0.05, and these persist for times of \(21R/U_{\text{bulk}}\) and \(42R/U_{\text{bulk}},\) or \(\tau^+ = 810\) and 1590, respectively. In the \(Re_\tau = 2000\) LES of Chung & McKeon (2010), at \(y^+ = 98,\) a correlation coefficient of 0.14 persists for \(\tau^+ \approx 860\) and a coefficient of 0.072 persists for \(\tau^+ \approx 1270\) (when converted to viscous time units in which \(u_\tau\) is the only velocity scale). The initially rapid decay in the correlation that slows as time delay increases is observed for the
Figure 4.15: Space-time correlation for the present pipe simulation. (a) \( R_{uu}(\tau, r_x)/u_\tau^2 \) contour lines for \( y^+ = 50 \) (translated upward by 10) and \( y^+ = 101 \) at contour values of \{0.16, 0.32, 0.48, 0.64, 0.80\}. Correlations at the same levels calculated between filtered versions of the same fields are also included (dashed lines), using the same filter as for figure 4.14. The shallower long-dashed grey lines originate at \( (\tau = 0, r_x = 0) \) and have slope equal to \( \bar{u}(y)/U_{\text{bulk}} \). The steeper long-dashed grey lines have slope equal to unity and are positioned to pass through the major axes of the low-magnitude contour lines. (b) The unfiltered (solid line) and filtered (dashed line) correlations for \( y^+ = 101 \) at \( \tau = \{0, 7.2, 14.4, 21.6, 28.8, 36, 43.2, 50.4\}R/U_{\text{bulk}} \) (peaks occur at correspondingly increasing \( r_x/R \) values). The maximum for the \( \tau = 0 \) correlation is indicated by a circle.

correlation plotted as a function of \( r_x \) for a sequence of times in figure 4.15(b). The curves are spaced by the same time interval as that separating each of the contour plots in figure 4.13. Such rapid initial decay is consistent with Dennis & Nickels (2008), in which the correlation between velocity evolving in time and velocity estimated with Taylor’s hypothesis decays by nearly 40% for a convection distance of 0.05\( \delta \).

The long times associated with the lower correlation values suggest that the persisting correlation is associated with the large scales of motion, such as those seen in the filtered fields of figure 4.14 that retain their overall configurations for
substantial durations. Dennis & Nickels (2008) found that a larger fraction of the initial correlation was retained at long time delays when the fields were replaced by filtered versions. To study the VLSM scales for the present pipe flow, the same filter that was used for the visualized fields (retaining only scales no shorter than \( \lambda_x = 6R \) and \( \lambda_\theta = 0.4\pi \)) was applied before calculating the correlation. Figure 4.15(a) shows that the correlation contours for long time delays \( \tau \) (and also long spatial shifts \( r_x \)) approach those of the filtered version. Figure 4.15(b) confirms this for \( y^+ = 101 \) and shows that much of the correlation is removed for small time delays by retaining only these very large scales, but the correlation is virtually identical with and without filtering for long time separations.

For the present pipe at \( y^+ = 50 \), the correlation decays more rapidly from the peak (along the ridge of the space-time correlation) than for \( y^+ = 101 \). It has been suggested that very large-scale motions convecting at approximately the bulk velocity have footprints extending near the wall (further discussed in §4.7). If these are present along with smaller-scale motions convecting at velocities nearer the flow’s local mean, there is reduced correlation between the original field and the field at later times due to the contributions convecting at different velocities while determining a single spatial shift \( r_x \) that maximizes the correlation. The lower local mean at \( y^+ = 50 \) increases this effect, consistent with the faster decay. When the correlation is scaled by a value independent of \( y \) (such as \( u_2^2 \) or \( U_{\text{bulk}}^2 \)), the correlations decay to similar \( R_{uu}(r_x,y,\tau)/u_2^2 \) values for long time delays (figure 4.15(a)). This occurs despite the stronger peak at \( (r_x = 0, \tau = 0) \) for \( y^+ = 50 \) than for 101 (4.26 vs. 3.19).

The topic of convection velocity of the motions is further discussed in §4.7, but the inclination of the space-time correlation is one method of calculating convection velocity of flow quantity fluctuations (Choi & Moin, 1990; Kim & Hussain, 1993; Chung & McKeon, 2010; Lee & Sung, 2011). Kim & Hussain (1993) defines
propagation velocity based on this concept to be

\[ c_d(y) = \frac{r_{x,\text{max}}}{\tau}, \quad (4.10) \]

where \( r_{x,\text{max}} \) is the axial (streamwise) displacement that maximizes the correlation \( R_{uu}(r_x, y, \tau) \) for a given time delay \( \tau \). The \( r_{x,\text{max}} \) values are also the displacements for the maxima of each curve in figure 4.15(b). To compare convection velocities, lines with slopes corresponding to convection at the bulk velocity and local mean velocity are included on the contour plots. The lines for the local mean are placed at the origin, and the major axes of the contour ellipses for higher levels appear to closely match this. However, the axes of the contours become steeper at later times, and the contours match the slope corresponding to convection at the bulk velocity.

This supports the conclusion that motions of small to moderate scale (up to LSMs) account for much of the correlation at shorter times and convect at approximately the mean, but they decay so that the influence of the very large-scale motions dominates the correlations at long times. The associated convection distances are large, with correlation coefficient values of 0.05 maintained at \( y^+ = 101 \) for convection distances up to \( 40R \) at approximately the bulk velocity. Considering that the periodic pipe domain length is \( 30R \), this means that the correlation persists past a complete washout (based on bulk velocity). Since the space-time correlation involves an average over the entire periodic domain that conceptually repeats beyond this length, the long distance exceeding the periodic length does not directly pose any issue. However, it is not certain how the periodic boundary conditions that are imposed in \( x \) and the corresponding longest motions that the domain can accommodate could affect the behavior of correlations at long time (and correspondingly long spatial) separations. While the spatial wavelengths are limited to \( 30R \), the longer-scale motions could appear in the time variation of the axial mean in the simulation, although the periodic domain length effects could introduce some error.
Accompanying the observed faster decay of correlation coefficient for $y^+ = 50$ compared to $y^+ = 101$, the convection velocity corresponding to maximum correlation (4.10) also departs from the local mean more quickly. In figure 4.15(a), high space-time correlation value contours appear as concentric ellipses with the same inclination angle (Zhao & He, 2009), and the inclination angles for these examples closely correspond to the local mean velocities. For lower levels, the inclination angles approach the angle corresponding to approximately the bulk velocity. This occurs as the smaller-scale motions presumably lose correlation and the faster-traveling larger motions dominate the correlation. The change in inclination from local mean velocity towards higher velocity as the space-time correlation levels decrease is clearly seen in the logarithmic layer region of the higher Reynolds number channel flow of Chung & McKeon (2010), increasing in strength as the wall is approached, whereas they convect at a velocity relatively close to the local mean further above the wall (as shown in their figure 3). In channel simulations at lower Reynolds numbers, Kim & Hussain (1993) noted that the propagation velocity calculated using this definition based on space-time correlations of velocity remained constant for a range of relatively small $\tau$ values, except for the smallest motions for which yet smaller $\tau$ values were used. For such channel simulations, Choi & Moin (1990) calculated space-time correlations for pressure that also curve to higher convection velocities at long time delays. At locations of more intense shear yet closer to the wall than $y^+ = 50$, it appears that the convection velocity may exceed the local mean even for relatively small time delays (or, equivalently, high contour values), as observed by Lee & Sung (2011) for a turbulent boundary layer at $y^+ = 30$.

In summary, the contours indicate that the present pipe flow remains measurably correlated (0.05 coefficient, for example) for long times that correspond to long convection distances, but the significant decay of the correlation for such time delays is
attributable to both the differences in convection velocities for different scales (del Álamo & Jiménez, 2009) and the small motions of the flow fields changing significantly, thereby altering the flow relative to pure convection of the original motions (as Taylor’s hypothesis would assume and was previously considered in reference to figure 4.13). The very long motions traveling faster than the local mean are what have been shown by del Álamo & Jiménez (2009) to shift energy from longer wavelength to overemphasize the longer wavelength peak when Taylor’s hypothesis is used with the assumption that all motions convect at the local mean. This effect becomes more pronounced for the larger differences between the local mean and bulk (or centreline) velocity that occur for higher Reynolds number flows. This effect is detected in the comparison between our spectra and experiments (§4.4.2) and would be strengthened for higher Reynolds numbers.

4.7 Convection Velocity

The streamwise convection velocity of motions may be calculated by several different methods besides the inclination of the space-time correlation previously considered. A variety of methods are considered by del Álamo & Jiménez (2009). The frequency-wavenumber ($\omega$–$k_x$) spectrum, which is the Fourier transform (both temporally and spatially) of the time-space correlation, allows convection velocities to be determined for individual wavelengths (or sets). Besides this method, the convection velocity of an individual (spatial) Fourier mode $\hat{u}(k_x, k_\theta, y, t)$ expressed in terms of magnitude and phase as $|\hat{u}(k_x, k_\theta, y, t)|\exp[i\phi(k_x, k_\theta, y, t)]$ may instantaneously be determined from $c = -\partial_t\phi/k_x$ assuming the modes (which vary sinusoidally in both the streamwise and azimuthal directions) propagate in only the axial direction as $u(x - ct)$. In general, the velocity of each mode varies in time as the flow evolves, so averaging is necessary to obtain a stable velocity. We use a time average to calculate the required averages (denoted by $\langle \cdot \rangle$). del Álamo & Jiménez
(2009) define the average phase velocity of an individual mode as

\[ c_u(k_x, k_\theta, y) = -\frac{\langle \hat{u}^* \partial_t \phi \rangle}{k_x \langle \hat{u}^* \rangle}. \]  

(4.11)

This is exact for a monochromatic frozen wave, and the convection velocity computed by this definition minimizes the difference between the actual time evolution of \( u(x, t) \) and a frozen wave \( u(x - ct) \) (del Álamo & Jiménez, 2009). By seeking the convection velocity of frozen waves that minimizes the difference from the actual flow, del Álamo & Jiménez (2009) find that the overall convection velocity for a set of wavenumbers \( \Omega \) can be written in terms of the convection velocities of individual modes (defined in (4.11)) as

\[ C_u(y) = \frac{\int_\Omega c_u(k_x, k_\theta, y) |\hat{u}(k_x, k_\theta, y)|^2 k_x^2 \, dk_x \, dk_\theta}{\int_\Omega |\hat{u}(k_x, k_\theta, y)|^2 k_x^2 \, dk_x \, dk_\theta}. \]  

(4.12)

This definition based on minimizing error for the instantaneous total derivative results in strongly weighting (by \( k^2 \)) the high-wavenumber contributions.

For the present pipe flow, we focus on the set of VLSM modes of streamwise velocity that were retained for the filtered field (figure 4.14): \( n_x = 1–5 \) and \( n_\theta = 1–5 \), the first five modes of positive wavenumber in each of the homogeneous directions. As previously noted, the corresponding streamwise wavelengths of \( 6R \) and longer are well within the VLSM range, and the wide azimuthal modes are best for measuring the axial phase because the circumference becomes small as radius decreases, leading to uncertainty in the azimuthal phase that is assumed constant. Additional time measurements would improve the convergence of the convection velocity calculation, and the smoothness of the results is improved by omitting contributions at times when a mode contains very little energy (this affects the results by 3% or less). The computed streamwise convection velocities for this VLSM mode set are plotted (thick black line) for a range of \( y \) along with the mean velocity profile of the pipe in figure 4.16.
Figure 4.16: Comparison of axial convection velocities ($u_c$) calculated for the present pipe DNS. The thick solid line represents the velocity calculated by the definition of del Álamo & Jiménez (2009) in (4.12) for the set of $u$ modes with the five longest streamwise and five widest azimuthal wavelengths. The convection velocity is also computed from the shift corresponding to maximum space-time correlation for time delays of 7.2 (circles) and 14.4 (triangles) $R/U_{\text{bulk}}$. These are computed for $R_{uu}$ calculated from velocity fields that are unfiltered (open symbols with dashed line) and filtered with the same filter as used for the instantaneous contours and space-time correlation in figure 4.15 (filled symbols with dash-dot line). The mean axial velocity is also shown (dotted line).

The behaviour appears consistent with convection velocities reported by del Álamo & Jiménez (2009) for turbulent channel DNS. Their study included convection velocities of various Fourier spatial modes of streamwise velocity fluctuation, and they found that modes of long streamwise wavelength near the wall propagated significantly faster than the local mean velocity. With the convection velocity definition from which (4.12) was written for the present pipe flow, they calculated convection velocity for a set of modes with $\lambda_x, \lambda_z \geq 0.25h$. The average convection velocity in the near-wall region of a $Re_\tau = 550$ channel flow was also calculated to be faster than the local mean in the near-wall region, but not to the extent displayed for the present pipe simulation. Probably, this is principally due to the differing sets of wavelengths for the modes that are included in each convection velocity calculation.
del Álamo & Jiménez (2009) calculate the convection velocity in their channel based on their definition for the set of modes with $\lambda_x \geq 2h$ and $\lambda_z \geq 0.4h$ and the set of modes with $\lambda_x < 2h$ and $\lambda_z < 0.4h$. The convection velocity of the longer mode set is much greater than both that of the smaller mode set and the local mean velocity particularly when near the wall, but the longer modes are slightly slower (and also slower than the mean) when above the log layer. Near the wall, the convection velocity of our set of VLSM modes for the pipe is a significantly larger fraction of the bulk velocity than the convection velocity of the $\lambda_x \geq 2h$ and $\lambda_z \geq 0.4h$ set of channel modes in del Álamo & Jiménez (2009). Since their modes include somewhat shorter streamwise wavelengths than our set, this suggests that the fastest convection velocities occur for the longest modes. In (4.12), their definition strongly weights the convection velocity of a mode set by the short-wavelength contributions. Therefore, the considerably higher convection velocities near the wall for our mode set that includes only very long motions is not unexpected.

The increase in convection velocity of $u$ modes with increasing streamwise wavenumber is further supported by velocities calculated from $\omega-k_x$ spectra for the same flow of del Álamo & Jiménez (2009). For sets of modes with $\lambda_z \geq 0.25h$, the convection velocity near the wall increases with increasing streamwise wavelength for long wavelengths, although there is some scatter between different definitions (in their figure 2). Any trend to progressively greater convection velocities for the longest wavelengths (greater than $6R$) is difficult to discern from spatio-temporal ($k_x-\omega$ or $\lambda_x-\lambda_t$) spectra, such as those of Chung & McKeon (2010), in an LES simulation of a very long channel at $Re_\tau = 2000$. However, they clearly find that near the wall, the motions of long streamwise wavelength propagate significantly faster than the mean velocity. Using propagation velocities calculated by (4.10) for bands of streamwise wavenumbers in a $Re_\tau = 180$ channel, Kim & Hussain (1993) observed propagation velocity increasing with wavelength of the bands. For
the present pipe simulation, at $y < 0.15R$, the overall trend is that the weighted average convection velocity over bands of azimuthal wavenumber index $n_\theta = 1$ to 5 decreases with increasing axial wavenumber index $n_x$ values (decreasing wavelength). The $n_x = 1$ mode set convects fastest, and the 2 and 3 mode sets convect at nearly identical (but slower than 1) rates, followed by further decreasing velocities. Since $n_x = 1$ is the fundamental wavelength, it is not clear if the behavior of this mode could be affected by the periodic domain length of the simulation.

LeHew et al. (2011) calculated convection velocities for experiments at $y^+ = 34$ in a boundary layer by finding a line of local maxima in the $k_x-\omega$ wavenumber-frequency spectrum for streamwise velocity, and measured a different trend from the present results. In their figure 9(b), the convection velocity generally increased with increasing streamwise wavenumber (decreasing wavelength) while remaining always slightly above the local mean velocity, but this includes all spanwise scales and is nearer the wall with relatively low Reynolds number. They also include data from a boundary layer experiment of Krogstad et al. (1998) in which the convection velocity decreases with increasing wavenumber (decreasing wavelength), as is observed with the present simulation, but the experiment only includes relatively short wavelengths. McConachie (1981), in an experimental study of $k_x-\omega$ spectra in an $R^+ = 2600$ turbulent pipe at $y^+ = 70$, found that convection velocity monotonically decreased with increasing wavenumber for all scales included.

LeHew et al. (2011) also computed convection velocities of individual streamwise velocity modes decomposed in both spanwise and streamwise spatial directions using a different definition of convection velocity in which an average is computed over the temporal frequencies of a frequency-wavenumber spectrum. They found from the same experiment at $y^+ = 34$ that modes of long streamwise and wide spanwise extent were convected fastest when streamwise wavelength was between 2 and 5 boundary layer thicknesses $\delta$. Their results indicated that con-
vection velocity decreased even below the local mean as streamwise wavelength further increased for a given spanwise wavelength. In the present pipe DNS, the limited averaging time leads to uncertainty in comparing individual modes. The trend among modes with long streamwise wavelength is that the convection velocity decreases with decreasing azimuthal wavelength for the widest few azimuthal modes after the fundamental \(n_\theta = 1\) mode. The fundamental azimuthal mode has generally slower convection velocity. For a turbulent channel simulation, Jiménez et al. (2004) computed the convection velocity of \(\nabla^2 v\) among modes within a band of wide \(\lambda^+ = 100–500\) spanwise wavelengths, and found that convection velocity increased with increasing streamwise wavelength, particularly for the longest wavelengths (their figure 7c). In a turbulent channel at \(y^+ = 15\), del Álamo & Jiménez (2009) found that the convection velocity of \(u\) modes increases with increasing \(\lambda_x\) for a constant \(\lambda_z\) among the modes with long streamwise (> 2h) and spanwise wavelengths.

The velocities calculated by (4.10) for the spatial shift that maximizes the space-time correlation for a given time delay are also included in figure 4.16. They are included for time delays of \(7.2R/U_{\text{bulk}}\) and \(14.4R/U_{\text{bulk}}\), which are quite long and for which we found the correlations to mainly consist of contributions from very long motions. For this reason, velocities calculated in this manner are also included for space-time correlations calculated from filtered \(u\) fields (as in §4.6) to evaluate the difference. The filter reduces the variation in the convection velocity with \(y\), and the effect is small after the \(7.2R/U_{\text{bulk}}\) time delay. For longer time delays than \(14.4R/U_{\text{bulk}}\), the convection velocity becomes yet more uniform in \(y\). Calculating velocities using these time delays is different from the typical goal of minimizing the time delay (Kim & Hussain, 1993), which approximates the instantaneous convection velocity. The instantaneous convection velocity is also what (4.11) seeks. However, we have restricted our calculations with (4.11) to a family of very large-

129
scale motions that convect at relatively similar velocities and are expected to retain their overall structure for a long time, so longer time delays are acceptable for calculating phase velocity. We have similarly restricted the filtered calculation with (4.10) to the same wavelengths as when using (4.11), with the slight difference that $k_\theta = 0$ modes are only included for the former. The differing definitions affect the velocities determined, but using the same sets of modes makes the results comparable, and figure 4.16 indicates that the convection velocities agree relatively closely for both definitions.

The relatively constant (in $y$) convection velocities noted are suggestive of motions that remain spatially coherent in $y$. The evidence from the space-time correlation also suggests that they have large spatial scales in $x$ and $\theta$, and that they persist for long times. Large motions extending near the wall have been noted by researchers, as summarized in del Álamo & Jiménez (2009), including del Álamo & Jiménez (2003) and del Álamo et al. (2004) for turbulent channel simulations. They have also been observed as ‘footprints’ extending near the wall that are strongly correlated with motions further from the wall (Hutchins & Marusic, 2007a; Mathis et al., 2009a). Although the structural forms of these motions in the present pipe flow are beyond the scope of this investigation, one structural proposal consistent with the evidence is that of a hierarchy of layers, as with the hairpin packet paradigm (Adrian, 2007; Marusic & Adrian, 2012), in which structures extend upward from the wall to various height scales and influence everything below their upper extents. Such structures may convect at the mean velocity at their upper (furthest from the wall) extent, as supported by Adrian et al. (2000b) finding that hairpin head cross-sections above relatively uniform regions of low-momentum fluid on average convect at the local mean velocity of the heads. Lee & Sung (2011) support this finding for turbulent boundary layer flows in their figure 11(b). Organized groups of hairpins and the associated very long coherent motions would
then propagate faster than the local mean near the wall. Using proper orthogonal
decomposition of streamwise/wall-normal planes of a turbulent channel flow, Liu
et al. (2001) found evidence of long structures far above the wall that affect the fluid
near the wall and presumably would travel faster than the mean velocity near the
wall. The relatively constant velocity with respect to \( y \) of these motions observed
for the present pipe flow is near the bulk velocity. This suggests that the motions
propagate slightly slower than the velocity at the centreline, and figure 4.16 con-
firms this. This was also observed in the channel LES of Chung & McKeon (2010)
near the centreline.

4.8 Net Force Spectra

Net force spectra were introduced by Guala et al. (2006), and their physical mean-
ing in wall-bounded shear flow was further discussed by Balakumar & Adrian
(2007). For a turbulent pipe flow, the mean flow profile is described by the
Reynolds-averaged axial momentum equation in cylindrical coordinates assuming
only the axial velocity component has a nonzero mean:

\[
\frac{\partial U}{\partial t} = \frac{-1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial U}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( ru'v' \right),
\]

(4.13)

where \( \frac{\partial U}{\partial t} \) represents the total derivative of Reynolds-averaged axial velocity
\( U \); \( u' \) and \( v' \) are velocity fluctuations in the \( x \) axial and \( -r \) wall-normal directions,
respectively; and \( \frac{\partial P}{\partial x} \) is the mean pressure gradient. For analogy to channel
flows, it is convenient to cast the equation as a function of wall-normal coordinate
\( y \):

\[
\frac{\partial U}{\partial t} = \frac{-1}{\rho} \frac{\partial P}{\partial x} + \left( \nu \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial U}{\partial y} \right) + \left( \frac{\partial -u'v'}{\partial y} + \frac{u'v'}{R-y} \right).
\]

(4.14)

The terms are also split to be in a form similar to the corresponding equation for
turbulent channels, which is included in Balakumar & Adrian (2007). The second
term included in each parenthesis does not occur in Cartesian coordinates, and these
are the only differences from the equation for channels. The terms involving the Reynolds shear stress $\overline{u'v'}$ may be Fourier decomposed to evaluate the scales that contribute to the net forces affecting the mean velocity profile.

Guala et al. (2006) studied the $\frac{\partial(\overline{-u'v'})}{\partial y}$ term when examining net force spectra for turbulent pipe flows, and this is the only Reynolds stress term for channels (Balakumar & Adrian, 2007). It is decomposed as the integral of the co-spectrum over all wavenumbers:

$$\frac{\partial(-u'v')}{\partial y} = \int_{0}^{\infty} \frac{\partial(-\Phi_{uv})}{\partial y} dk_x.$$  \hspace{1cm} (4.15)

Premultiplication by $k_x$ allows this to be presented on a logarithmic-linear plot as a function of $k_x$ with the area under the curve to be interpreted as net force, as energy is interpreted from premultiplied spectra.

The net force spectra presented in Guala et al. (2006) and Balakumar & Adrian (2007) were obtained via thermal anemometry. Overall, the net force (4.15) is positive below the location of peak Reynolds stress and negative for $y$ above this location, where its net effect is a retardation in the mean flow. All of the hot-film measurements were obtained above the peak, and the net force spectra showed that when the contributions to this term are decomposed, these generally negative terms may include positive contributions associated with long wavelengths. Such contributions are observed at wavelengths greater than $6R$ relatively near the wall (such as $y/R = 0.15$) and at sufficiently high Reynolds number (they were not observed at any of the heights measured for the $Re_\tau = 531$ channel of Balakumar & Adrian (2007)). The observation that spectra obtained using Taylor’s hypothesis (and assuming all motions convect at the local mean velocity) overemphasize energy in the long-wavelength peak region raises questions of how these effects might impact on the sign changing behaviour of the force spectra observed at long wavelengths for the thermal anemometry measurements. The DNS also furnishes finely spaced
data in $y$ to compute accurate derivatives (with the resolutions described in §4.2), whereas the experimental accuracy of the derivatives is limited. Near the wall, experimental net force spectrum measurements are not available for comparison, as it is difficult to experimentally measure the Reynolds shear stress (the nearest measurement of Guala et al. (2006) was $y/R = 0.15$).

The net force spectrum maps presented in figure 4.17 are shown with $y$ scaled both linearly (a) and logarithmically (b). The linear axis highlights the behaviour above the near-wall region and emphasizes the fractions of the pipe radius for which various contributions dominate. Line plots are included to the right of (a) for comparison to the aforementioned experimental results. We plot the sum of both net Reynolds force terms (whereas Guala et al. (2006) focused on just the first), but it is found that the first term dominates the features visible in the plots.

The contours for the present DNS indicate that the net Reynolds force contains a strong region of positive $-k_x(\partial \Phi_{uv}/\partial y)R/u_r^2$ in the buffer layer, identified by A, and this is expected from the rapid increase of $-u'v'$ (figure 4.3) in this region. The colour contour levels are chosen to clearly identify the regions of various sign, but it should be noted that these levels result in significant contour saturation for some regions. While contour levels range from $-1$ to 1, the values of the present DNS for $-k_x(\partial \Phi_{uv}/\partial y)R/u_r^2$ range from $-1.1$ to $15.6$, with the large positive values concentrated in the region below $y^+ = 20$, and values for the spectrum of both net Reynolds force terms range from $-1.8$ to $15.5$. Below the top of the buffer layer at $y^+ = 30$, indicated by a dashed black line in figure 4.17(a), all scales except the very smallest ($\lambda_x < 0.15R$) accelerate the mean flow. No experimental net force spectrum measurements have been taken sufficiently close to the wall to examine this strongly accelerating regime.

The motions (termed ‘main turbulent motions’ by Balakumar & Adrian (2007)) with wavelengths shorter than about $0.5R$ (with this classification presumably de-
Figure 4.17: Colour contours of the sum of both net Reynolds force spectrum terms $k_x R \left[ -\partial \Phi_{uv}(\lambda_x, y)/\partial y + \Phi_{uv}(\lambda_x, y)/(R - y) \right] / u_r^2$ for the present DNS. (a) includes coloured arrows at $y$ values for similarly coloured lines in the plot to the right: red: $y/R = 0.07$ ($y^+ = 49$); black: 0.15 (101); blue: 0.30 (204). Vertical dashed black lines on the map indicate $y^+ = 30$, $y^+ = 47$ (peak Reynolds stress), and $y/R = 0.2$; horizontal lines are $\lambda_x = 3R$ and $0.5R$. The upper line plot is a net force balance, in which the solid grey line is $-(1/\rho) \partial P/\partial x$, solid light blue is $\partial(-u'v')/\partial y$, dash-double-dot green is $u'v'/(R - y)$, dashed magenta is $\nu \partial^2 U/\partial y^2$, dash-dot orange is $-\partial U/\partial y$. They sum to zero. The dotted light blue line also depicts the component of $\partial(-u'v')/\partial y$ with $\lambda_x \geq 6R$. (b) contains $uv$ spectrum contour lines (as in figure 4.9b) overlaid for $-k_x \Phi_{uv}(\lambda_x, y)/u_r^2$ of 0 : 0.033 : 0.267.
ependent on Reynolds number) accelerate the mean flow from the wall up to the level of zero net force (second dashed line). This region is identified as B in the contour maps. An exception is a very weak decelerating contribution from the shortest wavelengths. The large-scale motions, with their bounding wavelengths of $0.5R$ and $3R$ indicated by horizontal dashed lines, accelerate the mean flow below $y^+ = 30$, but decelerate the mean flow above this location. (The short-wavelength boundary of LSMs was suggested to be nominally $0.1\pi$ times the outer length scale by Balakumar & Adrian (2007), but the distinction was not sharp.) The region of peak negative net force for LSMs above the buffer layer is identified by C and significantly weakens above $y/R = 0.1$.

The positive net force continues significantly above the buffer layer for wavelengths somewhat longer than the $3R$ nominal VLSM boundary. This region is identified by D and persists to approximately $y = 0.17R$ (or $0.25R$ for only the first term), slightly above the nominal top of the logarithmic layer, but becomes weak. The $y = 0.25R$ position also corresponds to the approximate location at which the axial convection velocities of the large modes with $\lambda_x \geq 6R$ match the local mean velocity (figure 4.16), so it appears these very long motions that travel faster than the local mean accelerate the mean flow (i.e., are associated with a positive net force). Their contributions are particularly strong near the wall where the difference in convection velocity between the longest motions and the local mean is very large.

The $y^+ = 49$ force spectrum line provides an example of the behaviour very near the peak Reynolds stress point. This position is remarkable in that the net force includes a positive peak for the shortest wavelengths, a negative region corresponding to LSM and VLSM wavelengths, and a very weak positive region at the longest wavelengths, i.e., this line passes through regions B, C, and D. The peak $-\overline{u'v'}$ Reynolds stress occurs where the net Reynolds force $\partial(-\overline{u'v'}/\partial y)$ term of
(4.14) is zero, as shown by the solid light blue line in the net force balance line plot of figure 4.17(a). This is equal to the spectrum of the first net Reynolds force term integrated over all wavelengths (4.15), so the peak Reynolds stress is therefore associated with the positive net Reynolds force contributions of all scales shorter than approximately 0.55R. These create an accelerating net force that exactly balances the decelerating net force supported by the scales longer than 0.55R, i.e. the LSMs and VLSMs, except for the very longest motions (longer than 7.5R).

At the higher location of \( y^+ = 101 \), available experimental measurements are generally consistent with the present simulation. Here, the wavelengths shorter than 0.5R have switched to decelerating the flow, with a peak region of this behaviour identified as E. Thus, all wavelengths decelerate the flow except for the very longest (associated with D). This, as illustrated by the \( y^+ = 101 \) line (black), matches the behaviour of the experiments in the logarithmic layer (such as at \( y/R = 0.15 \)), with negative values for high wavenumbers and a positive region for \( \lambda_x/R > 6 \). The positive region is weak for the DNS, and this is probably related to Reynolds number, as Balakumar & Adrian (2007) find no positive peak for their lowest Reynolds number (though it could exist at a lower \( y \) value than was measured). The spectrum contribution of the second net Reynolds force term also diminishes this, and the somewhat larger contributions from the first term alone are plotted in figure 4.18. There is also a very weak positive peak at the very highest wavenumbers that is not observed in experimental spectra and is not visually significant on the line plot for the present DNS.

The \( uv \) spectrum superimposed on the force spectrum with logarithmically scaled \( y \) shown in figure 4.17(b) sheds further light on the features observed. Since the first term of the net force spectrum is the partial derivative of the \( uv \) spectrum with respect to \( y \), the regions dividing the positive and negative net force contributions are approximately where the \( uv \) spectrum remains constant with \( y \) (and the
\(uv\) spectrum contour lines are parallel to the horizontal axis representing \(y\). The strongly positive net force near the wall is associated with the increase in the \(uv\) spectrum with \(y\) that is relatively uniform with respect to wavelength, but an indentation occurs for \(\lambda_x \approx 0.2 R\) and \(y^+ \approx 20\). As the spectrum in this region approaches a shape more similar to concentric ellipses, the associated force peaks to create the region labeled as B. The \(uv\) contributions peak in magnitude around \(y^+ = 30\) for LSM wavelengths, and the relatively rapid decline in magnitude with increasing \(y\) is associated with net force peak C. The protrusion in the \(uv\) spectrum for long wavelength that increases with \(y\) until \(y/R\) approaches 0.2 is associated with the positive net force region B. Above the \(y\) value at which peak \(-u'v'\) occurs for each wavelength, the net force is negative as \(y\) increases. The negative net Reynolds force is particularly strong at C because it opposes the significant positive net Reynolds force of B (and a smaller contribution from D) in this region near the overall \(-u'v'\) peak. E is offset to higher \(y\) and supports much of the negative net Reynolds force, whereas it is more uniformly distributed over a range of wavelengths for higher \(y\). Thus, subtle features of the \(uv\) spectrum become more important when the net force spectrum is examined.

In the higher \(y\) regime previously discussed in which the \(uv\) spectrum magnitude decays with increasing \(y\) for every wavelength, the behaviour matches observations for experimental spectra. In figure 4.17(a), the \(y^+ = 204\) line (blue) is typical of the experimental net force spectra obtained above the logarithmic layer (nominally indicated by the horizontal dashed line at 0.2) with the net force spectrum negative for all wavenumbers. The spectrum map confirms that this behaviour is consistent throughout the region sufficiently far from the wall.

The additional term \((u'v'/(R-y))\) relating to Reynolds stresses in the Reynolds-averaged axial momentum equation in cylindrical coordinates (4.14) was included in the net Reynolds force plots, but the observations above can be made from the
first term alone. The role of the second-term contributions is now considered. When far enough from the wall that viscous effects involving the mean axial velocity gradient are negligible, the Reynolds stress is proportional to radius: \(-\overline{u'v'} = u_2^2(r/R)\). Since this net force term is the Reynolds stress divided by the radius, the magnitude of this term is constant except in the near-wall region. When decomposed by streamwise wavelength, the distribution by wavelength remains approximately constant as \(y\) varies for \(y\) near the log-layer region and above. In this region (with values taken at \(y/R = 0.5\)), the peak at \(\lambda_x = 1.5R\) decays to half magnitude at \(\lambda_x = 0.4R\) and \(6R\). Summing the Fourier-decomposed Reynolds net force terms yields behaviour that remains qualitatively very similar to only the first term (figure 4.17), and the peaks identified for just the first term remain. The difference most clearly visible is that the negative force region between the peak Reynolds stress location and the pipe centreline is somewhat stronger. The magnitude of the overall contribution from the second term can be assessed by integrating the net force map contributions over all wavelengths, which simply recovers the terms of the net force balance. These are plotted at the top of figure 4.17(a) for \(y/R\) between 0 and 0.2, above which these terms continue their asymptotic behaviour. The solid light blue line is the first \((\partial(-\overline{u'v'})/\partial y)\) net Reynolds force term, and the dash-dot-dot green line is the second \((\overline{u'v'}/(R - y))\). The second term is zero near the wall and approaches a negative asymptotic value within \(y^+ = 25\), while the first term becomes very large and positive, then becomes negative at the peak Reynolds stress point and asymptotes to the same value as the second term. Thus, the first term dominates the near-wall and buffer region, while both terms contribute equal amounts of net force away from the wall, but the wavelength distribution of the second term’s contributions remains fairly consistent.

The remaining terms of the net force balance (4.14) also asymptote for large \(y\). The first viscous term \(\nu \partial^2 U/\partial y^2\) (dashed magenta lines) becomes very strongly
negative near the wall and largely balances the strongly positive first net Reynolds stress term in this region. The magnitudes of these terms are much larger than these or any other terms when away from this region, with peak magnitudes of approximately $45u^2_\tau/R$. For this reason, the vertical scale of the net force balance plot in figure 4.17 truncates the maximum scale to clearly display the terms when not in the near-wall region. The second viscous term $-\left[\nu/(R-y)\right]\partial U/\partial y$ (dash-dot orange lines), which exists due to the cylindrical coordinates, is nearly zero except when $y^+ < 20$, and there it is much smaller than the other viscous term. While the balance between the first Reynolds force term and the first viscous force term dominates the near-wall region, further from the wall the positive net force due to the axial pressure gradient balances the first and second viscous force terms, both of which asymptote to the same value $(-u^2_\tau/R)$.

These Reynolds force terms may be viewed as the combined effect of motions of all length scales (as decomposed in the net force spectrum map). To assess the effect of the very large scales, the contribution to the first net Reynolds force term for $\lambda_x \geq 6R$ is also included in the net force balance plot (dotted blue line). This length is longer than the nominal VLSM cutoff of $3R$, but is designed to emphasize the net accelerative force identified by Guala et al. (2006) for very long scales. Above $y^+ = 30 (y/R = 0.044)$, these contributions are relatively weak for the present pipe simulation, but the previous discussion suggests that this would strengthen with increasing Reynolds number. Some of the shorter scales included in $\lambda_x \geq 6R$ contribute negative net force for this term, which diminishes the acceleration measured for these wavelengths. At greater $y$ values than the $0.2R$ shown, this term becomes negative and as strong as $-0.32u^2_\tau/R$.

The $\partial(-u'v')/\partial y$ net Reynolds force term reveals further information when it is Fourier decomposed in both $x$ and $\theta$. Given $S_{ij}(k_x;k_\theta;y)$ as the two-dimensional
Fourier transform of \( R_{ij}(r_x; r_\theta; y) \), we define the one-sided cospectrum as

\[
\Phi_{ij}(k_x; k_\theta; y) = S_{ij}(k_x; k_\theta; y) + S_{ij}(-k_x; k_\theta; y) + S_{ij}(k_x; -k_\theta; y) + S_{ij}(-k_x; -k_\theta; y)
\]

\[
= 2\text{Re}\left\{ S_{ij}(k_x; k_\theta; y) + S_{ij}(k_x; -k_\theta; y) \right\}. \tag{4.16}
\]

Therefore, in two dimensions, the net Reynolds force term is decomposed as

\[
\frac{\partial (-u'v')}{\partial y} = \int_0^\infty \int_0^\infty \frac{\partial (-\Phi_{uv})}{\partial y} \, dk_x \, dk_\theta. \tag{4.17}
\]

The discrete form is analogous to that of the one-dimensional version in §4.4.1. \( \Phi_{ij}(k_x; k_\theta; y) \) and its \( y \) derivative are premultiplied by \( k_x k_\theta \) for plotting contours. Contours of the first term of the two-dimensional net force spectrum are presented in figure 4.18 for \( y \) positions matching the arrows in figure 4.17 as well as an additional position nearer the wall. The conventional one-dimensional \( \lambda_x \) spectra (shown in figure 4.17(a) for both net Reynolds force terms) are also included above each contour plot, and these are interpreted as the two-dimensional net force spectrum contours integrated in \( \lambda_\theta \). One-dimensional \( \lambda_\theta \) spectra are also included to the right of each contour plot, and these are the result of integration over \( \lambda_x \).

These two-dimensional net force spectra provide a simpler description of the motions that contribute to the one-dimensional \( \lambda_x \) net force spectra, which have been observed to occur in a somewhat complex set of regimes. The two-dimensional net force spectrum is the \( y \) derivative of the two dimensional \( uv \) spectrum. For reference, contour lines of the actual \( uv \) spectrum are superimposed on each net force spectrum. The net Reynolds force is a sum of both of these terms, with the \( uv \) spectrum contribution weighted to decrease as the wall is approached.

Contours of the \( uv \) spectrum for \( y^+ = 101 \) are qualitatively similar to the \( uu \) spectrum in figure 4.12, except for a shift to smaller \( \lambda_\theta \) for long \( \lambda_x (> 0.5R) \). The net force spectra for all \( y \) values are similar in that each possesses an ellipse of negative net force contribution below (i.e., at shorter azimuthal wavelength than) an ellipse
of positive net force contribution. The division between the regions is located at the azimuthal centre of the $uv$ spectrum. Since the $y$ derivative that produces the net force spectrum can be viewed as the difference between the $uv$ spectra of two neighbouring $y$ locations, this pattern is consistent with the shift to larger $\lambda_\theta$ with increasing $y$ that is apparent in the two-dimensional $uv$ spectra shown. There is also a clear but less pronounced increase in $\lambda_x$ wavelength with increasing $y$. Comparing the one-dimensional axial and azimuthal $uv$ spectra of figures 4.9(b) and 4.11(b) illustrates the differences in scale growth with $y$, and the azimuthal scale growth in
terms of $\theta$ is greater than in terms of $s$. The strengthening and weakening of $-u'v'$ with increasing $y$ also affects the net force spectrum.

Interpreting these net force spectra in terms of the various scales accelerating or decelerating the mean axial flow, for a given streamwise wavelength $\lambda_x$, the smaller azimuthal scales decelerate the flow while the larger azimuthal scales accelerate the flow. Given a hierarchy of scales as previously discussed (§4.7), this is consistent with large scales at higher $y$ (which are large also in the azimuthal sense) accelerating the mean flow profile at lower $y$. This general behaviour continues for a considerable distance above the pipe wall, although the more subtle behaviour of the net force distribution in $\lambda_x$ for a given $\lambda_\theta$ changes. Comparing the frames of figure 4.18, while $y^+ = 50$ includes the two ellipses above and below each other with similar streamwise wavelength extents, positive or negative net force regions for other locations protrude into the low-$\lambda_x$ region of the opposing net force region. The combined effect integrated over all azimuthal scales leads to the regimes of the $\lambda_x$ spectra that were observed. The changes in $\lambda_x$ spectra behaviour with $y$ arise as a result of different spanwise scales of negative and positive net force dominating. The $\lambda_\theta$ net force spectra shown to the right of each frame are more consistent in their pattern.

4.9 Conclusions

The DNS pipe flow simulation has been validated against experiments by comparisons with first- and second-order statistics, which show good agreement. Comparisons of streamwise energy spectra between the DNS and experiments using Taylor’s hypothesis reveal similar behaviour to that discussed by Monty & Chong (2009) and del Álamo & Jiménez (2009), in that long-wavelength peaks in premultiplied spectra have greater magnitude for the experiments, although the Reynolds number of the DNS limits the strength of this conclusion. The premultiplied stream-
wise energy spectrum provides evidence that a long-wavelength peak may be be-

ginning to form, most convincingly above where a logarithmic layer would be ex-

pected. The peak was observed by searching for a dip or flattened region in the

premultiplied spectrum, and the match between the wavelengths observed for the

peak at longer wavelength and the corresponding peak in spectra obtained from

experiments (figure 4.1) corroborates the possibility that the same phenomenon oc-

curs in both. Comparing spectrum maps of the simulation with experiment shows

that the y (or radial) position of the outer site associated with the long-wavelength

peak matches the location found in experiments.

This very large-scale energy behaviour is detected in premultiplied one-

dimensional spectra throughout a range of y spanning the logarithmic region, and

evidence of a rudimentary peak or shoulder region exists in a much wider range.

Several observations from the DNS characterize the range: in the buffer layer, the

shorter-wavelength peak dominates, but a weak yet distinguishable peak relative to

linear decay of the premultiplied spectrum with decreasing $k_x$ appears. By $y^+ = 30$

(figure 4.6a), this relative peak has developed into a flat region of the premultiplied

spectrum. Above the logarithmic layer, the original peak location is somewhat flat

by $y^+ \approx 130$, but a peak appears at longer wavelength. Due to the domain size, only

the four discrete values of $\lambda_x = 30R, 15R, 10R, 7.5R$ fall in the range where the long

wavelength peak occurs, and the lack of spectral resolution obscures the true wave-

lengths of the maxima. To within the spectral resolution of the DNS, the values for

the wavelength of the long-wavelength maximum agree with experiments, support-

ing the possible existence of long-wavelength peaks in DNS. While the wavelength

at the maximum of a long-wavelength peak serves to conveniently characterize the

scale of the very large-scale motions, a maximum is not necessary to see their sig-

nificance. The energy contained in the VLSM range and the visualizations of very
Table 4.3: Mean fractions of turbulent kinetic energy and Reynolds shear stress with wavelengths that are within the VLSM, LSM, and shorter ranges for the present pipe simulation. The values are computed from a linear average from the pipe axis to the wall.

<table>
<thead>
<tr>
<th>Streamwise Wavelength</th>
<th>⟨⟨uu⟩⟩</th>
<th>⟨⟨vv⟩⟩</th>
<th>⟨⟨ww⟩⟩</th>
<th>⟨⟨uv⟩⟩</th>
</tr>
</thead>
<tbody>
<tr>
<td>VLSM: λ_x &gt; 3R</td>
<td>0.44</td>
<td>0.13</td>
<td>0.13</td>
<td>0.35</td>
</tr>
<tr>
<td>LSM: 0.3R ≤ λ_x ≤ 3R</td>
<td>0.51</td>
<td>0.69</td>
<td>0.72</td>
<td>0.61</td>
</tr>
<tr>
<td>λ_x &lt; 0.3R</td>
<td>0.05</td>
<td>0.18</td>
<td>0.14</td>
<td>0.04</td>
</tr>
</tbody>
</table>

long low-speed streaks in the logarithmic layer unequivocally demonstrate the significance and existence of VLSMs.

While the importance of large- and very large-scale motions in contributing $uu$ energy and $uv$ shear stress has been accepted based on experiments using Taylor’s hypothesis, the effects of Taylor’s hypothesis require that these conclusions be re-examined. DNS results indicate that VLSMs carry large fractions of $uu$ energy and $uv$ shear stress for pipe flow at this Reynolds number, and the overall conclusion of the importance of VLSMs remains supported. The mean fractions of turbulent kinetic energy and Reynolds stress for each wavelength range are summarized in table 4.3, in which the fractions are averaged over all radii without weighting by circumference. In the region above $y^+ = 70$, wavelengths longer than $3R$ account for greater than 40% of $uu$ energy, and greater than 30% of $uv$ shear stress. These values are significantly less than the 65–70% of $uu$ energy and 50–60% of the $uv$ shear stress contributions measured by Guala et al. (2006), due, in part, to the lower Reynolds number of the DNS. Yet, there clearly remains substantial energy in the VLSMs. The spectrum maps with superimposed cumulative fraction isolines clarify how energy is distributed among different scales for all radii.

Having established the importance of the very large-scale motions in this flow simulation and their energy spectrum properties relative to experiments, this study characterized a number of other important properties of these motions, including
statistical measures of their dynamics. Space-time correlations indicate relatively rapid changes in the flow and different scale motions convecting at different velocities. Both phenomena would cause Taylor’s hypothesis with a single convection velocity to produce significant error in estimating a spatial field, even for structures barely within the VLSM range (3R long). Yet, a significant amount of energy is contributed by very large-scale motions of the flow that remain correlated for very long times and convect at approximately the bulk velocity. At $y^+ = 101$, for example, 5% correlation of streamwise velocity fluctuation remains after the bulk flow convects by 40R. The eddies smaller than LSMs propagate at velocities nearer the local mean, while the LSMs and VLSMs tend to propagate at velocities approaching the bulk velocity, even near the wall. The results illustrate the significant limitations that exist in inferring multiple-point statistics from single-point measurements, particularly for long scales.

Net force spectra have been calculated from the centreline down to the wall, and the results support the trends observed in experimental measurements of Guala et al. (2006) and Balakumar & Adrian (2007), while extending the results down to the wall. In general the net Reynolds force, given by the divergence of the Reynolds stress, is a vector field whose effects on the mean flow are much easier to understand than the Reynolds stress. The net force spectrum presents a new picture of how turbulent stresses influence the mean flow profile by accelerating the flow near the wall and decelerating the flow near the centreline. Decomposing the net Reynolds force into its spectral components makes it possible to ascertain the roles played by eddies of different scales and types. Figure 4.17 shows complicated behavior at various scales. The VLSMs accelerate the mean streamwise velocity for sufficiently low $y$ values (such as $y/R < 0.15$). The net force spectra indicate that motions with approximately VLSM wavelengths, which have been found to travel much faster than the local mean near the wall, accelerate the flow in this region, while the shorter
motions decelerate the flow. The smaller wavelengths coincide with known lengths of quasi-streamwise vortices and first-generation hairpin vortices, each producing second-quadrant events (Adrian et al., 2000b; Zhou et al., 1999). Approaching the pipe axis, the motions with VLSM wavelengths travel somewhat slower than the local mean, and the net force spectra indicate that these decelerate the flow, as do the smaller scales. The two-dimensional force spectra in figure 4.18 indicate rather simple and qualitatively similar behavior at all y-locations. For each streamwise wavelength there is a wavelength in the spanwise direction at which the net force contribution vanishes. This point always separates longer spanwise wavelengths that produce positive (accelerating) net force from shorter wavelengths that produce negative (decelerating) net force.
Chapter 5

STRUCTURE OF LARGE AND VERY-LARGE SCALES IN TURBULENT PIPE FLOW SIMULATION

The physical structures of velocity and vorticity are examined in a recent direct numerical simulation of fully developed incompressible turbulent pipe flow (Wu et al., 2012). The Reynolds number is $Re_D = 24,580$ (based on bulk velocity), and the Kármán number is $R^+ = 684.8$. The periodic domain length of 30 pipe radii $R$ is sufficient to examine long meandering motions of negative velocity fluctuation that are commonly observed in wall-bounded turbulent flows and correspond to the large fractions of energy present at very long streamwise wavelengths ($\geq 3R$). In particular, we study how long meandering motions are composed of smaller motions. The associated vortical structures are identified and compared with structural models, and we observe prominent features to match the vortex packet paradigm, although direct observations of the vortices are less convincing. We characterize the spatial arrangements of long meandering motions and find they possess dominant incline angles that are revealed by 2D and 3D two-point spatial correlations of velocity. The merging of negative streamwise velocity fluctuation structures with increasing distance from the wall is observed and compared with azimuthal scale growth statistics and observations in turbulent pipe experiments. These are interpreted with respect to the motions observed in the flow and, in particular, radially-inward ejections located near the wall.

Acknowledgments: The computer program used in this study was developed by the late Dr. Charles D. Pierce of the Center for Turbulence Research at Stanford. XW was supported by the NSERC Discovery Grant and the Canada Research Chair Program (CRC) in Aeronautical Fluid Mechanics. The calculations were performed at the High Performance Computing Virtual Laboratory (HPCVL). Ad-
ditional computations were performed using the Arizona State University High Performance Computing facilities. JRB and RJA gratefully acknowledge the support of the National Science Foundation with NSF Award CBET-0933848.

5.1 Introduction

The recent direct numerical simulation (DNS) of a turbulent pipe flow by Wu et al. (2012) has shown direct evidence, without the use of Taylor’s hypothesis, that very long scale motions (VLSMs) of wavelength greater than 3 pipe radii ($R$) contribute over 40% of the streamwise turbulent energy, and greater than 30% of the $uv$ (streamwise-wall normal) shear stress in that flow. That study focused mainly on statistical aspects and energy spectra of the pipe DNS. The purpose of the present study is to determine the structure associated with the very long-scale motions. Measurements of large fractions of streamwise turbulent energy and shear stress associated with VLSMs in pipe experiments (Kim & Adrian, 1999; Guala et al., 2006) have brought about discussion of their structural aspects. Numerical simulations of pipe flow allow the opportunity to observe very long structures at high resolution and in three dimensions without the limitations of experiments, such as the effects of using Taylor’s hypothesis, as discussed in Wu et al. (2012).

Based on premultiplied energy spectra from thermal anemometry measurements indicating two distinct peaks in turbulent pipe flows, Kim & Adrian (1999) identified the motions associated with the longer wavelength peak as VLSMs. This peak occurs from roughly the top of the buffer layer to wall-normal heights of typically $y/R \approx 0.25–0.4$ in which the peak wavelengths exceed three pipe radii $R$ and typically extend up to $14R$ (Kim & Adrian, 1999). Kim & Adrian (1999) conjectured that VLSMs were a consequence of spatial coherence in the positions of hairpin packets. Evidence of hairpin packets, coherent organizations of streamwise-aligned hairpin vortices that grew as ramps with downstream position
and were associated with uniform momentum zones of low streamwise velocity beneath them, was emerging at the time in wall-bounded shear flows based on studies of turbulent boundary layers (Adrian et al., 2000b) and channels Zhou et al. (1999). Kim & Adrian (1999) associated the largest hairpin packets with bulges commonly observed in boundary layers, which had streamwise lengths on the order of the shorter-wavelength peak observed in the spectrum. Thus, in this concept, hairpins organized into packets and packets aligned in a streamwise sense to establish VLSMs. Additional eddy types beyond packets were not needed to explain the wide range of long length scales observed in the spectra, although the mechanism by which alignment occurred was left as an open question which could involve another flow mechanism. Further evidence for the concatenated packet model appeared in Guala et al. (2006) in a smoke wire visualization photograph from the pipe flow of Lekakis (1988) that showed smoke wavering azimuthally with a streamwise wavelength somewhat greater than $2R$. Viewed from the side, two distinct regions exist containing groups of inclined structures consistent with the presence of hairpins (Adrian et al., 2000b).

The VLSMs observed in pipes appear similar to those observed in channels, which are believed to contain structural elements similar to those of boundary layers. Turbulent boundary layer and channel flows have generally received greater attention than pipes for studying structure. Besides the hairpin packet structure observed in boundary layers (Adrian et al., 2000b), the study of Hutchins & Marusic (2007a) brought significant attention to the streamwise velocity structure of very long motions (which they termed ‘superstructures’) that scale in outer units and exist through the logarithmic region. With hot-wire rake measurements in laboratory boundary layers and a spanwise array of sonic anemometers in an atmospheric boundary layer, both of which required the use of Taylor’s hypothesis, Hutchins & Marusic (2007a) confirmed that the log layers of these flows are populated with
very long meandering features with lengths over $20\delta$ (boundary layer thicknesses). Using turbulent channel DNS at $Re_\tau = 934$, they also found that superstructures extend as ‘footprints’ down to the near-wall region. They suggested that these structures may also be the VLSMs observed in pipe flows, but with internal geometries having less meandering than in turbulent boundary layers, leading to longer length scales observed in the internal geometries.

Monty et al. (2007) extended the boundary layer study of Hutchins & Marusic (2007a) to pipes and channels using hot-wire rakes. They studied the spanwise/azimuthal correlations and the scale growth indicated by this quantity as wall-normal position increases. Their hot-wire traces using Taylor’s hypothesis indicated that long, meandering low-speed regions flanked by high-speed regions in the spanwise/azimuthal direction were also present in the logarithmic regions of channels and pipes. They found that the logarithmic layers of turbulent boundary layers, pipes, and channels have qualitatively similar structures.

Recently, several experimental studies have examined the structure in turbulent pipe flows. Bailey et al. (2008) and Bailey & Smits (2010) focused on interpreting structure based on two-point correlations calculated from spectra measured with a pair of hot wire probes positioned for various azimuthal arc length separations ($\Delta s$) and radial positions. They also inferred streamwise scales by applying Taylor’s hypothesis. Bailey et al. (2008) used the two-point correlations to investigate how the azimuthal scales of large- and very large-scale (in an estimated streamwise sense) structures varied with wall-normal position for a wide range of Reynolds numbers. They noted the azimuthal two-point correlations of streamwise velocity were reasonably independent of Reynolds number. The negative correlation coefficient peak, existing at $\Delta s \approx 0.5R$, was weaker for $y/R = 0.1$ than for locations further from the wall. Based on analyzing cross-spectra, they associated this effect with the contributions to the azimuthal correlations from motions with VLSM and
LSM streamwise scales. The azimuthal correlations contributions associated with VLSMs were relatively strong near the wall compared to those of LSMs but less so at higher \( y \) values. They also inferred that the azimuthal scale (in terms of arc length) of LSMs is initially much narrower and grows more rapidly than that of VLSMs for \( y \) between 0.1 and 0.3, but the width growth rates reduce considerably and the widths become more similar by \( y/R = 0.5 \). Based on these different wall-normal behaviours, Bailey et al. (2008) suggest that, if the VLSMs are created by streamwise alignment of hairpin packets as suggested by (Kim & Adrian, 1999), only the oldest and largest hairpin packets align to create VLSMs. This is consistent with the Kim & Adrian (1999) discussion of \( 2R \)-long LSMs aligning and other evidence that the \( 2R \)-long bulges are the largest packets. Bailey et al. (2008) also note the different azimuthal scales suggest the possibility that LSMs and VLSMs could be independent entities, with VLSMs possibly arising from linear or nonlinear instabilities.

Correlations similarly obtained for the same pipe flow configuration at \( Re_D = 1.5 \times 10^5 \) were used by Bailey & Smits (2010) to calculate azimuthal correlation contributions corresponding to motions with VLSM and LSM streamwise wavelengths and to generate POD modes. In general, the radial-azimuthal correlations \( R_{uu}(r,r',\Delta \theta) \) (with streamwise separation \( \Delta x = 0 \)) had substantial magnitudes for less than about half the pipe circumference based on correlation contour plots, with the strong positive correlation near the reference probe position azimuthally surrounded by regions of negative correlation symmetrically located on either side. Bailey & Smits (2010) noted that this lack of correlation for \( |\Delta \theta| > 90^\circ \) suggests “minimal interactions occur between motions on opposite sides of the pipe.” By decomposing the radial-azimuthal correlations at \( \Delta x = 0 \) to the contributions associated with VLSM and LSM streamwise wavelengths, the VLSMs contribute much to the correlations at larger azimuthal angles from the reference position, whereas
LSMs are generally associated with much of the positive correlation at narrower $\Delta \theta$ values. For reference probe radii $r$ corresponding to wall-normal $y$ values ranging from $0.1R$ to $0.5R$, the correlations associated with VLSMs at $\Delta x = 0$ indicate velocity fluctuations remain correlated for $r'$ values extending near the wall. Conversely, correlations associated with LSMs indicate motions remain correlated approaching the wall only when the reference probe is located sufficiently near the wall, and little or no correlation was detected for probe locations further from the wall. They term these LSM motions ‘detached’ and distinguish types of LSMs in this manner. Bailey & Smits (2010) suggest that the sizable difference in scales between correlations associated with LSMs and VLSMs for reference probe locations near the wall supports the idea of Bailey et al. (2008) that VLSMs may not be simply alignments of LSMs that occur near the wall; and they note that the more similar scales between VLSMs and LSMs further from the wall could mean VLSMs spanning from near the wall to high above form from alignments of detached LSMs located far above the wall. By performing POD, they observe a lack of clear delineation between eigenspectra associated with VLSM and LSM and conclude that the two motions are interrelated. They also note the possibility of a linear mechanism creating traveling waves similar to those observed in pipe transition (Eckhardt et al., 2007), but more evidence would be necessary to support the existence of similar mechanisms, as there are a number of notable differences between transition and this fully turbulent regime.

POD calculations from experimental measurements of pipe flow were also performed by Hellström et al. (2011). Instead of hot wire probes, they used particle image velocimetry (PIV) to capture velocity vectors in a radial-azimuthal plane. With a series of such planes closely spaced in time, the velocity fields of three-dimensional volumes were approximated using Taylor’s hypothesis. Focusing on a low Reynolds number ($Re_D = 12500$) turbulent pipe flow, they found that recon-
structions with the 10 most energetic POD modes “capture all the principal characteristics of the VLSM. This suggests that VLSMs are constructed of the most energetic POD modes that, when superimposed, give the impression of long meandering structures.” The most energetic POD modes shown consisted of straight, streamwise-aligned segments of positive and negative streamwise velocity fluctuation with various azimuthal widths. They also noted that ‘the superposition of only the four most energetic modes will recreate meandering structures that appear to be much longer than any of its constituent modes.’ Hellström et al. (2011) noted that these POD modes appeared similar to a sum of two helical response modes obtained from the linear stability analysis of McKeon & Sharma (2010), which calculated modes fitting a particular form that would experience maximum amplification. Hellström et al. (2011) found this to support the linear mechanisms associated with the existence of these propagating response modes proposed by McKeon & Sharma (2010).

Große & Westerweel (2011) used a similar PIV measurement technique, but, instead of performing POD, studied the structures present in the pseudo-three-dimensional velocity fields, as well as various statistics, for turbulent pipe flows with Reynolds numbers ranging from \( Re_D = 10\,000 \) to 44\,000. This experiment also provided clear evidence of the very long structures of streamwise velocity. Große & Westerweel (2011) concluded that their measurements indicated the presence of low-speed and high-speed regions extending up to several pipe radii in streamwise length based on applying Taylor’s hypothesis. They also observed these structures possessing strong coherence (i.e., similar shapes) for a wide range of radii ranging from \( y/R = 0.05 \) to 0.5. Instantaneous streamwise-normal planes of velocity provided insight into how structures of retarded streamwise velocity merge in an azimuthal sense with radial distance above the wall. They also studied the azimuthal merging of the high and low speed streaks and the corresponding azimuthal length.
scales as $y$ increases using statistical quantities. This included dividing arcs into sectors in which velocity was within a certain threshold and computing probability densities of streak widths, which they found to be more strongly concentrated at short widths near the wall and more evenly distributed over a range of widths nearer the pipe core.

As noted above, the quasi-three-dimensional pipe fields obtained with radial-azimuthal PIV measurements rely on Taylor’s hypothesis to infer the streamwise spatial variation from two-dimensional fields closely spaced in time. Significant differences can occur in the longest motions between instantaneous fields and those obtained by applying Taylor’s hypothesis (Dennis & Nickels, 2008; del Álamo & Jiménez, 2009; Wu et al., 2012). In addition, this technique presents issues for measuring velocity in the region very near the wall. Hellström et al. (2011) discarded the $y/R < 0.1$ region due to optical refraction issues. Große & Westerweel (2011) found significant difference in velocity fluctuation statistics relative to DNS for $y/R < 0.1$ at their highest Reynolds number, likely due to insufficient resolution for this region in the experiment.

Given these experimental limitations, DNS data are particularly well-suited to examining the structure of very long scale motions without the use of Taylor’s hypothesis while resolving the smallest relevant motions. Although structure has been observed in several previous pipe simulations performed at lower Reynolds numbers (for example, Eggels et al., 1994; Duggleby et al., 2009), relatively few have been performed for $Re_D > 10,000$. Wu & Moin (2008) performed pipe simulations for $Re_D = 5300$ and 44,000 and generally focused on statistics but made several visualizations of streamwise velocity. At $Re_D = 44,000$, they noted the presence of much “fine-grain structure.” In a constant $\theta$ plane, they observed a large number of worm-like elongated high-momentum structures with very narrow azimuthal
dimension.’ The domain length of this simulation limited the study of very long structures.

The present DNS has a Reynolds number high enough to include significant energy at VLSM scales (wavelengths $\lambda_x \geq 3R$) and a domain length of $30R$ that is long enough for examination of very long structures, with attention given to how various smaller scales relate to the VLSMs that are significant in energy spectra. Although DNS is subject to numerical accuracy, computational domain size, and streamwise periodicity issues, the present simulation has been verified by generally good agreement with experimental statistics (Wu et al., 2012).

This study characterizes the VLSMs in the pipe flow simulation, their relation to smaller structures, their organization, and their similarities to experimental observations and structural concepts for wall-bounded turbulent shear flows. This study first visualizes the velocity streaks in a streamwise-azimuthal cylindrical surface, describes their characteristics, and statistically quantifies their lengths (§5.3). We next visualize the velocity structures’ radial extents (§5.4) and observe the near-wall footprint attached to low-momentum regions. To answer how these velocity structures relate to vortical structures, the associated vortical structures are also visualized (§5.5) and the organization of the smaller-scale vortical structures are discussed in the context of the hairpin vortex packet paradigm. Conditional averages are then employed to study the overall organization pattern of the very long velocity streaks (§5.7). Finally, we examine the mean azimuthal scale growth with wall-normal position and how it relates to instantaneous structures (§5.8).

5.2 Computational Details

In the present study, the unit length-scale is pipe radius $R$, the unit velocity-scale is $U_{\text{bulk}}$, which is defined as the ratio of mean mass flow rate and pipe cross-sectional area. The unit time scale is therefore $R/U_{\text{bulk}}$. The Reynolds number based on
An overbar denotes ensemble averaging, superscript + refers to normalized quantities by friction velocity $u_\tau$ for velocity, and by viscous wall unit $\nu/\tau$ for distance. Additional details of the simulation and its validation are described in Wu et al. (2012).

5.3 Long Streaks of Streamwise Velocity Fluctuation

As noted in the introduction, long streaks of streamwise velocity ($u$) fluctuation are ubiquitous in canonical wall-bounded shear flows. A cylinder at $y/R = 0.15$ and $y^+ = 101$, towards the top of where a logarithmic law could be expected, is clearly visualized when rolled out to be viewed as a plane (Monty et al., 2007), and is analogous to constant $y$ planes viewed in channel and boundary layer flows. Contours of streamwise velocity fluctuation at an instant in time (figure 5.1) appear qualitatively similar to those obtained from pipe experiments using Taylor’s hypothesis (Monty et al., 2007; Große & Westerweel, 2011; Hellström et al., 2011), with long, approximately streamwise-oriented streaks of low- and high-velocity fluctuations...
visually dominant. The appearance of the low-velocity streaks is similar to that at $y/R = 0.10$ in the $Re_D = 44\,000$ pipe flow of Große & Westerweel (2011), which is similar to a nearer-wall location at which they noted meandering with frequent ‘joining and splitting of streaks.’ At these radii far greater than those of the core region, the streaks in the present DNS are similar to those at similar heights in Große & Westerweel (2011), in their observation that streaks do not persist across the centerline (that is, travel azimuthally by $180^\circ$). Monty et al. (2007) observed several examples of low-speed streaks traveling $180^\circ$ around the circumference at $y/R = 0.15$, but this was at significantly higher Reynolds number ($Re_\tau = 3472$ and $Re_D = 152\,000$). The limited azimuthal resolution of the hot wire rake may also have made discerning streaks more difficult. For the present simulation, the entire simulation domain, shown in figure 5.1(a), includes a number of long negative-$u$ streaks, but it is somewhat ambiguous where a streak begins and ends, particularly when viewed without fine details clearly shown.

Two examples of low-speed streaks are shown in greater detail in figure 5.1(b,c). The presence of multiple scales of motion, with many fine-scaled turbulent fluctuations, is clear. However, more distinct upstream and downstream breaks in a connected organization of low-momentum fluctuations become apparent when viewed at this higher resolution. The left and right boundaries of figure 5.1(b) correspond to the apparent breaks in long streamwise motions. In this subfigure, there appears to be a long low-speed motion between $x/R = 12$ and 18 and another at greater $s/R$ (azimuthal location) between $x/R = 15$ and 20. The break identified near $x/R = 18$ is much clearer and marks the beginning of a low-speed motion that extends far past the subfigure. Both subfigures (b) and (c) contain low-speed regions that appear to be wavering in an azimuthal sense, and this is consistent with observations in the logarithmic layers of other wall-bounded turbulent shear flows Hutchins & Marusic (2007a). Within the wavering streak, several of the strongest segments
Figure 5.1: Axial velocity fluctuation $u'/U_{\text{bulk}}$ contours of a streamwise-azimuthal cylinder at $t = 252R/U_{\text{bulk}}$ and radius where $y/R = 0.15$ and $y^+ = 101$. (a) displays the entire simulation domain, with two prominent long, streaky, low-speed regions (b) and (c) shown in greater detail. (d) contains even finer detail of a very fine-scaled motion from (b). Two relatively straight low-speed streaks in (a) are identified as A and B.

of negative fluctuation appear to be more streamwise-aligned, such as the regions from $x/R = 8.2$ to 9 and from $x/R = 12$ to 13 in subfigure (c). Viewing further detail of the small-scale fluctuations in (d), these smallest distinctive motions appear to consist of slightly-streamwise elongated negative velocity fluctuation peaks that decay in magnitude rapidly with distance from the peak location. While this will be discussed in greater detail, the streak appears to be composed of these somewhat randomly oriented but similarly dimensioned small fluctuations, suggesting concatenations of small motions.

One method of characterizing the lengths of the streaks is measuring the lengths of contiguous regions with velocity fluctuation below a threshold value, such that a structure that is contiguous but wavers is identified as a single structure. Based on quasi-three-dimensional velocity measurements in a turbulent boundary layer,
Dennis & Nickels (2011b) extracted isosurfaces of negative velocity fluctuation at a particular threshold value (10% of the local mean velocity) and measured their streamwise lengths to create a histogram representing the frequencies at which they exist. Große & Westerweel (2011) calculated histograms of azimuthal widths of velocity structures, and found that introducing a threshold was also necessary.

For the present study of structures’ streamwise lengths, clusters of points with velocity fluctuations below a specified fraction of the bulk velocity (independent of y) are extracted from a streamwise-azimuthal plane, and the streamwise lengths of the resulting point clusters are used to calculate the statistics of streamwise lengths. The results are expected to be similar to those for three-dimensional clusters, although the latter would also permit bridging of discontinuous segments at a particular wall-normal location by attached continuous segments at other radii. To illustrate the contiguous regions identified, filled contours of negative fluctuations stronger than a threshold value $u'_{\text{thr}}$ of $-0.10U_{\text{bulk}}$ are displayed in figure 5.2. Overall, these contours create the visual impression of scattered strong small-scale negative fluctuations with some touching and forming longer contiguous regions, others almost touching but forming shorter contiguous regions, and many small-scale negative fluctuations scattered and disconnected but often organized along streak-like lines. This suggests that what comprises a streak is significantly dependent on the threshold value chosen.

Histograms of the corresponding low-speed streak lengths are presented for $y^+ = 101$ in figure 5.3 with the same contour threshold level of $u'_{\text{thr}}/U_{\text{bulk}} = -0.10$, for which contours are shown in figure 5.2. In identifying clusters of points comprising streaks of low-velocity fluid, the streamwise and azimuthal periodicities of the simulation geometry are taken into account when extracting contiguous regions. The frequencies of occurrence are counted for bins encompassing ranges of one pipe radius in streamwise length. The mean number of structures with stream-
Figure 5.2: Axial velocity fluctuation contours of a streamwise-azimuthal cylinder at \( t = 252R/U_{\text{bulk}} \) and radius where \( y/R = 0.15 \) and \( y^+ = 101 \). Black regions are the regions of strong negative \( u' \) fluctuation with velocity below the contour threshold level of \( u'_{\text{thr}}/U_{\text{bulk}} = -0.10 \), which is approximately the rms fluctuation magnitude at this radius.

The streamwise length that falls within each bin per field is displayed in figure 5.3(a). Every streamwise length is included, while Dennis & Nickels (2011b) focused on structures greater than 2 boundary layer thicknesses long. The shorter structures occur much more frequently than longer ones, so the axis is displayed on a logarithmic scale. Clearly, the vast majority of contiguous low-speed regions for this threshold value have streamwise lengths less than \( R \). As \( \langle n_{\text{str}} \rangle = 1 \) indicates an average of one structure within the corresponding bin occurs per instantaneous \( x-\theta \) cylinder, the histogram indicates that only structures with length less than \( 8R \) occur more than once per field, on average.

As discussed, the lengths are somewhat subjective due to the choice of threshold. Dennis & Nickels (2011b) noted that ‘changing the level within a reasonable margin does not alter the distribution greatly.’ To address the effect of threshold, figure 5.3(b) includes contour lines of \( \langle n_{\text{str}} \rangle \) histogram values (logarithmically spaced) for structure length bins also as a function of threshold value. Thus, figure 5.3(a) is a bar chart representation of the horizontal slice of figure 5.3(b) at which \( u'_{\text{thr}}/U_{\text{bulk}} = -0.10 \). In general, the occurrence frequencies of the longest structures are most sensitive (in terms of percent change) to the threshold values. The length bin for which \( \langle n_{\text{str}} \rangle = 10^1 \), for example, remains much more constant
with changing threshold, and is at approximately $l_s = 4$ in the vicinity of the threshold noted above, so contiguous streaks of length $4R$ and shorter generally exist most frequently with several longer streaks also likely per field. In the context of a different quantity (relating to vortical motions), del Álamo et al. (2006) discussed the phenomenon of the clusters of connected points satisfying a varying threshold merging into a few complex, confusing objects as the threshold value is reduced. In figure 5.3(b), the contour lines representing the $\langle n_{str} \rangle$ values of $10^1$ and $10^2$ peak at longer structure lengths as threshold magnitude is reduced. For high threshold magnitudes, as the threshold magnitude is reduced, more and more contiguous regions of negative fluctuation appear and others connect to form longer regions.
Figure 5.4: Histograms of streamwise length of contiguous regions of negative $u'$ fluctuation as in figure 5.3, except $y^+ = 30$ (a) and (c) correspond to a threshold value of $u'_{\text{thr}}/U_{\text{bulk}} = -0.10$. These peaks occur when regions of moderate length ($R–3R$) connecting with other regions to become longer than this length begins to occur more frequently than shorter regions connecting or enlarging to become regions of this length. For the longest length bins, combining of shorter structures is the dominant effect and the frequency monotonically increases with decreasing threshold magnitude. At the lowest threshold of 0, which corresponds to the connected regions at which the streamwise velocity fluctuation is negative at any magnitude, several such structures often span the entire periodic domain length. The resulting structures are irregular and difficult to interpret, so for this reason the histograms are not calculated for too small of threshold magnitude.

Although relatively small numbers of long structures ($l_x > R$) are extracted, this statistic does not fairly represent their importance. Due to these structures’ greater areas/volumes, each structure occupies a larger fraction of the simulation domain.
that each short length structure. In an extreme case in which the threshold is so low that all negative fluctuations coalesce into a single connected structure in each field, the average number of structures per field $\langle n_{str} \rangle$ would be one, but the area represented by this one structure would be a sizable fraction of the $x-\theta$ cylindrical surface area ($\langle A_{\text{clust}} \rangle / A$). For this reason, comparing the average area fraction of the cylinder occupied by each bin of structure lengths more accurately reflects their importance than their mean frequencies of occurrence. By depicting this quantity, the comparison of lengths does not require the logarithm to be taken, and the mean area fractions are presented on a linear scale in figure 5.3(c) and with linearly spaced contour levels in figure 5.3(d). The bars in figure 5.3(c) indicate the mean fraction of the total cylindrical surface area that low speed streaks of each contiguous length contribute. For example, the third bar indicates that low speed streak structures of length $2R$ to $3R$ account for $1.8\%$ of the area in the cylinder when the clustering threshold is $u'_{\text{thr}}/U_{\text{bulk}} = 0.1$. The sum of all area fraction bars is the total fraction of the area occupied by regions negative fluctuations stronger than the threshold value ($u' < u'_{\text{thr}}$). The trends of area fractions are similar to those of the logarithm of number of occurrences.

The structure lengths are also similarly calculated nearer the wall at $y^+ = 30$ in figure 5.4. The histogram uses the same threshold value of $u' = -0.1U_{\text{bulk}}$. At $y^+ = 101$, this was approximately the magnitude of the local $u'$ rms value of $0.0995U_{\text{bulk}}$. At $y^+ = 30$, the local $u$ rms value is $0.133U_{\text{bulk}}$, so the threshold corresponds to a relatively weaker fluctuation when scaled by rms $u'$ fluctuation, thereby promoting the extraction of longer structures.

To roughly compare with the results of Dennis & Nickels (2011b) for their $Re_\tau = 2460$ turbulent boundary layer, a $10\%$ fluctuation from the local mean is equivalent to a threshold of $u'_{\text{thr}} = -0.092U_{\text{bulk}}$ for the pipe at $y^+ = 101$, which is similar to the $-0.1$ value shown in the bar chart histogram. Using the $-0.1$ thresh-
old at $y^+ = 101$ and excluding the structures shorter than $2R$, structures with lengths of \{2-3, 3-4, 4-5, 5-6, 6-7\}R account for \{65, 23, 7, 2, 2\}% of the extracted low-speed streak structures. For Dennis & Nickels (2011b), instead nondimensionalizing by boundary layer thickness $\delta$, the analogous values are \{44, 19, 14, 11, 10\}%. In both cases, less than 5% of structures are longer than these bins. The difference in frequency distribution is likely due to a combination of the different flow geometries, Reynolds numbers, nondimensionalizations (the comparison assumes $R$ and $\delta$ are equivalent), and inclusion of structures nearer the wall in the three-dimensional method for the boundary layer. To include the connected structures nearer the wall, the percents of structures longer than $2R$ in each bin at $y^+ = 30$ are \{51, 22, 11, 6, 3\}% based on the $-0.1$ threshold as displayed in figure 5.4(a). In general, the results for the pipe are broadly similar to those for the boundary layer experiment of Dennis & Nickels (2011b) and support the same trends.

While this is one method of characterizing the low-speed structure lengths, the most common statistical indications of structure lengths in turbulent flows are energy spectra. It is on this basis that VLSMs are traditionally defined as motions with streamwise Fourier wavelengths of $3R$ and greater ($\lambda_x \geq 3R$) (Guala et al., 2006). As noted in Wu et al. (2012), the present pipe simulation energy spectra indicates the flow contains a significant fraction of $u$ energy in the VLSM streamwise wavelengths, with approximately 44% of the energy associated with $\lambda_x \geq 3R$ when averaged over all radii. While experimental pipe spectra in the logarithmic layer at sufficiently high Reynolds number formed a distinct peak at long wavelength that corresponded to the VLSMs and led to the spectrum appearing in a bimodal form (Kim & Adrian, 1999), at least some of this peak is introduced by error due to applying Taylor’s hypothesis for the experiments (del Álamo & Jiménez, 2009). Although this region of the spectrum is flatter in DNS for which true spatial spectra are calculated and more consistent with a theoretical $k_x^{-1}$ behavior at long wave-
lengths (del Álamo & Jiménez, 2009). Wu et al. (2012) found that a very weak peak may possibly be forming (with a much higher Reynolds number DNS necessary to definitively determine if this is the case), but in any case the spectrum was reasonably represented with a bimodal form.

The issue remains of how this concentration of wavelengths represented by the spectra may be related to the concentration of scales revealed by the contiguous low-speed region extraction algorithm described above. Since Fourier wavelengths represent the wavelengths of periodicity, they involve both positive and negative fluctuations if they are arranged in a streamwise periodically alternating fashion. It is reasonable that the strongest Fourier wavelengths would be associated with this period relatively independent of the exact form of the fluctuation, and additional Fourier components would assume a less dominant role. This is suggested by the case of calculating the Fourier series representation of a square wave and triangle wave in which the dominant wavelength would be that of the periodic signal length for both cases. If one considers a negative fluctuation of a certain length immediately neighboring, in a streamwise sense, a positive fluctuation, then the dominant wavelength would be twice the length of the negative structure. This assumes that the length of positive and negative fluctuation structures are the same. In boundary layers, the distribution of high-speed fluctuation streaks is relatively similar to that of their low-speed counterparts, but somewhat weighted toward shorter streamwise extent, according to the results of Dennis & Nickels (2011b).

In the present flow’s premultiplied energy spectrum (Wu et al., 2012), the energy in the logarithmic layer region begins to decay rapidly with increasing wavelength at approximately $\lambda_x = 10R$, which suggests (in the previously discussed scenario) periodically occurring low-velocity structures of length $5R$. The statistics of extracted structures in figure 5.3(c) indicates that this structure length is approximately the length at which the areas represented by such structures is small among
the scales longer than $2R$. The complexity of structure organization makes the link between the energy spectra and extracted low-speed structures, but the statistics of extracted structures generally supports the importance of structures with lengths corresponding to VLSMs. However, organization and wavering of structures also clearly affects their associated energy spectra, as explored in depth by Hutchins & Marusic (2007a). They showed that a relatively simple streak produces energy at a range of length scales and the distribution is affected by meandering. Therefore, additional aspects of the structure dimensions and the organization of structures are next considered.

5.4 Radial Extent of Velocity Structures

While the velocity fluctuations were shown in figure 5.1 for $y^+ = 101$ ($y/R = 0.15$), the top of where a logarithmic region could be expected, the radial extent of the structures is also significant. In a turbulent channel simulation, Hutchins & Marusic (2007a) noted the ‘footprints’ of streamwise velocity that extend down very near the wall. In a turbulent pipe flow, it is possible for the behaviour to be different due to the curved geometry of the wall, although the correlations of Bailey & Smits (2010) suggest VLSMs remain well correlated far down to the pipe wall. An example pipe field in figure 5.5 demonstrates the correlation between streamwise velocity fluctuations ($u'$) for cylinders of various radii. A section of this pipe is cut in half radially (cut along a constant $\theta$) and visualized by colour isosurfaces of streamwise velocity fluctuation in figure 5.5(a). It is viewed with the pipe wall nearest the viewer, and the velocity fluctuations nearest the viewer have the characteristic azimuthal widths for near-wall low speed streaks ($\lambda^+_s \approx 100$). Near the vertical center, where the viewer is looking radially inward to the pipe axis, it is clear that fluctuations very rarely have alternately signed fluctuations located radially above structures present above the wall (into the page), suggesting that the structures remain well correlated.
as the radius varies. Since the velocity fluctuations are uniformly scaled by $U_{\text{bulk}}$, the weaker motions located toward the pipe core are not visible, as inevitable merging occurs such that scales become larger. The visualized structures are viewed from a side perspective towards the lower and upper regions of figure 5.5(a), and their radial extents are visible from this perspective. A distinctive diamond-shaped arrangement of low speed streaks (which does not appear to occur particularly commonly) centred at $x/R = 10$ is apparent in figure 5.5(b) at $y^+ = 80$. Nearer the wall, at $y^+ = 20$, the diamond pattern remains discernible, but with additional fine scale structure. The arrangement of the colour isosurfaces in figure 5.5(a) also represents this diamond-shaped structure arrangement, with blue low-speed fluctuations surrounding the diamond’s centre of yellow-orange high-speed fluctuations. Further into the core, at $y^+ = 270$ or $y/R = 0.39$ in figure 5.5(d), visually apparent elements of the diamond pattern remain discernible. This example indicates structures of velocity motions often remain correlated over wide radial extents. This also suggests that low-speed streaks in the logarithmic region are strongly associated with similar footprints also extending near the wall.

With the view instead toward the pipe axis and looking outward toward the wall, figure 5.6 depicts isosurfaces of negative streamwise velocity fluctuation with a weaker $u'/U_{\text{bulk}}$ level to emphasize the structures more distant from the wall. The region visualized is that of figure 5.1 in which low-speed streaks A and B were identified. The $y^+ = 101$ surface of greyscale contours visualized there is also included on figure 5.6. This surface blocks the dark blue near-wall structures from cluttering the higher velocity structures. The visible isosurfaces indicate that the negative velocity fluctuations penetrate far toward the pipe centreline. Low and high velocity streaks may be visualized with isosurfaces representing fixed values of several possible quantities, including $u'$ fluctuations scaled by the local fluctuation rms or local mean $\bar{u}(y)$, as in the case of Dennis & Nickels (2011b). The present visualiza-
Figure 5.5: Example streamwise velocity fluctuation at $t = 324R/U_{\text{bulk}}$. Isosurfaces for $u' = -0.18U_{\text{bulk}}$ (blue) and $0.18U_{\text{bulk}}$ (yellow-orange) for a segment of a pipe azimuthal-half looking perpendicular to the cut plane and with the pipe axis in the distance are shown in (a). Grey contours (black to white for $u'/U_{\text{bulk}} = -0.2$ to 0.2) are shown for: (b) $y^+ = 20$, (c) $y^+ = 80$, and (d) $y^+ = 270$.

Fluctuations use $u'$ normalized by $U_{\text{bulk}}$, a normalization independent of $y$. Relative to the present method, normalizing by the local mean would tend to emphasize the structures near the wall where the mean velocity is small and fluctuation intensities are large, whereas normalizing by the local rms would strengthen motions far from the wall where intensity is weaker. In a turbulent boundary layer, Lee & Sung (2011) plot isosurfaces of negative velocity at a specific value independent of wall-normal position. It should be recognized that fluctuations that weaken in accord with the lowering intensity in the core may persist significantly further from the wall than depicted in figure 5.6.

The long structures revealed in figure 5.6 are typical of the negative velocity structures occupying the pipe. The visualization indicates the motions observed in the log-layer region penetrate deeply into the core, with the overall trend of ramp-
Figure 5.6: Three-dimensional isosurfaces of negative u fluctuation \( \left( u'/U_{\text{bulk}} = -0.15 \right) \) for the pipe section containing low speed streaks A and B in previous figures \( (t = 252R/U_{\text{bulk}}) \). The cylinder shown previously at \( y/R = 0.15 \) with grey contours is included. The isosurfaces are colored according to radius, ranging from deep blue near the pipe wall to green at the centreline.

like inclination indicated by the yellow lines for streak A. The structures below approximately \( y^+ = 101 \) have a strongly swept (inclined at a shallow wall-normal angle) appearance probably due to the very strong shear in this region. The structures visible above this level appear to be inclined at a steeper wall-normal inclination (more similar to 45 degrees). Further from the wall than the \( y^+ = 101 \), the strong negative fluctuations appear more broken up than the continuous streak A visualized in figure 5.1, with the three-dimensional structures appearing somewhat like a concatenation. It should be noted that the isosurfaces represent the stronger motions comprising the streak visible in the contours at \( y^+ = 101 \). The overall picture is one of streamwise series of generally ramp-like negative-velocity regions that extend from near the wall to deeply into the core approaching the pipe axis. The structures also appear to widen as scale growth occurs with distance above the wall, but this is less obvious because the fluctuations tend to simultaneously weaken. Streak
B in figure 5.6 is viewed from approximately directly above, and this perspective indicates that relatively little wavering occurs in an azimuthal sense relative to the structure’s shape at $y^+ = 101$, consistent with the observation of similar structures throughout a wide variation in radius for figure 5.5. Further characteristics of these structures are visualized in greater detail by viewing various cross-section planes.

5.5 Visualization of Structures

Structures in wall-bounded turbulent flows have traditionally been studied using vector fields and vortical quantities (vorticity and swirl) in various two-dimensional planes. To permit comparisons with preceding studies, similar planes are examined and similar analysis techniques are now applied to the present simulation. Observations from these views are then applied to the perspectives provided by three-dimensional views of important structures.

5.5.1 $x–r$ Planes

The hairpin packet model was developed using measurements of boundary layers in a streamwise-wall normal plane. While the basic elements of hairpin-shaped vortical structures in wall-bounded turbulent flows had been proposed previously (Theodorsen, 1952), Adrian et al. (2000b) proposed that hairpins organized in long growing trains of various sizes internal to the boundary layer. This was based largely from evidence in experimental measurements of velocity in streamwise-wall-normal planes of turbulent boundary layers. These planes also indicated that boundary layers were composed of large zones of nearly constant streamwise momentum (Meinhart & Adrian, 1995). These irregularly shaped time-varying zones were often inclined and existed in layers with clumped regions of concentrated spanwise vorticity separating the zones in the wall-normal direction (Meinhart & Adrian, 1995). Adrian et al. (2000b) identified these concentrated spanwise vorticity regions with vortex cores of hairpin heads. They did this based on formulating
a ‘hairpin vortex signature’ with several features of a velocity field associated with an idealized hairpin vortex. An idealized hairpin vortex is a horseshoe-shaped vortex loop with concentrated vorticity in a tube that begins near the wall oriented in a streamwise direction in ‘legs’ that begin to incline upward from the wall. As the vortex tube reaches a distance from the wall, it forms a neck as it turns to the spanwise direction. The wall-normal inclination reduces until it reaches a peak wall-normal height and is oriented entirely in the spanwise direction to form the head. It then continues downward into another neck segment and aligns in the negative streamwise direction to return to the wall. In the ideal symmetric case, the shape of this half is a spanwise mirror image about a streamwise-wall-normal mid-plane at the head. The vorticity is oriented such that both legs eject fluid upward at the midplane and the head ejects fluid below upward and backward (opposite the downstream streamwise direction) at the midplane. The vorticity at the hairpin head therefore has the same orientation as the strong near-wall shear and the mean shear, but is concentrated into a core. Studies of simulation fields suggest that versions lacking the symmetry of the idealized hairpin, but sometimes possessing a ‘cane’-like geometry, may be more commonly observed than symmetric hairpins. Evidence also suggests that asymmetric hairpins develop more quickly (Zhou et al., 1999).

A ‘hairpin vortex signature’ captures the salient features associated with a velocity field plane passing through the spanwise center (the plane of symmetry) of an idealized hairpin vortex. Adrian et al. (2000b) states this signature consists of the following: ‘(i) a transverse (i.e. spanwise) vortex core of the head rotating in the same direction as the mean circulation; (ii) a region of low-momentum fluid located below and upstream of the vortex head, which is the induced flow associated with the vorticity in the head and neck; (iii) an inclination of this region at approximately 35–50° to the x-direction below the transverse vortex and more nearly tangent to
the wall as the wall is approached.’ Adrian \textit{et al.} (2000b) also notes a fourth feature that may occur: ‘Frequently, a fourth-quadrant Q4 event \((u - U > 0, v \leq 0)\) is observed to oppose the Q2 event, forming a stagnation point and an inclined shear layer upstream.’ Q2 and Q4 refer to quadrants that fluctuations may be assigned to depending on the signs of \(u\) and \(v\) fluctuations, with Q2 corresponding to upward and opposing the flow and Q4 corresponding to downward and with the mean flow.

After observing hairpin vortex signatures frequently appear in groups with similar arrangements, Adrian \textit{et al.} (2000b) detailed an organizational pattern describing frequently occurring characteristics of these groups. Termed ‘hairpin vortex packets’, these streamwise-aligned series of hairpins were observed in streamwise-wall-normal planes as ramps of vortex core cross-sections (corresponding to slices through hairpin heads) with growth angles of, on average, \(12^\circ\) wall-normal with downstream distance. The regions below these ramps have retarded momentum, with the various layers corresponding to uniform momentum zones observed by Meinhart & Adrian (1995). Within the ramps of vortex heads, the vortices are spaced roughly \(200^+\) (viscous units) apart in the streamwise direction (Marusic & Adrian, 2012). Thus, the presence of ramps with growth angles of approximately \(12^\circ\) containing vortex cores (and hairpin vortex signatures) separated by \(200^+\) in the streamwise, with the ramps of cores separating layers of uniform momentum zones, is indicative of the flow containing the patterns expected for hairpin vortex packets. The frequent appearance of this pattern in two-dimensional streamwise-wall-normal planes is strong evidence of the presence of hairpin vortex packets in the flow. It should be noted that these structures correspond only to long low-speed regions, with the signatures based on planes passing through the centres of these structures. Random streamwise-wall-normal planes collected from experiments could pass through either low-speed or high-speed fluctuation streaks, so in
this case it was to be expected that these patterns would occur frequently, but only in a fraction of the planes.

While thought to be generally applicable to wall-bounded turbulent flows, the hairpin packet model was developed largely from observations of turbulent boundary layers. Observations in turbulent channel simulations also provided support for the presence of hairpin packets (Zhou et al., 1999; Adrian & Liu, 2002; Adrian, 2007), although other researchers have concluded the vortices appearing in relatively high Reynolds number channel simulations have structure in an average sense that ‘coincides with a large-scale hairpin eddy’ but instantaneous examples are much more complex than simple vortex loops (del Álamo et al., 2006). The existence of hairpin packets in pipe flows was supported from flow energy spectra considerations by Kim & Adrian (1999), who advanced the hypothesis that the packet organization of streamwise-aligned trains of hairpin vortices provides an explanation of the significant energy at long streamwise wavelengths and the long streamwise correlation tails that are observed in hot-wire pipe flow measurements. Although for a different flow, this hypothesis also gained support from the theoretical work of Marusic (2001), which showed that randomly scattering synthetic packets instead of hairpins as the basis for a structural model of turbulent boundary layers better reproduces the long tails for streamwise correlation of streamwise velocity fluctuation observed in experiments.

Therefore, streamwise–wall-normal planes are extracted from the present pipe DNS and analyzed for hairpin vortex signatures and the patterns associated with packets. By applying similar analysis as was performed in Adrian et al. (2000b), it can be determined if the existence of packets in the present pipe flow is similarly supported. Limited work has been previously performed on studying similar planes in experimental pipe flow, including Urushihara et al. (1993) before the hairpin packet model was developed and Große & Westerweel (2011) in a more so-
phisticated three-dimensional experiment that uses Taylor’s hypothesis to infer the streamwise spatial behaviour. Große & Westerweel (2011) displayed an example low-speed region in which they observe vorticity consistent with the signature of a hairpin vortex head above the low-speed region. In contrast to experimental studies in which only random planes are available, the three-dimensional instantaneous pipe fields of the present DNS allow planes’ azimuthal positions to be specifically chosen along the centres of low speed regions where these features are expected to occur.

The streaks selected for examination are chosen from $x-\theta$ cylinder surfaces at $y^+ = 101$, as shown in figure 5.1, in which numerous long ($> 3R$) wavering continuous dark regions of low-velocity fluid are present. The two examples that are here described in depth are the low-speed streaks identified as A and B in figure 5.1. The precise $y^+$ location at which structures are selected is not critical because of the generally strong correlation between nearby radii, as discussed in §5.4, but $y^+ = 101$ leads to selecting low-velocity structures extending at least to the uppermost height of a logarithmic layer. Examples are selected by choosing several such streaks from each field contour at $y^+ = 101$ that are strongest in magnitude, longest, most streamwise-aligned (such that a single plane passes through a long region) and least wavering. For example, in 7 separate fields at different times, 30 such examples are selected, but clearly many more such streaks exist. As noted in §5.3, it is often somewhat subjective whether a given region constitutes a single streak or several separate streaks. The length of each relatively straight streamwise-aligned example often exceeds $3.5R$ ($2400^+$), so in one single field with 7 such examples, the total streamwise length of the example planes within low-speed streaks is $33.9R$ or $23\,200^+$. This is the equivalent length of 46 PIV vector fields in Adrian et al. (2000b), each of length $500^+$, and all of these present examples are known to pass through a strong low speed region with significant streamwise length.
To view the necessary details of vortices in the vector patterns of streaks A and B, the planes must be divided into shorter streamwise segments. A and B are chosen at random and are not more or less convincing of structural organization than other streaks. The segments included in figures are typical and chosen to fairly illustrate the features of interest or lack thereof. Since the simulation is performed on a nonuniform grid, the vector field planes are interpolated onto a coarser, PIV-like uniform grid spacing (equal in $x$ and $r$) to make them more similar to PIV. To show the overall features of relatively long regions, a somewhat coarse spacing of 6.75 in wall units is used, and a finer resolution shows better details of the cores. For comparison, the experimental pipe measurements of Urushihara et al. (1993) had a uniform grid spacing of $\Delta x^+ = \Delta y^+ = 4.6$, and the streamwise extent of the measurements was nearly 500 in wall units.

Figure 5.7 displays several vector field segments from streak A, presented as in Adrian et al. (2000b), with vectors indicating the actual (non-fluctuating) velocity with a constant convection velocity subtracted instead of the Reynolds-decomposed fluctuation. Galilean transformation has the advantage that the Navier-Stokes equations are invariant with respect to it. Regions of constant streamwise momentum are also more easily identified. This and other issues of interpreting turbulent fields in wall-bounded shear flows are discussed by Adrian et al. (2000a). Contours of swirling strength $\lambda_{ci}$ are used to identify vortices because it measures local swirling motion while discriminating against vorticity associated with pure shearing motion (Zhou et al., 1999; Chakraborty et al., 2005). $\lambda_{ci}$ is defined as the imaginary part of the complex eigenvalues of the velocity gradient tensor $D = \nabla u$, in which $u$ is here taken as the actual (non-fluctuating) velocity vector. For experiments in which velocity fields are only available in a plane with only velocity components in that plane measured, only two-dimensional $\lambda_{ci}$ may be calculated for only the two available velocity components. However, all three components are used to
calculate \( \lambda_{ci} \) herein, unless otherwise noted. Azimuthally-oriented heads are the vortex structures of interest for which core cross-sections would appear in these streamwise-wall-normal planes. Since \( \lambda_{ci} \) is nonnegative and does not indicate the direction of rotation, it is useful to sign \( \lambda_{ci} \) by \( \omega_\theta \), the azimuthal component of vorticity \( \omega = \nabla \times \mathbf{u} \). Swirling strength contour peaks indicate the centres of vortex cores at the heads, and these are used to determine the constant streamwise velocity to subtract from the field for clearly visualizing a selected vortex. Adjusting the velocity of the frame of reference so streamlines become circular (and therefore the centre has zero velocity in this frame) about a region of intense vorticity is another method of selecting the translation velocity (Adrian et al., 2000b), and this agrees with the method by which Kline & Robinson (1989) define vortices.

Figure 5.7(a) is one segment of a plane passing through streak A with three-dimensional \( \lambda_{ci} \) contours signed by \( \omega_\theta \) also shown. The concentrated region of strong prograde (i.e., with the same vorticity sign as the mean shear, Wu & Christensen (2006)) swirl indicated by the leftmost arrow (red) is the location at which the streamwise velocity of \( 0.83 U_{\text{bulk}} \) is selected to subtract from each of the velocity vectors and clearly visualize this vortex. Other vortex cores (indicated by orange arrows) would require slightly different frame velocities for optimal visualization, but the velocity selected for the leftmost identified region is approximately correct. It should be noted that these identified cores are complex, with other apparent vortex segments nearby. The yellow-to-red contour colours correspond to retrograde (opposing the mean shear) vortices, or more frequently, vortices oriented largely in the plane for which \( \omega_\theta \) is weak and changes sign with small changes in \( x \) and \( y \) position. While the region of vortex core identified by \( \lambda_{ci} \) is contiguous, the signing may cause parts of this region to change signs. When such vortex segments pass through the viewing plane with significant inclination to the normal orientation, they exhibit a non-circular contour pattern at their intersection with the plane.
Figure 5.7: Streak A at $t = 252R/\U_{\text{bulk}}$ with $u - 0.83U_{\text{bulk}}$ vectors at coarse (6.75\textsuperscript{+}) spacing and (a,b) colour contours of $\omega_\theta$-signed 3D $\lambda_{ci}$ or (c) contours of $\omega_\theta$. Blue-green is negative, prograde vorticity and yellow-red is positive, retrograde vorticity.

177
By comparison to the hairpin vortex signature described above, the leftmost identified vortex in figure 5.7(a) includes a vortex core of the type in condition (i), but the core revealed by the $\lambda_{ci}$ contour suggests it is either not normal to the plane or is touching another vortex that is not. There is clearly low-momentum fluid below and upstream it, as (ii) indicates, and the inclination of the vectors appears consonant with the average angle specified by (iii), with the vectors becoming tangent to the wall as the wall is approached. Thus, this is generally consistent with the hairpin vortex signature, except for a distorted vortex core. The identified core is centred at $y/R \approx 0.16 \ (y^+ \approx 110)$, near the top of where a logarithmic layer might be expected to occur. Interestingly, there appears to be a stagnation immediately downstream, but in the section displayed there is none visible upstream.

The other two core candidates identified by arrows (orange) in figure 5.7(a) also include complex core shapes, with other nearby vortices apparently touching. This close proximity appears to also induce vertical components of velocity where $\lambda_{ci}$ suggests the core centres should be located. There are clearly regions of retarded negative velocity below and upstream these cores, but the furthest downstream core ($x/R \approx 4.05$) is a sizable distance above the wall at $y/R \approx 0.2 \ (y^+ \approx 137)$, and the negative fluctuation region below this core does not appear to extend to the wall. There are, however, other complex structures below as the wall is approached. Thus, the example cores in figure 5.7(a) are generally similar to the proposed hairpin vortex signature, but these examples are more complex and cluttered by other motions than the idealization. They also appear generally more complex than examples shown in the boundary layer experiments of Adrian et al. (2000b), which frequently show what appear to be simple vortex cores corresponding to heads.

The right two core candidates marked by arrows (orange) appear to be arranged in an approximate ramp, with additional regions of swirl that are consistent with other vortex cores present along the ramp indicated by the dashed line. This line
has a growth angle relative to the streamwise direction of 12°, and this agrees with the 12° reported in Adrian et al. (2000b). If there are four main vortex heads along the indicated ramp from $x/R = 3.55$ to $x/R = 4.25$, this would indicate a mean spacing of $160^+$ between vortices, in reasonable agreement with the roughly $200^+$ spacing suggested by Marusic & Adrian (2012) (the spacings may tend to be larger for higher Reynolds numbers and higher $y^+$ locations than those of the present example).

Another ramp early in its growth appears in the downstream continuation of this plane section, shown in figure 5.7(b). The approximate downstream extent of the ramp in figure 5.7(a) is depicted with the dashed line, but another dashed line is added for the ramp growing beneath the older ramp. This shorter ramp growing at an angle of 10° appears to contain two or three embryonic vortex cores, as well as a taller, more highly developed vortex core indicated by an arrow. This is probably the extent of the packet. Other vortices further downstream are arranged in an unclear pattern.

These layers of ramps are consistent with the uniform momentum zones discussed by Meinhart & Adrian (1995) and Adrian et al. (2000b), but do not contain as many hairpin signatures in a row, probably because of the lower Reynolds number in the present simulation. The layers are also consistent with the model of Marusic & Adrian (2012), which describes an idealized scale hierarchy with generations of hairpin packets. In this view, the first fully formed hairpins occur at about $100^+$. As they grow taller in packets, additional younger generations of packets grow below them. The limited Reynolds number $R^+ = 684.8$ of the present simulation presents the issue that the fully formed hairpin’s height corresponds to $y/R = 0.15$, nearly the upper limit of the approximate logarithmic layer where self-similar behaviour is to be expected, leading to limited possibilities for observing hairpin packets. In turbulent boundary layers, evidence of hairpin packets extends
above the logarithmic layer (Adrian et al., 2000b), but this behaviour could be different for turbulent pipe flows.

Near the wall, strong regions of azimuthal vorticity not associated with swirling motions (vortex cores) are present. Figure 5.7(c) displays the same planar region as figure 5.7(b), but colour contours are of azimuthal vorticity instead of swirling strength. These contours reveal a continuous layer (with nothing implied about azimuthal width) of azimuthal vorticity inclined along the ramp line identified from figure 5.7(b) emerging from the wall. This continuous region appears to terminate or pinch off at a location downstream the near-wall ramp origin by approximately 0.44R (at x/R = 4.44), closely touching a concentration of vorticity (x/R = 4.5 with a pointing red arrow) that has a more distinct vortex core velocity pattern. The translation velocity of this frame corresponds to the velocity at the centre of this core, and it is also 0.83U_{bulk}. There do not appear to be other vortices clearly belonging to this ramp further downstream. The ramp of vorticity leading up to this core appears consistent with the well-documented inclined shear layers (which are of relatively short spanwise or azimuthal width) identified in wall-bounded turbulent shear flows (Robinson, 1993; Johansson et al., 1991; Jeong et al., 1997).

Ramp-like velocity structures appear to be associated both with near-wall shear layers, apparently extending up to y^+ ≈ 70, and with vortical structures consistent with hairpin packets, extending greater distances from the wall. These are clearly visualized in contours of streamwise velocity to reveal uniform momentum zones, using a similar technique as Adrian et al. (2000b) (in their figures 14–17). For their turbulent boundary layer fields, they plotted contour level lines for streamwise velocities (true velocities, not fluctuations) at chosen intervals from a fraction of the free stream velocity U_{∞}. One example was plotted at (u − 0.82U_{∞})/u_{τ} = \{-4.8, −2.4, −2.0, 4.4\}. In figure 5.8, contour lines are drawn for the present pipe at (u − 0.88U_{bulk})/U_{bulk} = \{-0.8, −0.6, −0.4, −0.2, 0, 0.2, 0.4\}. For comparison, U_{∞}
Figure 5.8: Uniform momentum zones visualized by contours of \((u - 0.88U_{\text{bulk}})/U_{\text{bulk}} = \{-0.8, -0.6, -0.4, -0.2, 0.0, 0.2, 0.4\}\). The 0.4 line corresponds to the boundary between the lightest shade of grey and white. (a) visualizes the first half of streak A with the ramps drawn in figure 5.7 also included as long dashed thick black lines, and (b) visualizes the first half of streak B. Thick lines represent contours of azimuthal vorticity (dashed for prograde negative orientation), and thin lines represent three-dimensional \(\lambda_{ci}\). For a boundary layer might be most analogous to the mean centreline velocity of \(1.259U_{\text{bulk}}\) for the present pipe flow. Figure 5.8 also includes azimuthal vorticity \(\omega_\theta\) contours to visualize the shear layers and vortex cores oriented normal to the plane which are known to separate layers of uniform momentum zones (Meinhart & Adrian, 1995). In addition, contours of three-dimensional swirling strength \(\lambda_{ci}\) to indicate which regions of concentrated vorticity correspond to swirling motion (vortex cores).
Figure 5.8(a) includes the same regions visualized in figure 5.7 with the ramp lines also included. While the ramp lines were drawn solely based on the vortices observed in figure 5.7, they clearly correspond to the upper extents of uniform momentum zone ramps in figure 5.8(a). The contours agree with those of Meinhart & Adrian (1995) in strong vorticity regions indicating boundaries between layers of uniform momentum zones. However, some strong layers of vorticity also correspond to ramp-like boundaries of uniform momentum as revealed by the contour lines. The majority of these shear layers (without swirl indicating vortex cores) are limited to within about \( y^+ = 70 \) of the wall. Examples in figure 5.8 are common, with one inclined up to a tip at \( x/R = 3.75 \) and \( y^+ \approx 70 \) in figure 5.8(a).

However, figure 5.8 also indicates some regions of azimuthal vorticity without detectable swirling strength for \( y^+ > 70 \). Many such regions bridge between locations of azimuthal vorticity at which significant swirling strength is also present. This suggests that azimuthal vorticity without swirling strength at the same locations (and thus probably associated with detached shear layers) is present in organized arrangements with vortex tubes. This is not inconsistent with the hairpin vortex packet paradigm, and Adrian et al. (2000b) suggested that inclined shear layers could be expected upstream of a hairpin vortex where the Q2 ejection associated with the hairpin could collide with Q4 vectors. However, the colliding Q2 and Q4 events need not necessarily be associated with a shear layer, and other mechanisms could also explain the presence of these shear layers. Pirozzoli et al. (2010) decomposed the vorticity fields of a supersonic turbulent boundary layer simulation into vorticity associated with tube-like (swirling, as with vortex tubes) and sheet-like (purely shearing) contributions. From calculating the turbulent kinetic energy associated with each contribution, they found that the sheet contributions on average significantly dominated near the wall (especially \( y^+ < 50 \) where the hairpin model does not apply), but vorticity in vortex tubes made more comparable contributions.
further from the wall. The more recent study of the correlation between motions associated with sheets and tubes (Pirozzoli, 2011) indicates the motions are strongly associated and characteristics of four configurations are described. In the present pipe flow, the uniform momentum zone boundaries and ramp-like motions above approximately $y^+ = 70$ are clearly associated with vortex cores (swirling motions), so the present results are consistent with Meinhart & Adrian (1995) and Adrian et al. (2000b) in this regard. There does appear to be additional clutter that seems not as prevalent in Adrian et al. (2000b). For example, a higher-momentum zone penetrates near the wall for $x/R \approx 4.2$ and downstream, with a complex arrangement of nearby vortex cores and azimuthal vorticity, including some retrograde, in figure 5.8(a).

Overall, the changes in ramp height (radial extent) that correspond to the presence of individual vortex cores is consistent with the concept of very long scale negative $u$ fluctuations being composed of a concatenation of shorter motions. While the vortices themselves are each associated with short motions that arrange in a pattern such that longer ramps exist, the ramps are also shorter than the overall extracted streaks and are arranged in streamwise-concatenated patterns, with no information of azimuthal arrangement provided from this perspective. As each part of figure 5.8 corresponds to approximately the first half of the low speed streaks identified as A and B in figure 5.1, a number of these ramps on the order of $R$ in length are required in succession to compose the full streak length.

For comparison, an example from a particle image velocimetry experimental measurement of a $Re_D = 50 000$ ($R^+ = 1300$) turbulent pipe flow by Urushihara et al. (1993), is reproduced from Adrian et al. (2000a) in figure 5.9. Using the technique to identify vortices by changing the velocity of the frame of reference until the velocity vectors appear to have circular streamlines, a set of vortex cores were identified and labeled A-H. The vector field is shown with the appropriate
frame velocity to identify vortices relatively far from the wall. Additional analysis performed by Adrian et al. (2000a) on this field indicates that these are the largest concentrated regions of two-dimensional $\lambda_c$, so there are approximately 8 significant prograde (and no significant retrograde) vortex cores in this measured field with length $L_x^+ = 470$ and height $L_y^+ = 280$. There also is a significant region of azimuthal vorticity bridging vortex cores D and G, with possibly a smaller (unnamed) core in between. This may be an indication of a ramp. Otherwise, this field seems to have fewer vortices present above $y^+ = 100$ than the present simulation. Other visualizations of this flow in Urushihara et al. (1993) indicate azimuthal vorticity also appears in shear-layer-type structures extending upward from the wall to approximately $y^+ = 60$, with a detached region of concentrated vorticity consistent with a vortex core above and downstream the layer (their figure 6(a,b)). Additional examples of apparent inclined shear layers appear in other examples. Further above the wall, it is difficult to observe the existence of ramps of vortex cores growing at approximately $12^\circ$ to the wall, but one may exist in their figure 6(c,d), albeit with somewhat steeper growth angle. The limited streamwise extent of these images of $L_x^+ = 350$ and relatively small set of samples are not sufficient to make any strong conclusions, but the behaviour indicated by these pipe experiment velocity fields appears generally consistent with observations from the present DNS field, with the present pipe DNS perhaps appearing somewhat more complex with more retrograde vortices and with vortex cores in less circular cross-sections but possibly touching more frequently.

For streamwise–wall-normal planes in identified low-speed regions of the present DNS field, many more examples have been examined, and the examples discussed are typical of the behaviour. For example, in the streak A previously identified, of 10 possible vortex cores that are the best candidates for hairpin heads, only two are circular in cross-section. The features of these less convincingly match
Figure 5.9: Urushihara et al. (1993) experimental measurement of the $Re_D = 50000$ ($R^+ = 1300$) turbulent pipe flow, reproduced from Adrian et al. (2000a). A constant streamwise velocity of $0.75U_{cl}$ is subtracted. The centres of the visible vortex cores are identified by dots (identified D and F-H), as well as those identified with $0.5U_{cl}$ subtracted (A-C and E).

The idealized hairpin vortex signature than five other of the candidates (with more complex core shapes). Of these, two appear to be in a clear ramp organization with other head-like swirling regions suggestive of hairpin packets. Additional study of many more example long negative-fluctuation regions indicates that ramps are a relatively common arrangement, but there also are other less organized arrangements that don’t perceptibly conform to an identifiable pattern. For example, figure 5.8(a) contains several clear ramps consistent with the expected growth angle of approximately 12°, but in figure 5.8(b) it is difficult to find such arrangements of vortices. Thus, there is a significant amount of variation. To objectively quantify the average behavior of structures, statistics based on the presence of a vortex head provide an important perspective beyond visualizing individual instantaneous structures.
5.5.2 Estimate of Conditional Average for Swirl Event in the x–r Plane

As azimuthally-oriented vortices are commonly observed at the upper boundaries of uniform momentum zones, it is useful to compute the average fluctuating velocity field given one such vortex. Using $x$–$y$ planes measured via PIV for a turbulent channel, Christensen & Adrian (2001) estimated this average conditioned on the presence of an event of two-dimensional swirl ($\lambda_{ci}$) about a spanwise axis. Even if three-dimensional data is available, using three-dimensional $\lambda_{ci}$ directly as an event is not suitable because it implies nothing about the orientation of the swirling motion. One simple method to specify an event requiring swirling about the spanwise (or azimuthal) axis that would correspond to a vortex head is to condition based on two-dimensional $\lambda_{ci,2D}$ calculated in the streamwise–wall-normal plane. Christensen & Adrian (2001) includes further details of the linear stochastic estimate (LSE) technique that estimates this field with greater statistical convergence based on the unconditional correlation than would be possible by calculating a conditional average directly. The estimate is given by

$$
\langle u'_j(x', y') \mid \lambda_{ci,2D}(x, y) \rangle \approx \frac{\langle \lambda_{ci,2D}(x, y) u'_j(x', y') \rangle}{\langle \lambda_{ci,2D}(x, y) \lambda_{ci,2D}(x, y) \rangle} \lambda_{ci,2D}(x, y), \tag{5.1}
$$

which is closely related to the two-point correlation

$$
R_{iu}(r_{x,y}) = \frac{\langle \lambda_{ci,2D}(x,y_{ref}) u'(x + r_{x,y}) \rangle}{\sigma_i(y_{ref}) \sigma_u(y)}. \tag{5.2}
$$

Root-mean-square values of quantities are represented by $\sigma$, and the average represented by $\langle \cdot \rangle$ is computed by averaging over $x$, $\theta$, and time, which is consistent with the averaging performed for other quantities in Wu et al. (2012). The result of this procedure is an estimate of the fluctuating velocity field $u'_j$ that occurs when a point at the specified radius (or $y$) in the pipe has a particular value of $\lambda_{ci,2D}$. This estimate may be expressed as a scaled version of the two-point correlation in which the two velocity components
are expressed as $u(r_x, y) \approx R_{du}(r_x, y)\sigma_u(y)\left[\lambda_{ci,2D}(0,y_{ref})/\sigma_{\lambda}(y_{ref})\right]$ and $v(r_x, y) \approx R_{dv}(r_x, y)\sigma_v(y)\left[\lambda_{ci,2D}(0,y_{ref})/\sigma_{\lambda}(y_{ref})\right]$, relative to an event at streamwise displacement $r_x = 0$ and $y = y_{ref}$. For comparison with Christensen & Adrian (2001), the LSE for the present pipe is performed with an event of two-dimensional $\lambda_{ci,2D}$ that is unsigned, and therefore $\lambda_{ci,2D} \geq 0$. Christensen & Adrian (2001) notes the magnitude of the LSE is proportional to the event strength, so the character of the LSE to the conditional average remains the same for any non-trivial event, regardless of magnitude.

For a $\lambda_{ci,2D}$ event specified at $y_{ref} = 0.15R$ (and $y^+_ref = 101$), the linear stochastic estimate of velocity fluctuation associated with the conditional average is plotted in figure 5.10(a). To visualize the vectors over a long streamwise extent, they are interpolated onto a uniformly-spaced grid with significantly coarser resolution than the computation. As with Christensen & Adrian (2001), the vectors are plotted with uniform length (independent of velocity magnitude) to reveal the weaker motions. The presence of the ramp growing with an angle consistent with the angles observed in hairpin packets is similar to the results of Christensen & Adrian (2001) for their turbulent channel experiment with the event at $y/h = 0.15$, and a significantly longer downstream domain is displayed for the present pipe. However, no vector patterns clearly corresponding to hairpin heads along the ramp downstream the event are observed for the present pipe, in contrast to those Christensen & Adrian (2001) found in their results as evidence of hairpin packets with preferred hairpin spacings.
Figure 5.10: Linear stochastic estimate based on an unsigned 2D swirl event located at $y/R = 0.15$ and $y^* = 101$. Velocity vectors with constant vector magnitudes to reveal the ramp pattern in (a), vector lengths proportional to the velocity magnitude reveal the strength of the motion in (b), and $\lambda_{\psi,2D}$ is estimated instead of velocity in (c). This LSE is closely related to the two-point correlation, and contour lines for correlation coefficient values of 0.2 decreasing in increments of 0.01 are drawn, with 0.00 omitted and negative values shown with dashed lines.
There are several possible explanations for the lack of additional vortex cores appearing along the ramp in the present results. The resolution of figure 5.10(a) makes the finest-scale motions difficult to discern, but the vectors were examined at finer resolution, and no clear vector patterns similar to those of a vortex (hairpin head) were observed. In Christensen & Adrian (2001), the vortex patterns for the additional swirling motions identified upstream and downstream the event were generally smaller than the swirling motion associated with the event. In figure 5.10(a) some instances of vectors appearing to converge in upward motion are apparent upstream the event (for $r_x < 0$), but the appearance of these features is related to the resolution of the vectors. The ramp that is visible occurs at locations in which the streamwise component of velocity fluctuation changes sign (crosses zero). Since the vectors each have unit length and there is a weak upward velocity calculated for these regions, the vectors point directly upward where $u$ approaches zero. This occurs in a broader region in the ramp downstream the event, and the resolution is sufficient to better visualize this motion. In the upstream region, $u$ having a value very close to zero only occurs at relatively few of the points along the ramp on which the vectors are shown, and this leads to the clusters of vectors that appear to converge upward with relatively consistent streamwise spacing on the upstream side of the event.

Another possible explanation for the lack of additional vortex cores in the present pipe results is that the vortices that appear in other studies would be weakened with greater statistical convergence, and the two-point correlations for the present pipe DNS are converged to a greater degree. As previously noted, the ramp occurs where the estimated $u$ component is zero, but the vectors are uniform length so their directions are a result of the $u$ and $v$ velocity components in these regions. Therefore, they are very sensitive to any noise in these correlations surrounding these locations. The relative weakness of the velocity magnitude in these regions
is indicated by figure 5.10(b), in which the vector lengths are proportional to the velocity magnitudes. To compare convergence with the experiments of Christensen & Adrian (2001), it is noted that they collected 3500 separate (statistically independent) fields that are each \( h \) (one channel half-height) long in streamwise extent. For the present pipe simulation, about 90 fields widely enough spaced in time to be essentially statistically independent in terms of vortex details were used for this calculation. All of the azimuthal planes for each field with length of \( 30R \) are included in the calculation, but each azimuthal plane for a given field is not independent (there is strong correlation between neighbouring azimuthal planes). From Wu et al. (2012), \( |R_{uu}(\Delta \theta)| < 0.05 \) for all \( |\Delta \theta| > 1 \) (radian). Requiring this spacing between each plane to be assumed independent as a conservatively small estimate, there are 6 statistically independent \( x-y \) planes per 3D field (6 statistically independent azimuthal angles). This corresponds to 16 200\( R \) of independent streamwise length, as compared to 3500\( h \) for the channel experiment of Christensen & Adrian (2001). Assuming equivalence between \( h \) and \( R \), this suggests the present pipe results are significantly more statistically converged than the channel experiment results. In addition, the exact periodicity of the DNS also improves the convergence of the correlation, as every possible periodic streamwise shift is implicitly included in the statistics.

Lee & Sung (2011) computed a similar LSE for a turbulent boundary layer DNS with a swirl event at \( y/\delta = 0.18 \) and observed the presence of swirling vortex cores along the ramp that grew with distance downstream the event location. This demonstrates that such packet evidence can also observed in numerical simulations. Since the pipe includes the statistically-homogeneous \( x \) coordinate in the averaging, the lack of this homogeneity in the boundary layer may decrease the statistical convergence, but no clear comparison of the degree of convergence between these flow simulations can be made. Lee & Sung (2011) indicate that the spacings to
the additional vortex cores nearest the event are approximately $0.3\delta (160^+)$ and that these are in general agreement with the channel values of Christensen & Adrian (2001), $0.3-0.4h$, as well as with other values in literature. Christensen & Adrian (2001) also noted that the spacings between vortex cores observed within the ramp remained very consistent (when scaled in outer units, by $h$) between $Re_\tau = 547$ and 1734, providing evidence that these patterns are indicative of organization in the flow and are not random noise. It should be noted that Christensen & Adrian (2001) observed the swirling patterns appeared smeared in the streamwise direction and suggested that this could be a result of the variations in streamwise spacings between vortices for each of the various packets.

Another possibility to explain the absence of clear additional vortices in figure 5.10(a) is the differences in the flow due to the different geometries (channel, boundary layer, and pipe) and due to the numerical simulation in contrast to experiment. While observed in channel experiments (Christensen & Adrian, 2001) and boundary layer simulations (Lee & Sung, 2011), no other results are presently available for these features in turbulent pipe flows. While there is a possibility that the difference could be due to the pipe geometry, it could also be due to the streamwise periodicity enforced in the periodic pipe domain (but not in the experimental channel or boundary layer simulation) or another aspect of the numerical solution. As previously noted, Christensen & Adrian (2001) suggested variations in streamwise spacings between vortices could cause the smearing noted in the streamwise direction for the vortices in the LSE. Simply a more uniform distribution of spacings (lack of preferred vortex locations) for vortices within packets of the present pipe flow relative to the other flows may thereby smear the vortices to a greater degree in the streamwise direction, possibly to the point that they do not appear as distinct vortices. A similar highly-converged calculation for a pipe flow experiment would
be necessary in order to conclusively determine the cause of the lack of distinct additional vortices in the present LSE.

Aside from the lack of other vortices, the ramp in figure 5.10(a) is generally consistent with the results of other studies of wall-bounded shear flows. Compared to the present 11° ramp growth angle, Christensen & Adrian (2001) (in a channel experiment) found a growth angle of 13–14° and Lee & Sung (2011) (boundary layer DNS) also observed a growth angle of 13–14°. Hambleton et al. (2006), in a PIV experiment of a turbulent boundary layer, calculated the LSE based on a condition specified with a modified version of two-dimensional swirling strength in which only the $\lambda_{ci,2d}$ at locations of prograde out-of-plane vorticity (and at which swirling strength was greater than a weak threshold) was included. This discriminated against retrograde vortices, but the ramp angle measured remained similar to other results at approximately 13°. Especially near the wall, prograde vortices are prevalent (Wu & Christensen, 2006), but specifying simply $\lambda_{ci,2d}$ as a condition would include some contribution from retrograde vortices. For the present pipe DNS, the LSE yields similar results when signed $\lambda_{ci,2d}$ is used instead to specify the condition. It is interesting to note, as is apparent in Hambleton et al. (2006), that the ramp lines (along which the streamwise velocity fluctuation changes sign) for the present DNS appear to connect to the upper side of the event vortex for the upstream ramp and to the lower side of the event vortex for the ramp downstream the event. For the present pipe DNS, this appears slightly stronger when signed $\lambda_{ci,2d}$ is used (not shown). Hambleton et al. (2006) did not indicate the presence of any additional vortices along the ramp (in addition to the event vortex), although some evidence was present of additional vortices in a lower streamwise–spanwise plane that was also obtained, in addition to the vortices expected to be due to the legs of the hairpin with its head at the event. In the present pipe DNS, the length downstream at which the ramp in the LSE extends is remarkable, and this is consis-
tent with very long streamwise motions, although the magnitudes of such motions are very weak in this estimate. Perhaps as a consequence of the details of other vortices being washed out by the averaging, the ramp appears as a shear layer.

Figure 5.10(b) includes a subset of figure 5.10(a) near the swirl event, but with the vector lengths proportional to the velocity fluctuation vector magnitudes. This presentation indicates the strengths of the various regions, and shows that this estimate of a conditional average matches the hairpin vortex signature description of Adrian et al. (2000b), in that there is an inclined line of Q2 vectors that are much stronger than the vectors above the core, for instance, and a relatively long streamwise region of retarded fluid momentum below the core. In addition to the effect of the vortex head associated with the event, this retarded fluid may also be associated with other coherently organized vortices (as in a packet), though the details of such vortices are eliminated by the averaging.

While the presence of distinct additional vortices are not observed in this estimated conditional average, the conditional average of \( \lambda_{ci,2d} \) conditioned on a \( \lambda_{ci,2d} \) event would indicate the preferred locations of vortices in this plane. Contours of the two-point correlation \( R_{\lambda\lambda} \) (using \( \lambda_{ci,2d} \)) for which the LSE is a rescaled version indicate a weakly greater probability of other azimuthal vortex cores occurring along a line inclined at approximately 10° in figure 5.10(c). The strongest correlation contour levels are related to the average core diameter of the vortex specified by the event, as the condition only specifies a single point so it averages over all points of strong \( \lambda_{ci,2d} \) fluctuation in vortex cores. As indicated by these contours, the cores, on average, are spatially compact. Immediately upstream and downstream these high contour levels, regions of lower or negative \( \lambda_{ci,2d} \) fluctuation exist, which indicates a relative rarity of other vortices directly touching in these regions. The relevant features are the regions of positive fluctuation upstream and downstream the event location, indicating a higher frequency of vortex cores.
(with orientations corresponding to hairpin heads) appearing in these regions. The highest contour levels may be inclined at slightly steeper angles than the 10° indicated, and the downstream contour appears centred about approximately 0.3R from the event, which is consistent with the vortex spacings reported by Christensen & Adrian (2001). These contour levels are, however, weak at correlation coefficient values of 0.02 and 0.01, so this implies a weak preference for these locations. It should be noted that calculating $\lambda_{ci,2d}$ is nonlinear, and therefore the LSE of $\lambda_{ci,2d}$ conditioned on $\lambda_{ci,2d}$ is not equivalent to computing $\lambda_{ci,2d}$ of the velocity fluctuation field obtained by LSE for a $\lambda_{ci,2d}$ condition. Thus, the LSE for velocity fluctuation in figure 5.10(a,b) could wash out details of the vortices while the LSE for $\lambda_{ci,2d}$ in figure 5.10(c) does not.

5.6 Vortices and Structures in 3D Volumes

Having established the presence of vortices above long structures of low streamwise velocity and having found that the character of these structures is generally consistent with hairpin vortices, attention is next turned to the properties of three-dimensional vortices and how they relate to low-streamwise-velocity structures present in the pipe flow. As a simple indicator of the presence of vortices bounding the long low-speed regions azimuthally, figure 5.11(a) displays velocity fluctuation contours at constant wall-normal distance $y$ with three-dimensional $\lambda_{ci}$ contours also included. This region is also the region from which streaks A and B were identified. Visually, vortical motions (with cores identified by $\lambda_{ci}$) appear to be strongly associated with the azimuthal boundaries between low velocity (dark) and high velocity (light) regions, consistent with the idea that trains of hairpins induce negative velocity fluctuation to the fluid within their boundaries, and that the plane shown slices through the necks of these hairpins. However, this view is insufficient to indicate whether the vortices are similar in geometry to idealized hairpins with
pairs of vortex cores on either side of the low-velocity regions connected by relative-
ively simple vortices that also correspond to cores above the low-velocity regions as hairpin heads or if the vortices appear in more complex arrangements. To evaluate these possibilities, the vortices and their relations to low-velocity structures are examined in three dimensions. It should also be noted that there is a relative absence of vortex cores indicated in other regions, so this supports the emphasis placed on understanding the vortical structures surrounding the negative-velocity regions.

Visualizing the negative velocity fluctuation isosurfaces to display streaks A and B as in figure 5.6, figure 5.11(b,c) also includes isosurfaces of $\lambda_{ci}$ (from the full three-dimensional velocity gradient tensor) to display the relatively strong vortices. The arrangements indicate that vortices frequently surround the low-speed regions in three-dimensions, as expected from the two planar views studied. Vortex strengths on average weaken with increasing distance from the wall above the logarithmic region, and consequently the amount of vortices visible with swirling strength exceeding the visualization threshold decreases. This is consistent with the relative lack of vortices visible above the negative $u$ fluctuation isosurfaces. This could be one reason that it is difficult to directly observe complete, ideal (full loop) hairpin shapes. The vortices visible in figure 5.11(b) have geometries consistent with hairpin legs that are largely streamwise-oriented near the wall and turn upward (wall-normal) as they grow up to the neck region, but heads are often not easily discerned. At $x/R \approx 4$, there appears to be a vortex that contains a more convincing head-like geometry. Beneath this vortex, this perspective clearly displays the ramp-like growth of the negative $u'$ fluctuation region for streak A previously noted, and this region is indicated by the dashed line. Other ramp-like regions of negative $u'$ fluctuation are also visible in figure 5.11(b,c). In figure 5.11(c), lines indicate the boundaries corresponding to the section of a cylinder centred at the pipe.
Figure 5.11: (a) A section of figure 5.1 at $y/R = 0.15$ and $y^+ = 101$ with red contour lines of three-dimensional $\lambda_{ci}$ indicating vortex cores that are concentrated at the edges of low speed regions. The section surrounds streaks A and B, which are also visualized in figure 5.6. Red $\lambda_{ci}$ isosurfaces surrounding cyan $u' = -0.21U_{\text{bulk}}$ isosurfaces are visualized in the regions identified for (b) streak A and (c) streak B. In (c), the additional boundary line at constant radius is at $y/R = 0.15$ and $y^+ = 101$, which is the radius shown in the flattened cylinder section of (a).
axis with a radius of $r/R = 0.85$, marking the location of $y/R = 0.15$ and $y^+ = 101$. It is observed that the vortices resembling hairpin legs mainly occur below this height, whereas above this location, the segments of vortices connected to these legs are largely oriented wall-normal upward and other vortex segments with connectivity to other vortices that cannot be determined from this visualization also occur with more random orientations. These visualizations show that these short segments would require a lower threshold for these vortices to be more clearly examined, but uniformly lowering the swirling strength isosurface threshold would add clutter to the region near the wall.

To more clearly visualize the vortices further from the wall and understand the relation of these structures to the long streaks of negative velocity fluctuation, figure 5.12 displays isosurfaces of $\lambda_{ci}$ normalized by its root-mean-square (rms) value, $\sigma_{\lambda}(y)$. As the rms value decreases with increasing $y$ for radii approaching the core region, the weaker vortices are emphasized by this normalization. This procedure of normalizing vortex identification quantities to expose the vortices relatively far from the wall has been employed in other studies of turbulent flows (Nagaosa & Handler, 2003; del Álamo et al., 2006). This viewing angle clearly indicates the presence of vortices on the azimuthal boundaries of the negative-velocity-fluctuation streaks and also shows that vortices tend to congregate at the same streamwise and azimuthal positions even as $y$ changes, although clear connections between vortices are difficult to discern even when the vortices further from the wall are brought out more strongly. The trains of vortical structures arranged with streamwise coherence are consistent with the general notion of hairpin packets, and figure 5.11(b,c) suggests that growing (in a wall-normal sense) ramps of vortices occur with significant frequency.

As previously noted, turbulent boundary layers were significant in developing the hairpin packet model (Adrian et al., 2000b; Adrian, 2007). Among direct nu-
Figure 5.12: Section surrounding streak A (previously identified) displaying white isosurfaces of normalized $\lambda_{ci}/\sigma_{ci} = 3$ with a plane at $y^+ = 59$ coloured by streamwise velocity fluctuation (blue is negative and red in positive) and black contour lines of $\lambda_{ci}$ on this plane to indicate the presence of vortex cores. To focus on roughly the logarithmic law region and avoid confusion from other structures near the core, the domain extends only up to $y/R = 0.34$ and $y^+ = 235$. 

198
numerical simulations of fully turbulent wall-bounded flows, several recent DNS studies of turbulent boundary layers by Wu & Moin (2009a), Wu & Moin (2009b), Wu & Moin (2010), and Wu (2010) have included the clearest visual evidence of the presence of dense forests of hairpin vortices populating the flow fields, with relatively pristine and symmetric hairpin vortices commonly present. This may be related to these simulations allowing the flow to undergo a realistic transition process with no streamwise periodicity (in contrast to channel and pipe flow simulations). However, other turbulent boundary layer simulations using different methods do not report observing clear hairpins (Jiménez et al., 2010) or do not observe them at the highest Reynolds numbers (Schlatter et al., 2010). Evidence of vortices consistent with hairpins arranging into packets has also been reported in other turbulent boundary layer simulations (Ringuette et al., 2008; O’Farrell & Martin, 2009). Dennis & Nickels (2011a) found their pseudo-three-dimensional measurements of a turbulent boundary layer to reveal structure consistent with hairpin packets in an averaged sense, but instantaneous structures varied from the idealized structures. Their figures 6–8 provide an interesting comparison of hairpin-like vortices hugging low-speed structures that they observed to the relationship revealed between vortices and low-speed structures in visualizations of the present flow. In addition, evidence of packets in turbulent channel DNS was also reported by Adrian & Liu (2002). While hairpins were particularly clear at the upper extent of the boundary layer for Wu & Moin (2010), it is generally more difficult to visualize hairpins at lower layers because they are frequently tangled with other vortices and covered by taller layers of vortices. Wallace et al. (2010) visualized such vortices closer to the wall in this same flow by plotting vortex lines and successfully extracted vortices consistent with those extracted using the $Q$ criterion (vortex identification quantity). The study of Chakraborty et al. (2005) indicates that popular vortex identification quantities including $\lambda_{ci}$ and $Q$ yield similar results in turbulent flows. While a de-
tailed discussion of the possible effects to vortical structures with simulations of different canonical wall-bounded turbulent flows is beyond the scope of the present study, the application of plotting vortex lines for these flows suggests that it would be informative to also analyze fields of the present pipe simulations via this method.

5.6.1 Core Tracing

The vortex line method for identifying vortices successfully applied by Wallace et al. (2010) had been previously used in other wall-bounded flows, such as Moin & Kim (1985) for a turbulent channel simulation. They noted that vortex lines are defined in three-dimensional space by

\[
\frac{dx}{ds} = \frac{\omega}{|\omega|},
\]

(5.3)

where \( s \) is distance along the vortex line. Vortex lines \( x(s) \) were drawn by selecting a starting location \( x_0 \) and integrating (5.3). Moin & Kim (1985) noted that the results were very sensitive to the choice of \( x_0 \), but suggested that if \( x_0 \) were placed inside a vortical structure, then the resulting line would probably be confined to the core for a long distance \( s \). They selected the starting positions based on planar plots displaying velocity vectors and vorticity. They also used sets of closely-located vortex lines and suggested that the lines remained adjacent to each other when in a hairpin but diverge farther away. Overall, they found this method to “clearly display the three-dimensional structure of the hairpins.” However, Robinson (1991) found that vortex lines in the vicinity of true vortices can produce very misleading results, including hairpin-shaped vortex lines without the presence of hairpin-shaped vortices. Robinson (1991) found that “unless the starting point for the integration of the [vortex] line is chosen almost precisely within the vortex core, the [vortex] lines will trace out well-defined upright hairpins.”

While \( \lambda_{ci} \) is a scalar quantity obtained from the imaginary part of the complex conjugate pair of eigenvalues when they exist for the velocity gradient tensor, the
real eigenvector that exists under the same conditions is known to trace the ‘spine’ of the vortex tube being identified (Zhou et al., 1999; Zhou, 1997). Gao et al. (2011) found that a significant difference in direction between the vorticity vector and this real eigenvector existed in wall-bounded turbulent flows. For example, a straining motion in the direction of the mean shear (about an azimuthal axis) that is itself not associated with swirl but is added to a swirling motion would lead to a more azimuthal orientation for the vorticity vector than for the real eigenvector that is only sensitive to the swirling motion. Pirozzoli et al. (2008) also studied the differences in directions of vorticity vectors and real eigenvectors in a turbulent boundary layer flow. Zhou et al. (1999) discussed possible drawbacks of using vortex lines after noting that Bernard et al. (1993) observed in DNS of channel flow that vorticity vectors were more inclined than the vortex structures.

Despite these issues, the more recent work of Wallace et al. (2010) found that, when the points from which vortex lines were started were chosen judiciously, vortex lines do appear to identify vortices when the lines appear in spatially coherent bundles. Wallace et al. (2010) also noted that this method has the benefits of not requiring any detection threshold and allowing a particular vortex to be specified. Since it has been observed for the present pipe flow that the vortices tend to weaken as the wall-normal distance increases above the logarithmic-layer region, the independence from needing to specify a fixed threshold or selecting a normalization that varies with y makes the vortex identification less subjective. As the definition in (5.3) makes clear, the vortex lines depend only on the direction of vorticity and are independent of vorticity magnitude. Wu (2010) also used vortex lines to show the existence of hairpins in another boundary layer simulation.

An example of tracing vortices around a section of negative-velocity-fluctuation streak A is shown in figure 5.13. To focus on the region of interest, the core region of the pipe is not shown, and an angular arc of approximately 36° is shown. In the
constant-y cylinder surface included with colour contours of velocity fluctuation and contour lines of $\lambda_{ci}$ (also clearly shown in figure 5.12), the peaks in concentric contour circles of strong $\lambda_{ci}$ marking vortex cores located at the boundary of the negative $u'$ fluctuation streak are used as starting points ($x_0$ positions) for computing vortex lines. This is a different choice that that of Wallace et al. (2010), in which vortex lines were started at locations identified within vortex heads. Lines originating at the presently suggested starting locations are expected to follow legs down to the wall and, if the vortex geometries closely resembled those of hairpins, to continue away from the wall following the neck, turn following the head, and turn downward to follow the opposing leg until near the wall. Vortex lines do not terminate unless they reach the end of the domain, so the vortex lines are terminated when the swirling strength $\lambda_{ci}$ falls below a chosen value. This value is chosen by creating vortex lines based on closely spaced bundles of starting points within the identified core of a vortex (by $\lambda_{ci}$ contours on the cylinder above the wall) and determining the value of $\lambda_{ci}$ for which the lines remain closely bundled. When the swirling strength is too low, the vortex lines diverge apart as they are no longer following a vortex. While this step does introduce a threshold, it is a much weaker threshold swirling strength than for the isosurface visualization and the line termination is not sensitive to this threshold. The advantage of this approach is that the strong vortices of interest are included and are followed as they weaken, while other weaker vortices that would clutter the view are not visualized, as they would be if isosurfaces with a very low threshold were shown for the field.

Figure 5.13(a) includes bundles of vortex lines (light blue) originating at the cores of strong swirling strength surrounding the negative $u'$ fluctuation streak on the $y^+ = 59$ cylinder section, and red isosurfaces of $\lambda_{ci}$ at a constant $\lambda_{ci}/\sigma_\lambda$ value are also shown to compare vortices identified in this manner with the paths of the vortex line bundles. Examining this field from other angles shows reasonable agreement
in the regions where the swirling remains strong following the isosurface tube path, but other regions with isosurfaces (enclosing regions stronger than the isosurface threshold) appear above the cylinder for which the isosurfaces do not extend down to the cylinder section. Of the vortex line bundles originating at the cylinder section, some do not have $\lambda_{ci}$ isosurfaces extending significantly above this wall-normal location, indicating that the vortices weaken rapidly with $y$.

To visualize the relationship of these vortices with the negative $u'$ fluctuation streak, figure 5.13(b) shows the same vortex lines (recoloured from light blue to red) as in figure 5.13(a), but the isosurfaces (blue) are of negative $u'$ fluctuation. Figure 5.14 shows the same field from an angle that better displays the vortex bundles bounding the negative velocity fluctuation in an azimuthal sense. The vortex bundles often appear directed approximately radially upward from the wall until the local swirling strengths become weak and the lines diverge, instead of remaining bundled inside strong vortices turning to form vortex heads. This perspective clearly indicates that these vortices bound the negative $u'$ fluctuation streaks in an azimuthal sense, but the vortices generally do not appear to have heads, and the legs on opposing sides of the streaks are not clearly connected across the streaks.

Since heads are not frequently found in the present flow by tracing vortex lines from the leg or possible neck region but evidence of vortex cores with head-like orientations bounding the upper extents of negative velocity regions was found from streamwise–wall-normal planes in §5.5.1, the vortices associated with these features can be visualized using vortex lines originating from these cores. As previously noted, originating vortex lines at these head locations was the approach employed by Wallace et al. (2010) for studying hairpins in turbulent boundary layers. Although vortices identified in this manner in the present pipe simulation often are roughly azimuthally-oriented across the azimuthal width of the negative-velocity fluctuation region but then apparently diffuse without inclining toward the wall.
Figure 5.13: Section surrounding negative-velocity fluctuation streak A displaying (a) vortex lines to trace cores compared to scaled $\lambda_{ci}$ isosurfaces and (b) $u'$ isosurfaces with vortex lines. The region near the core is removed to better visualize near the wall. Vortex lines originate on the $x$–$\theta$ cylinder at $y^+ = 59$ from cores based on strong $\lambda_{ci}$ indicated by contour lines. Color contours of $u'$ on this cylinder section are also shown, ranging from blue (strongly negative) to red (strongly positive).
Figure 5.14: Section surrounding negative-velocity fluctuation streak A displaying $u'$ isosurfaces with vortex lines, as in figure 5.13 but from a viewing angle to compare the azimuthal positions of the vortex line bundles with those of negative velocity fluctuations. The region near the core is removed to better visualize near the wall. Vortex lines originate on the $x$-$\theta$ cylinder at $y^+ = 59$ from cores based on strong $\lambda_{ci}$ indicated by violet contour lines. Grey contours of $u'$ on this cylinder section are also shown, ranging from dark (strongly negative) to light (strongly positive).
as a strong vortex core, in some cases they do return to the wall in a full-loop, hairpin-like geometry. One such example is included in figures 5.13 and 5.14 as the orange bundle of vortex lines. The vorticity along the bundle weakens when the vortex lines intersect the $y^+ = 59$ plane, and this is why this bundle was not identified by originating vortex lines at the strongest cores at this plane. No other similarly-convincing examples of comparable wall-normal height were found for this particular streak.

These results appear consistent with vortex shapes observed in other studies performing DNS of channels and turbulent boundary layers at relatively high Reynolds numbers. Motivated by earlier proposals and refinements of an attached eddy model for turbulent motions bounded by a wall (Townsend, 1976; Perry & Chong, 1982; Perry et al., 1986, 1995), del Álamo et al. (2006) extracted vortices from an $Re_\tau = 1900$ turbulent boundary layer simulation and divided the contiguous vortices extending near the wall, which they termed attached, from detached vortices that do not extend within $y^+ = 20$ of the wall. Of the attached vortices, they also selected those for which the vortices extended wall-normally above $y^+ = 100$ and termed these the ‘tall attached clusters’. They found arches, as expected for idealized hairpin shapes, to sometimes occur, but the vortex geometries were generally more complex in many configurations. In their DNS of a supersonic turbulent boundary layer, Pirozzoli et al. (2008) observed “hairpin-shaped vortices, occasionally arranged in packets”, but also noted that “hairpin vortices with a well-defined distinction between counter-rotating legs and a spanwise head are rather rare.” Dennis & Nickels (2011a) experimentally measured flow in a turbulent boundary layer (sub-sonic) and found some near-‘archetypal hairpins’ in their pseudo-three-dimensional fields, but generally found that the vortices were “much less idealised, and could be described as arches, canes and indeed, plenty of other obscure shapes.”
As previously discussed, the vortices in the present pipe simulation include entities similar to legs that rise up from the wall but do not clearly form heads that then connect to opposing legs and entities similar to heads that either diffuse instead of descending to the wall or weaken as they form a neck turning to the wall. This appears consistent with the complex wall-attached vortices extracted by del Álamo et al. (2006) that frequently did not include a clear arch (head) and the smaller detached vortices that they also observed. These detached vortices may also have included the vortex segments associated with head-like vortex cores as identified in $x-r$ planes of the present pipe simulation. It appears that features of hairpin vortices (i.e., legs, heads, etc.) exist in the present pipe simulation but are not as ideally or simply connected as in the vortices of the turbulent boundary layer simulation of Wu & Moin (2010). While vortex geometries must be inferred when drawing conclusions from only two-dimensional planes, leading to uncertainty in the true geometries of vortices if no three-dimensional information is available, the adherence of other features of the flow structure provided evidence agreeing with the relatively simple vortex forms inferred from planar experimental data in Adrian et al. (2000b). Similar experiments in pipes, discussed with regard to figure 5.9, appear to reveal simpler velocity field patterns than similar fields of the present simulation. It is possible that the simulation could contain more complex structures than the flow in experiments or that the structures in experiments are more complex than would be anticipated from viewing planes. There are a number of factors that could potentially affect structures in simulations, including periodic boundary conditions (as more ideally-shaped hairpins were more readily observed in the simulation of Wu & Moin (2010) that included a non-periodic inflow, even at the highest Reynolds numbers included) and effects of simulation resolution on how near-wall vorticity may roll up into vortices (as few as 3–6 grid points may span across a vortex core diameter for which $\lambda_{ci} > 2\sigma_\lambda$). Conversely, overall statistics
match closely between simulation and experiments, as shown in Wu et al. (2012), strongly suggesting the behaviours of both are very similar. Another possible factor is Reynolds number, as self-similar structures (hairpin vortices) are proposed to occur in the logarithmic layer region, and the Reynolds number of the present simulation results in a relatively thin layer approaching adherence to a logarithmic law. If multiple generations (resulting in multiple wall-normal layers) of packets are necessary, with fully-formed hairpins suggested to occur when heads reach $y^+ \approx$ by Marusic & Adrian (2012), the approximate top of the logarithmic layer being expected to occur at $y/R = 0.15$ corresponds to $y^+ = 100$ in the present flow, allowing little space for multiple layers. However, Adrian et al. (2000b) found ramps consistent with hairpin packets to grow significantly above the logarithmic layer in turbulent boundary layers.

5.7 Streamwise-Azimuthal Organization of Structures

Having considered the form of structures of streamwise velocity fluctuation with various streamwise lengths in the pipe simulation and their relationship to vortical structures, with the long negative velocity streaks resembling the structures associated with hairpin packets (though the individual vortices may vary considerably from the idealized hairpin model), the organizational patterns of the long streaks are now considered. One method of extracting structure is through conditional averages, as approximated by a LSE in §5.5.2 to obtain the average velocity field associated with a point of azimuthal swirling motion. Since long streaks of negative $u'$ velocity fluctuation are very important structures observed to commonly exist, this suggests a conditional average based on a strongly negative $u'$ fluctuation would be informative. However, there clearly is a significant amount of variation among such structures that are observed in the flow, and this must be considered when interpreting the results of an average.
5.7.1 Conditional Average: General $u'$ Event and Two-point Correlation

For events specified as specific $u'$ values, the conditional averages of the velocity field in an $x$–$\theta$ cylinder for a specified $y$ or radius (not explicitly indicated) $\langle u'_j(x', \theta') \mid u'(x, \theta) \rangle$ can be approximated with linear stochastic estimates by

$$\langle u'_j(x', \theta') \mid u'(x, \theta) \rangle \approx \frac{\langle u'(x, \theta)u'_j(x', \theta') \rangle}{\sigma_u^2} u'(x, \theta),$$

(5.4)

which, for the streamwise velocity component, is closely related to the two-point correlation

$$R_{uu}(\Delta x, \Delta \theta) = \frac{\langle u'(x, \theta)u'(x + \Delta x, \theta + \Delta \theta) \rangle}{\sigma_u^2}.$$  

(5.5)

Thus, the LSE is simply a scaled version of the two-point correlation.

5.7.2 Other Studies of Correlation in Wall-bounded Turbulence

A number of other studies have analyzed streamwise–spanwise planes in describing turbulence structure. One with a particular focus of understanding long, wavering superstructures of streamwise velocity and their effects is Hutchins & Marusic (2007b). They noted the distinctive X pattern in the two-point correlation and showed that sinusoidal wavering in superstructure streaks could cause this.

Other studies include Ganapathisubramani et al. (2006) in turbulent boundary layers, with two-point correlation results indicating parallel negative $u$ correlation streaks separated from the positive streak by $2\delta$ spanwise distance. Volino et al. (2007) also extensively studied two-point correlations in turbulent boundary layers. In streamwise–spanwise planes of a turbulent boundary layer simulation, Lee & Sung (2011) calculated two-point correlations and also conditional two-point correlations conditioned on the sign of streamwise velocity fluctuation. Delo et al. (2004) used two-point correlation of a passive scalar (smoke in a visualization) to investigate structure in a low-Reynolds-number turbulent boundary layer.
Elsinga et al. (2010) used two-point correlation of swirling strength $\lambda_{ci}$ to investigate the streamwise-spanwise organization of vortices in a supersonic boundary layer. Hutchins et al. (2011) calculated conditional averages with attention given to the wavering of structures and skin friction events.

5.7.3 Conditional Average: General $u'$ Event and Two-point Correlation Results

A series of two-point correlations described in §5.7.1 for several $y$ positions are displayed in figure 5.15. These examples have contour levels chosen to clearly display low-level correlations. As shown by figure 5.1, $u'$ fluctuations contain many fine scales that would be expected to lose correlation with small spatial displacements. This is also apparent from figure 15(b) of Wu et al. (2012), in which the total turbulent kinetic energy in $u'$ (which is related to the space-time correlation at zero displacement $r_x = 0$ and zero time delay) for $y^+ = 101$ is reduced to 19% of the unfiltered value when a filter is applied that removes the fine scales to only include very-large-scale motions. These observations suggest that, given a strong velocity fluctuation event, the average fluctuation strength would decay rapidly with displacement, but more rapidly for azimuthal displacement than streamwise displacement due to the commonly streamwise-aligned and streamwise-extended form of the fluctuations. Since strong fluctuations contribute significantly to the two-point correlation, it is expected to behave similarly. Figure 5.15(e) includes an axis to represent the two-point correlation $R_{uu}$ as a surface, and this presentation clearly shows the rapid decay. Therefore, if one wishes to focus on the larger scales of motion and their correlation, it is necessary to focus on smaller correlation coefficient values, since much of the correlation is associated with the very fine-scaled motion. For this reason, figure 5.15 includes lines at contour levels ranging up to correlation coefficient magnitudes of only 0.05. Colour contours in the region of small displacements are highly saturated.
Figure 5.15: Contours of two-point correlation coefficient $R_{uu}(\Delta x, \Delta s)$ in $x-\theta$ cylinder surfaces: (a) $y^+ = 20$, (b) $y^+ = 101$, (c) $y^+ = 250$, (d) $y^+ = 342$, (e) $y^+ = 80$. 

211
Figure 5.15 includes distinctive patterns of streaks in low levels of the two-point correlations. The streaks have long streamwise extents and are inclined relative from the streamwise direction. The question of what these patterns indicate is now considered. The most straightforward interpretation is suggested by the LSE introduced above, indicating that, on average, this is the pattern of the streamwise fluctuation given any $u'$ fluctuation event, so that this shifted to the event location and scaled by the appropriate velocity estimates the field. Clearly, no instantaneous velocity field appears like this, and the spike of velocity near the event indicates that a conditional average would require many $u'$ events to be specified to reasonably approximate the field. This effect of averaging is an indication of the great deal of randomness present, particularly at small scales, but the scales of these low-level correlation patterns suggest that they are associated with very long scale motions. While the small scales may contain much randomness, their form and organization may be strongly influenced by the large scales of motion, as the work of Mathis et al. (2009a) suggests.

The patterns shown in figure 5.15 depend somewhat on the sets of velocity fields included in the averages when they are not converged to a high degree. The features near the centre (small displacements) are very consistent even without a high degree of statistical convergence, and the overall character of the patterns at large displacement remain the same. In addition, as is shown in §5.8, experiments also provide evidence of the correlation patterns that occur at large $\Delta \theta$ on the other side of the pipe ($|\Delta \theta| > 90^\circ$) when $\Delta x$ is zero.

Viewed from the perspective of this function being a two-point correlation, the definition of the two-point correlation suggests that the main contributors from individual fields are point pairs with the displacement of interest at which the $u'$ magnitudes are large. This would suggest that the regions of peak strength within streaks would be large contributors. Since the two-point correlation is formed by an
average, both over different fields and over spatial positions within each field, it is difficult to relate the structures observed in the two-point correlation with structures that would commonly occur in particular instantaneous fields.

To more precisely identify the source of the low-level correlation patterns observed in the two-point correlation, refining the large contributors into a more specific event would be useful. Besides magnitude being one of the important qualities of strongly-contributing fluctuation motions, all motions within a region might be expected to contribute. An example to illustrate this would be two Gaussian bumps in space, for which the two-point spatial correlation would be another Gaussian bump, with the peak occurring at the displacement that is the vector connecting the centres of the contributing Gaussian bumps. However, the $u'$ fluctuations alternate signs. As another example, if sinusoidal functions in space were correlated, the correlation peaks occur at the displacements between peaks of the source functions. This motivates the centres of the functions of interest being points of interest.

The effect of sign on the $u'$ fluctuation event remains to be considered. The LSE, corresponding to the two-point correlation, includes motions of both positive and negative fluctuations in the statistical computation. Lee & Sung (2011) found nearly identical behaviours of conditional two-point correlations in streamwise–spanwise planes of a turbulent boundary layer simulation with the condition made on the sign of $u'$, except that there was different streamwise biasing based on the signs. Of instantaneous streaks in the present pipe flow, it appears the negative streaks are, on average, slightly stronger and slightly narrower azimuthally than positive streaks. Since negative streaks are also of interest because they are related to structures resembling hairpin packets, they therefore contain appropriate events upon which to focus.
5.7.4 Conditional Average on Long Negative $u'$ Regions

To refine what events are key contributors to the two-point correlation, attention is focused on a more specific event that is a likely candidate for being influential. Having motivated centres of strong $u'$ fluctuations as being events expected to be influential in contributing to the patterns observed and the noting importance of negative $u'$ fluctuations, the result of a conditional average conditioned on these events is next compared to the two-point correlation. In summary, the comparison is between an LSE of a velocity fluctuation, which is equal to the two-point correlation signed and scaled to the event specified, and a conditional average conditioned on the centre of a long negative $u'$ streak, which is a much more stringent condition than simply a velocity fluctuation. The latter (conditional average) is comprised of the average over a finite set of events, which result in shifted fields such that the average is defined relative to the event.

The events are chosen as the centres of mass of the contiguous negative $u'$ streaks extracted in §5.3. Only streaks longer than $2R$ are included for specifying events. At the chosen threshold of $u'_{thr} = -0.10U_{bulk}$, an average of 10 such events ($2R$- to $5R$-long clusters) occur per field. Motions of these lengths are generally seen as concatenations of smaller motions, as discussed earlier, but they are often relatively straight, in the sense of not changing azimuthal inclinations relative to the streamwise direction, although they may be uniformly inclined in this way.

In figure 5.16(a), this conditional average is compared to the two-point correlation which estimates the velocity field for a much more general condition of $u'$. The agreement in the locations of the streaks and locations at which signs change is very good, as indicated by solid lines enclosing red colour contours and dashed lines enclosing blue colour contours. This is a strong indication that the two-point
Figure 5.16: (a) Conditional average of streamwise velocity fluctuation $u'$ conditioned on events of centres of contiguous regions of negative $u'$ with lengths between $2R$ and $5R$ at $y^+ = 101$. This colour contour plot also includes contour lines of two-point correlation coefficient $R_{uu}(\Delta x, \Delta s)$ in $x-\theta$ cylinder surfaces, with solid black lines representing levels of 0.005 and 0.01, and dashed black lines representing levels of -0.005 and -0.01 (when the sign is changed to represent the LSE of a negatively-signed event) to indicate the boundaries at which signs change. Additional straight black lines are drawn to indicate dominant inclination angles. (b) displays a synthetic $u'$ field azimuthally inclined at $5^\circ$ from the streamwise direction, (c) displays its two-point correlation, and (d) displays the average correlation for (c) and its counterpart with the opposite direction of inclination.
correlation contains strong imprints of the patterns of the relatively long regions of negative $u'$ fluctuations.

The features of both the two-point correlation and the conditional average described above include X-like patterns extending from the location of zero displacement and long, inclined, streak-like regions. It is noted that inclined streaks are also present in instantaneous $u'$ field realizations (figure 5.1). Dominant azimuthal inclinations in the two-point correlation and conditional average are drawn as thick solid lines in figure 5.16(a), with the line extending from the zero displacement region and extending downstream inclined azimuthally $5.0^\circ$ from the streamwise direction. The line offset azimuthally above the zero-displacement region is inclined at $4.2^\circ$. It appears that these inclinations may be an imprint of dominant inclinations that are included in the flow, and this issue will be explored in further depth.

The symmetries of the two-point correlation and conditional average should also be noted. One mathematical property of this two-point correlation is mirror symmetry in which either domain half may be chosen (either $\Delta x \leq 0$, $\Delta x \geq 0$, $\Delta \theta \leq 0$, or $\Delta \theta \geq 0$) and then the other half is its mirror image. However, when this half is again divided in the other direction, there is no symmetry implied from this consideration. If a streamwise displacement half is first chosen, however, there should be mirror symmetry about the remaining azimuthal direction due to the lack of preferred azimuthal direction. The deviation from this may be an indication of the degree of statistical convergence of the two-point correlation. For the conditional average, no symmetries are mathematically imposed by the calculation, so the symmetries present are an indication of the degree to which this average has captured the symmetries naturally present in the flow (including the lack of preferred azimuthal direction). The inclined streaks in the flow have no preference of inclination in the positive or negative azimuthal direction, so it reasonable that they should occur with the same frequency in either direction such that the correlation
and conditional averages are symmetric about zero displacement in the azimuthal direction. This symmetry could be directly imposed by including the azimuthal mirror image of each $u'$ field since it is a component of an equally valid numerical solution to the Navier-Stokes equations.

### 5.7.5 Model of Streaks

Since the concept of long streaks of streamwise velocity fluctuation that may possess spanwise inclination relative to the streamwise direction being important structures has emerged, the properties of such structures relating to the $u$ two-point correlation are now considered. A useful approach to evaluate if a specific instantaneous structure would be associated with a correlation pattern is the approach of Hutchins & Marusic (2007a), which is to synthetically generate a velocity associated with a proposed instantaneous structure and then compute its two-point correlation to verify if it matches what is observed.

A single synthetic streak is generated with the form introduced by Hutchins & Marusic (2007a), in which the spanwise or azimuthal cross-section includes an approximately sinusoidally-shaped bump of fluctuation with oppositely-signed bumps on each spanwise side. The angle of inclination is chosen as $5^\circ$. The proper azimuthal width of a representative streak may be estimated by several means. Since the energy spectrum is, in some sense, a measure of the width of patterns, one might expect a peak in the azimuthal energy spectrum at a wavelength corresponding to twice the width of the streak of single $u'$ sign. The azimuthal two-point correlation of $u$ also could be used to estimate the width of the pattern by extracting twice the width from the zero-displacement peak to the $\Delta s$ for which $R_{uu}(\Delta x = 0, \Delta s) = 0$, since the azimuthal pattern is essentially alternating streaks. The number of streaks can also be visually estimated from example fields and the total azimuthal arc length divided by this to estimate the widths of the streaks. Though the estimates vary, it
ultimately is not of vital importance to the correlation pattern from this synthetic field, and the value could be further refined but it is not important for the present purpose. The $u'$ field synthetically generated with approximately estimated parameters is displayed in figure 5.16(b).

The two-point correlation is also calculated from this synthetic field and displayed in figure 5.16(c). The inclination is clearly apparent in this also and remains the same as that specified for the synthetic field. As previously noted, streaks are equally probable to be inclined in the positive and negative azimuthal directions, and all are included in an average of actual fields. For this reason, a synthetic field and its average is also computed with the same inclination but in the opposite direction, and the average of the correlations for the fields with oppositely-oriented inclinations are displayed in figure 5.16(d). This generates an X-shaped correlation pattern extending from the region of zero displacement. Due to the straightness of the inclinations and only two inclination angles used that are exactly the same magnitude, periodically-spaced peaks occur in this two-point correlation. The angle specified of the inclinations in each synthetic field is recovered by connecting the peaks of the same sign, such as when following one branch where the positive-correlation region at $\Delta x = 0$ and $\Delta s = 0$ bifurcates with increasing $\Delta x$ to the next peak that occurs. In reality, there are a variety of different inclination angles of streaks contributing to the statistics and each streak is not precisely straight. For this idealized case, this is somewhat analogous to McKeon & Sharma (2010) adding opposing modes of velocity that have a similar form (for a constant radius) to the present correlation functions for each inclination separately, although there are differences (this is a relatively isolated streak instead of streaks periodically filling the entire circumference) and the fundamental quantities are different (correlation in this case instead of velocity fluctuation). McKeon & Sharma (2010) found that
adding such modes yielded a pattern of alternating low- and high-speed streamwise-aligned regions. This is also discussed by Hellström et al. (2011).

5.7.6 Interpretation of Organization

The two-point correlation includes an X-shaped pattern of correlation, which Hutchins & Marusic (2007a) also discussed. They found that a sinusoidally-wavering (in a spanwise sense) synthetic u streak could generate this pattern. The results in §5.7.5 also indicate that inclined structures can generate similar patterns when averaged over many differently-inclined examples. The presence of inclined structures also generates weaker inclined streaks at further distances than the positive-correlation X in the two-point correlation. In actuality, there is a distribution of azimuthal inclinations that appear for relatively long structures of negative $u'$, and this would lead to a less distinct pattern.

While there is strong evidence of sinusoidally wavering structures in the hotwire-rake traces of Hutchins & Marusic (2007a) and the pipe flow visualization of Guala et al. (2006), in the present pipe simulation it is more difficult to see clear examples of long sinusoidally wavering structures. Examples of long contiguous or almost-contiguous structures with an overall inclination (particularly when viewed without focusing on the details) are clearly apparent in figure 5.2. The pipe flow experiment of Große & Westerweel (2011) also appears to contain many relatively streamwise-aligned shorter motions of streamwise velocity fluctuation. In the present pipe simulation, the lines drawn on both the two-point correlation and the conditional average for long negative-$u'$ motions (figure 5.16(a)) suggest that $5^\circ$ is a frequently-occurring or mean azimuthal inclination of such long structures. In an experimental hot-wire rake measurement of a turbulent pipe flow, Monty et al. (2007) noted that very long structures could rotate about the pipe axis by $180^\circ$ and meandering was also observed. To an approximation, long segments of approxi-
mately sinusoidal wavering could be split into relatively straight inclined segments, so it could be difficult to distinguish between the two organizations. Regardless, it appears that both scenarios can lead to two-point correlation patterns consistent with those observed in actual flows.

Besides the overall inclination, in figure 5.2 it is also clear that the shorter motions comprising these are frequently inclined differently in an azimuthal sense. The shorter-scale motions in figures 5.1 and 5.2 appear to be more streamwise-aligned than the overall structures they appear to comprise. This appears consistent with evidence from the two-point correlation, in which contour lines for moderate contour levels would be strongly streamwise-elongated (which can be drawn from figure 5.15(e)). The synthetic field with a single smooth, long, inclined streak therefore approximates the effect of a concatenation of smaller motions such that more fine details exist in structures that appear in the actual flow and their correlations.

Based on the multiple scales of $u'$ motions observed in figure 5.1, the following organization scenario emerges based on the evidence for the present pipe simulation, focusing on the negative $u'$ fluctuations: small, intense velocity fluctuations exist that are somewhat random in their shape and precise dimensions, but are frequently located near vortices and seem in an average sense to be consistent with what might be expected from a single hairpin vortex. They are somewhat streamwise-elongated, and examples of such fluctuations are shown in figure 5.1(d) with lengths on the order of $0.2R$ ($140^+$). This is consistent with the 100-200+ lengths of low-speed streaks for hairpins suggested by Marusic & Adrian (2012). These velocity motions appear to frequently organize in streamwise series that may be azimuthally inclined, but appear to often be streamwise-aligned. Examples can be seen in figure 5.1(b,c) as the relatively straight and horizontal segments of negative velocity fluctuation. The lengths vary significantly, but regions often appear straight for approximately $1R$ to $2R$. From this perspective, they appear to be streamwise-aligned
Figure 5.17: An idealized depiction of concatenated low-velocity streaks with hairpin vortices surrounding the streaks to represent approximately LSM-length and relatively straight streaks organizing into the long, meandering very long scale motions commonly observed. The perspective is looking down to a wall, with flow above the wall from left to right. Blue represents vortical structures and orange represents low-velocity fluid.

series of the smaller scales. This is consistent with the idea of hairpins concatenating into packets, and the 685–1370+ lengths are consistent with the 500–2000+ lengths suggested for hairpin packets in Marusic & Adrian (2012). These may also be related to the LSM scales originally identified with turbulent bulges in boundary layers (discussed in Wu et al., 2012). From examining \(x-y\) planes, there appear to also be ramps and other features consistent with packets, but the actual vortices and their organizations visualized in three dimensions appear distorted from the idealized model. In streamwise–azimuthal planes, these packet-like entities appear to concatenate, as suggested by Kim & Adrian (1999), but they also often have azimuthal offsets. This is apparent in figure 5.2, in which it appears that relatively straight regions of \(1R\) to \(2R\) length connect to each other or are closely organized to form regions that are much longer and often azimuthally inclined. The overall organization is depicted diagrammatically in figure 5.17. Depending on how offsets occur, these patterns could result in sinusoidal or straight (sometimes with inclination) VLSMs.

The histograms of figure 5.3 are now interpreted with respect to these structures of different scales. It appears that the 1–2\(R\) long regions that tend to be stream-
wise aligned and are identified with LSMs and packet-like entities merge together depending on the strength of each such structure, its geometry, and precisely how it is oriented with respect to other such structures. Due to these variations, as the threshold varies and they coalesce into longer objects, it is difficult to extract clearly delineated scales for each type of motion. The main organization of the LSMs is in streamwise arrangements often with azimuthal offsets. For this reason, as the threshold decreases more and more LSMs coalesce into these straight, sometimes inclined, VLSM structures. This variation in structures is consistent with the relatively broad length histograms that do not show clear peaks and the relatively broad premultiplied energy spectra, in which the peaks approximately indicate characteristic lengths.

5.7.7 Three-Dimensional Correlation

Similar to the two-dimensional $x-\theta$ two-point correlation, the three-dimensional two-point correlation may also be computed to display additional information about the structure. In this case, a $y$ reference location (radius) is chosen, but instead of displaying the two-point correlation at the same $y$ (as in the case of the two-dimensional $x-\theta$ presentation), the correlation values are shown for other $y$ locations (while the reference $y$ remains constant). Figures 5.18 and 5.19 display isosurfaces of three-dimensional two-point correlation for three reference heights. In each, the event or reference position is placed at the bottom wall of the pipe elevated by $y_{\text{ref}}$. Each two-point correlation is also displayed for two isosurface values, with the stronger level used to display the main character of the motions. This overall form is of a long streamwise-elongated region of positive correlation with two shorter (in part because they are weaker) parallel regions of negative correlation offset azimuthally. The weaker isosurface value provides a radial perspective of the weaker correlated motions that appear in figure 5.15. For these visualizations in
Figure 5.18: Three-dimensional two-point correlation $R_{uu}$ for reference (event) location of $y^+ = 30$. Correlation coefficient values of isosurfaces are (a,b) ±0.033 and (c) ±0.1.

which the isosurfaces fill the entire domain that has the same dimensions as the pipe, the domain is split at $\Delta x = 0$ and both halves are shown. The split at $\Delta x = 0$ also reveals the behaviour at that location.

The visualizations of figures 5.18 and 5.19 show that the azimuthal patterns remain relatively constant radially. The set of reference $y$ values indicate how the two-point correlation structure changes for three regimes of reference $y$, and the cut at $\Delta x = 0$ displays the behaviour of $R_{uu}(\Delta \theta, y, y_{ref})$. As the $x-\theta$ two-point correlation discussed in reference to figure 5.15 corresponds to the cylinder of the three-dimensional two-point correlation at which $y = y_{ref}$, the $x-r$ two-point correlation
Figure 5.19: Three-dimensional two-point correlation $R_{uu}$ for reference (event) locations of $y^+ = 101$ (a,b) and $y^+ = 250$ (c,d). Correlation coefficient values of isosurfaces are (a,d) $\pm 0.05$ and (d) $\pm 0.1$. 
\( R_{uu}(\Delta \theta, y, y_{ref}) \) is the plane revealed at this cut where \( \Delta x = 0 \), and the behaviour of azimuthal structure revealed by this plane is next considered.

### 5.8 Azimuthal Structure and Scale Growth

Having described the organization of the low- and high-speed streaks of streamwise velocity in streamwise-azimuthal planes, the organization patterns relating different radii are also important. As noted in §5.4, there is significant correlation between streamwise-azimuthal planes (or cylinders) at neighbouring radii, yet there is significant azimuthal scale growth as the wall distance increases (radius decreases), as observed from azimuthal energy spectra in Wu et al. (2012). It was also observed in the azimuthal two-point correlations, and these issues were explored for pipe experiments by Monty et al. (2007) and Große & Westerweel (2011).

While it is clearly evident in the \( x-\theta \) planes that the signs of velocity fluctuation alternate with \( \theta \) for a given \( x \) position, the structure of how these azimuthal regions of constant sign widen with \( y \) is less obvious. A simple perspective from which to observe instantaneous examples of azimuthal scale growth is azimuthal–radial planes of flow fields. Although streamwise variation information is not available from this perspective, clear patterns are visible in these planes.

While vector fields of the \( r-\theta \) planes of turbulent pipe flows have not been frequently studied, with the prominent exception of Große & Westerweel (2011), \( y-z \) planes have been more often examined in turbulent channel simulations. Aside from a number of studies focusing very near the wall, Hanratty & Papavassiliou (1997) studied vector fields in these planes of relatively low Reynolds number channel simulations and the fields of conditional averages based on Q2 events. They observed velocity vector patterns of streamwise-oriented vortices near the wall that they termed ‘wall eddies.’ They found that the “wall vortices appear to induce sheetlike jets that extend over very large regions of the channel” and observed a
structure of large Reynolds stress extend far from the wall past the channel centre-line. More recently, Mito et al. (2007) extended similar analysis to an $Re_c = 950$ channel simulation.

An example vector field in an $r-\theta$ plane for the present pipe simulation is displayed in figure 5.20. The appearances of this and other examples are qualitatively similar to those of Große & Westerweel (2011). Important features are radially-inward ejections from the walls, which are often accompanied by negative $u'$ fluctuations (blue) as slower near-wall fluid is transported inward to the core, and these regions correspond to Q2 motions in the flow. Near the wall, the azimuthal (arc length) scales of these motions appear near the expected $\Delta s^+ = 100$ dimensions. These ejections also appear to be between two counter-rotating quasi-streamwise vortex cores, as are expected to exist in the near-wall region. The behaviour of azimuthally alternating regions of $u'$ fluctuations continues up from the wall approximately to the $y^+ = 30$ circle drawn.

Above the near-wall region, the regions of negative $u'$ fluctuation transported upward from the wall are observed to merge, suggesting that significant azimuthal scale growth is occurring. If the azimuthally-wide regions of positive and negative $u'$ fluctuations are followed down to the wall, it is observed that much of the finer motions below these wide regions have a tendency to contain more $u'$ fluctuation of the same sign as the higher regions, consistent with the footprints previously observed. It appears some vortex cores are also cut with orientations normal to this plane. Much of the scale growth (merging) appears to have occurred once $y/R$ values have reached approximately the upper extent of the logarithmic layer. A circle at $y/R = 0.3$ is drawn to depict a region above the log layer. The fluctuations above that location appear more isotropic, but tall ejections beginning near the wall are seen to extend above $y/R = 0.3$. 

226
Figure 5.20: Example plane at fixed $x$ position for an instantaneous field of the present pipe simulation showing the in-plane velocity with vectors and the plane-normal velocity ($u'$) fluctuations by colour contours ranging from blue (negative) to red (positive), with presentation similar to Große & Westerweel (2011). (a) includes the entire pipe diameter with vectors interpolated on a coarse uniform grid, while (b) and (c) show details near the walls with vectors on finer grids. Arrows indicate $\Delta s^+ = 100$ arc lengths at the wall.
Linear stochastic estimation reveals the structure on average associated with a negative $u'$ fluctuation. Instead of applying the LSE in the $x$-$\theta$ plane, as in §5.7, it is now applied in the $x$-$r$ plane (as also performed by Große & Westerweel (2011)). Similarly, as shown in 5.7.1, the LSE is simply the scaled two-point correlation of velocity. In figure 5.21(a), the zero-level contours, indicating locations at which the streamwise velocity fluctuation in a field conditioned on a $-u$ event switches sign, are shown for a series of $y$ values for the reference position. As regions of opposite velocity fluctuation sign occur azimuthally offset on both sides of the event and these may have significant strength, dashed contour lines of these are also included. The correlation value of $-0.0004U_{\text{bulk}}^2$ corresponds to correlation coefficient values ranging between $-0.073$ and $-0.018$, depending on the reference location. The reference locations are indicated by coloured dots, and the contour lines are similarly coloured. For the zero-level contours, the lines show a surprising amount of agreement independent of the radial position of the event. This is evidence of towering regions of streamwise velocity fluctuation that penetrate deeply into the core while extending outward far to the wall, consistent with footprints.

Another noteworthy feature of figure 5.21(a) is the presence of two positively correlated regions on the opposite half of the pipe as the event. The correlation lines are also consistent with respect to event radial location, and these correspond to streaks also clearly visible in the three-dimensional correlation isosurfaces (§5.7.7), as this plot is essentially a slice at $\Delta x = 0$ of the three-dimensional two-point correlation. This can be seen from the profile at the cut of figure 5.19(a). Note that the signs are opposite between these plots because the three-dimensional plot is purely two-point correlation, whereas the LSE in figure 5.21(a) is computed for a negative $u'$ fluctuation. In an experimental two-point correlation of turbulent pipe flow, Bailey et al. (2008) also noted the presence of slight positive azimuthal correlation particularly at low Reynolds numbers, such as their measurements for $Re_D = 7.6 \times 10^4$. 228
Figure 5.21: (a) contains contour lines at two-point correlation $R_{uu}(\Delta x = 0, r, r_{\text{ref}}, \Delta \theta)$ values of 0 (solid lines) and $-0.0004U_{\text{bulk}}^2$ (dashed lines) coloured as the dots indicating the $r_{\text{ref}}/R = \{0.50, 0.63, 0.78, 0.88, 0.97\}$ reference points. The LSEs of $-u'$ events with colour contours of $u'$ (and levels incremented by 0.05 fractions of the event strength) and vectors for the other velocity components are shown for (b) $y_{\text{ref}}^+ = 20$, (c) 101, and (d) 341. (b) includes an upper subfigure for higher resolution of the event region.
Some weaker positive correlation is also visible up to $Re_D = 5.5 \times 10^6$. In the present plots, no positive correlation is discernible from the contours above approximately $y/R = 0.3$. The presence of these features is a strong indication that elements of the patterns observed in the low-level correlations are not merely noise or lack of statistical convergence. Note also that the azimuthal symmetry is not imposed, so the symmetry present is directly related to how well converged the correlation is.

These regions of figure 5.21(a) have boundaries shaped more parallel to the radial line at the centre of each region than shaped radially extending from the centre of the pipe cross-section (as a sector of a circle). The widths of the secondary regions of (small) positive correlation are nominally $40^\circ$. If one positions lines to pass through the pipe wall and the centres of the near-core boundaries for these secondary positive correlation regions, the lines extend not from the pipe axis, but from an origin located approximately $0.1R$ above the pipe axis. Very roughly, the regions are two $45^\circ$-wide positive correlation regions with $45^\circ$ of negative correlation in between.

Contour colour levels are chosen to represent relatively strong motions in figure 5.21(b–d), with colours saturating at half the event strength (which is a correlation coefficient value of 0.5) for streamwise velocity fluctuation. In each case, a counter-rotating pair of vortices appears. The result for the event specified near the wall at $y^+ = 20$ appears similar to structures observed in instantaneous fields such as figure 5.20, with a pair of vortex cores consistent with legs of quasi-streamwise vortices shown in figure 5.21(b) on a fine scale grid of vectors. The coarser-spaced vectors below it show the larger scale effects on the field, with vector lengths that reveal the weaker motions, and some correlated region of weaker upward ejection and negative $u'$ fluctuation continues far above the wall. For events at further distances from the wall, the LSE still indicates approximately the same pattern, with a symmetric pair of vortices, but this is probably a statistical result because vortices
are equally probable on either azimuthal side of a negative velocity fluctuation. The instantaneous fields suggest that the vortices rarely occur in pairs at these distances from the wall.

Although in a turbulent channel simulation, Toh & Itano (2005) discuss ejections near the wall and their interactions with large-scale structures of velocity. It appears the scenario they devised is relevant to the correlations between the intense near-wall ejection motions that appear in the pipe and the tall correlated regions of weaker ejection with merging in the present pipe simulation. There would, however, be an effect of the pipe geometry instead of channel.

It has been observed above that scale growth is represented in an averaged sense with the two-point correlation. The sign of $u'$ estimated for the negative $u'$ event changes as $\theta$ departs from the event location while $r(y)$ remains at the value at which the event was specified. Figure 5.21 indicates this change occurs in close proximity to the presence of one of the counter-rotating vortices. The width at which the sign change occurs or at which the correlation initially decays to a small positive value is therefore an indication of the local scale width where the event is specified. The two-point correlation with fixed $y$ is simply $R_{uu}(\Delta s,y)$. Monty et al. (2007) plotted these azimuthal widths at which the correlation coefficient $R_{uu}(\Delta s,y)$ decays to 0.05 for a range of $y$ values and compared with analogous spanwise widths for turbulent channel and boundary layer flows. The width is defined between the two locations at which the correlation decays to this level, such that the width is twice the $\Delta s$ value required to decay from $R_{uu}(0,y) = 1$ at zero displacement to a value of 0.05. The present pipe flow is compared with the flows Monty et al. (2007) and others examined, as well as several other relevant flows in figure 5.22.

Figure 5.22 includes data from the pipe experiment results of Große & Westerweel (2011), which were at Reynolds numbers comparable to the present simu-
Figure 5.22: Comparison of the azimuthal (arc length) or spanwise distance \( l_z \) between the points at which the correlation coefficient \( R_{uu}(\delta z, y) \) has decayed from the peak to 0.05. \( \delta \) is the appropriate outer length scale for each geometry: channel half-height, pipe radius, or boundary layer thickness. The data are coloured by geometry: pipes are red, channels are blue, boundary layers are green. The present pipe DNS data are represented by a thick solid red line. Additional points from the \( Re_D = 44 000 \) (\( Re_\tau = 1215 \)) turbulent pipe simulation of Wu & Moin (2008) are added as solid red diamonds. The experimental measurements of Große & Westerweel (2011) are included as dotted red lines with symbols: * is for \( Re_D = 10 000 \), + is for \( Re_D = 20 000 \), and × is for \( Re_D = 44 000 \). This figure is based on the comparison of Monty et al. (2007), which included a symbol for their pipe data for \( Re_\tau = 1000–4000 \) (▼). Turbulent boundary layer data shown includes Hutchins et al. (2005) at \( Re_\tau = 2800 \) as □, Hutchins & Marusic (2007a) \( Re_\tau = 19 960 \) as ▲, Krogstad & Antonia (1994) \( Re_\tau = 1850 \) as ◆, and Tomkins & Adrian (2003) \( Re_\tau = 2600 \) as ●. Monty et al. (2007) included \( Re_\tau = 3100 \) turbulent channel data (○) and compared with the \( Re_\tau = 934 \) turbulent channel simulation of del Álamo et al. (2004), shown as a dash-dotted blue line. The \( Re_\tau = 590 \) channel simulation of Moser et al. (1999) is displayed by the dashed blue line. Additional comparisons of Bailey et al. (2008) pipe data are also included: ■, \( Re_D = 7.6 \times 10^4 \), and ●, \( Re_D = 8.3 \times 10^6 \). Volino et al. (2007) turbulent boundary layer with \( Re_\tau = 1772 \) is also shown as ◦. The pairs of dashed and solid lines are the linear fits to data included in Monty et al. (2007).
lation, whereas most other experiments are at significantly higher Reynolds numbers. The present results are consistent with Große & Westerweel (2011) for similar Reynolds numbers, although slightly smaller in arc length widths, and more noticeably so as the pipe axis is approached. At sufficiently small pipe radii, the motions remain spatially correlated such that the two-point correlation coefficient never decays below 0.05, and therefore no length scale can be extracted. The line on the plot ends at this point, and there is good agreement between the present pipe simulation and the experiments with respect to the locations at which this occurs.

Monty et al. (2007) considered in detail the comparison of azimuthal and spanwise length scales for turbulent pipe, channel, and boundary layer flows. Although the goal of the present study is not to address the differences between scale growth in different geometries, several observations from the present comparison are relevant to issues raised in other studies. In Monty et al. (2007), although only one data point was present for pipe flows, it was more consistent with channels for its relatively low \((y/R = 0.15)\) position. Boundary layer length scales were appreciably shorter than for the other two flows, even below the logarithmic layer region and near the wall. Bailey et al. (2008) found the azimuthal length scales (in terms of arc length) to be similar to channels for \(y/R = 0.1\) and 0.2, but to decrease relative to the width observed in channel flows for \(y/R > 0.2\), which they attribute to the spatial constraints imposed by the pipe geometry. Only the highest and lowest Reynolds numbers of Bailey et al. (2008) are included in figure 5.22, but data points for the intermediate Reynolds numbers fall in between these extremes. Große & Westerweel (2011) compared their experimental measurements of pipes and found the length scales to be consistently slightly smaller for their flows than for other experiments of pipes at higher Reynolds numbers.

A casual comparison of the present pipe with the three types of flows displayed in Monty et al. (2007) might suggest that the behaviour is more similar to boundary
layers than channels, particularly when attention is focused on the region far from the wall, but a more careful analysis suggests that the behaviour closely resembles that of channels when near the wall. To illustrate this, a turbulent channel simulation with a relatively similar $Re_\tau$ value of 590 included on the plot matches very closely with the present pipe up to $y/R = 0.1$, and has diverged very significantly by $y/R = 0.2$ and above. Bailey et al. (2008) experimental data suggests that the width scale reaches higher values with increasing Reynolds numbers, with greater relative differences near the wall. This is also apparent from comparing the data points for the higher Reynolds number pipe simulation of Wu & Moin (2008) with those of the present pipe. It appears coincidental that the present pipe data has the appearance of behaving more similar to the boundary layer data. The width scales of the boundary layers continue to grow when far from the wall, albeit at a slower rate than for the channels, as discussed by Monty et al. (2007), whereas the pipes appear to level to a constant scale and then grow sharply until the correlation no longer decays to the threshold level. Thus, the overall behaviour is that this azimuthal scale of pipes behaves similar to that of a turbulent channel at comparable Reynolds number until roughly $y/R = 0.1$, and then the scale growth reduces due to the geometry, as noted by Bailey et al. (2008). The channels compared also show that the spanwise width scale grows with increasing Reynolds number, but it appears that this behaviour asymptotes for high Reynolds numbers as it does for pipes.

5.9 Conclusions

An examination of the structure in the present DNS of a turbulent pipe flow reveals that long meandering motions of streamwise velocity fluctuation are composed of smaller motions. The motions of streamwise velocity have been divided into three distinct types: short scales of approximately $0.2R$, longer scales of approximately
1–2R, and VLSMs. It appears that the scales of 1–2R have features that are consistent with hairpin packets, and there is evidence of vortical structures being associated with these motions. Careful examination of these vortical structures reveals that they contain hairpin-like features, but they are distorted from a pristine hairpin shape/signature and are similar to those observed in channel DNS. Through the use of several vortex eduction techniques, there is evidence that vortices commonly form into leg-like shapes that lift upward from the wall but instead of forming hairpin-like heads, they appear to diffuse indistinctly. However, these vortices frequently bound negative streamwise velocity fluctuation regions, and other vortices with approximately azimuthal orientations frequently exist along the upper extent of these regions. The behaviour in streamwise–wall-normal planes is similar to the uniform momentum zone behaviour identified in other wall-bounded turbulent flows, in which strong vorticity and core cross-sections of azimuthally-oriented vortices bound regions of relatively constant streamwise momentum. In addition, these vortex cores in this plane sometimes occur in ramp-like arrangements, consistent with hairpin vortex packets.

These LSM-like motions are observed to frequently align closely with the streamwise direction, but they also organize in approximately streamwise alignments, though often with azimuthal offsets between LSM-like motions. In this way, they create very long motions that are frequently azimuthally-inclined and somewhat wavering. In a general sense, the organization is consistent with the proposal of Kim & Adrian (1999), with LSMs concatenating to form VLSMs.

These very long meandering motions have spatial arrangements and dominant azimuthal inclination angles revealed by two-dimensional (streamwise–azimuthal) and three-dimensional two-point spatial correlations. While from the organization it appears that small motions concatenate into these patterns, it must also be allowed that linear modes suggested by McKeon & Sharma (2010) and Hellström
et al. (2011) form these patterns or arrange smaller motions. While the structures could grow from the wall and organize from the effects of the induced motions, Mathis et al. (2009a) and Mathis et al. (2009b) have shown that very large motions can modulate the presence of smaller motions, so the presence of additional large motions corresponding to linear modes could also induce the presence of the smaller scale motions identified. A more complex interaction is also possible, and a dynamical study would be necessary to further investigate these possibilities. The results have shown that much of the vortical motions are concentrated near and on the boundaries of the long low-speed motions.

The study also reveals that merging negative streamwise velocity fluctuation regions accompany azimuthal scale growth as the pipe axis is approached. These structures are frequently connected to ejections located near the wall, and the effects of the near-wall ejections are observed to penetrate deeply toward the pipe core as revealed by instantaneous examples and correlation statistics. This strong radial correlation is also consistent with the concept of footprints suggested by Hutchins & Marusic (2007a). As revealed by length scales of azimuthal two-point correlation of streamwise velocity, azimuthal scale growth in terms of arc length closely matches azimuthal scale growth in terms of spanwise length for turbulent channels up to $y/R = 0.10$, suggesting that the effect of pipe curvature has little effect up to this location, but then the scale growth becomes slower and remains approximately constant for the pipe.
Chapter 6

THREE-DIMENSIONAL POD OF TURBULENT PIPE FLOW SIMULATION
The present DNS simulation of the pipe was studied in terms of the single- and two-point statistics to verify the flow simulation, and energy spectra were computed for comparison with experiments, as well as other related aspects including convection velocities and net force spectra (chapter 4). Structure in this pipe simulation was also considered using standard methods that have been applied to other turbulent wall-bounded flows, and its organization of structures was explored using two-point correlations and other methods (Chapter 5). Having established these properties, the structure is analyzed using proper orthogonal decomposition (POD) in three dimensions.

6.1 Introduction
Full three-dimensional POD has not been frequently performed on turbulent pipe flows, particularly at Reynolds numbers significantly above the transitional Reynolds numbers where turbulent flow begins to be maintained. Recently, POD has been applied a set of pipe turbulent simulations to investigate the effects of several drag reduction strategies by Duggleby (2006), but the scope was restricted to relatively low Reynolds numbers. Several publications arising from this study include Duggleby et al. (2007), in which POD modes of a standard $Re_t = 150$ pipe simulation were examined and classified by their properties. Many of the modes were found to propagate at a relatively constant velocity with time, but the amplitude of the POD coefficient (the instantaneous contribution of each mode) varied rapidly for the modes. A relatively long simulation domain length of $20R$ was used. Duggleby et al. (2007) compared these modes with modes for a similar pipe simulation with the same applied pressure gradient but with azimuthal wall oscillations being simulated by applying a different time-varying boundary condition at
the walls. This had the effect of reducing drag, and the POD mode comparison focused on understanding the physical differences and mechanisms responsible for drag reduction.

The pipe POD study was later extended to compare POD modes for the \( Re_\tau = 150 \) pipe flow with those for \( Re_\tau = 125 \) and \( Re_\tau = 80 \) channels. Duggleby et al. (2009) found the energy more uniformly distributed (among modes) for the \( Re_\tau = 150 \) pipe flow compared to the \( Re_\tau = 125 \) channel flow, in which the first few most energetic modes contained a considerably greater fraction of the total turbulent energy. The physical reasons for this were discussed, and it was noted there were also differences in the amounts of energy represented by different classes of modes.

POD was performed on experimental pipe measurements at high Reynolds number relative to the present capability of simulations by Bailey & Smits (2010). The correlation or spectral density information necessary for the POD computation was collected with a pair of hot-wire probes positioned in various locations and with streamwise spatial information inferred using Taylor’s hypothesis. The resulting POD modes included only the streamwise component of velocity, and for a domain spanning a subset of the pipe radius.

POD was also performed on experimental measurements obtained via a different technique for a turbulent pipe flow at relatively low Reynolds number (\( Re_D = 12 500 \)) by Hellström et al. (2011). Particle image velocimetry measurements of all three velocity components in an axial-normal plane were closely spaced in time and used with Taylor’s hypothesis to infer the streamwise variation. An example field at \( y/R = 0.2 \) indicated the presence of very large-scale motions (VLSMs) in streamwise velocity fluctuations, as described by Kim & Adrian (1999) and Guala et al. (2006). POD modes were constructed with these quasi-three-dimensional data. Hellström et al. (2011) discarded the \( y/R < 0.1 \) region due to optical re-
fraction issues, so the POD modes were calculated on this smaller domain using the snapshot method with three-dimensional fluctuating velocity. A reconstruction with the ten most energetic POD modes indicated that these modes captured “all the characteristics of the VLSM.” They found that superimposing a small number of modes in the reconstruction is sufficient to establish the meandering character of the streamwise velocity fluctuation motions. They noted the similarities between these modes and the linear stability modes that McKeon & Sharma (2010) calculated for turbulent pipe flow. McKeon & Sharma (2010) showed that alternating streamwise-aligned segments of negative and positive streamwise velocity fluctuation could be constructed with a superposition of left- and right-going helical response modes from their linear stability analysis with a form similar to that of POD modes. Hellström et al. (2011) also found that the VLSMs reconstructed by the ten most energetic POD modes had significant radial extent, with VLSMs growing taller than the logarithmic layer.

The present study extends POD analysis of pipe simulations to higher Reynolds numbers using the present $Re_T = 685$ turbulent pipe simulation to visualize the forms of the most energetic modes. This study also seeks to compare these modes with analogous modes for the lower Reynolds number pipe flow in the POD analysis of Duggleby et al. (2007). Instead of the snapshot POD approach applied by Hellström et al. (2011), this study follows the approach of Duggleby et al. (2007) in which POD is calculated on the full domain for all three velocity components and the natural symmetries of the flow are imposed to maximize the statistical convergence for the data set available. By imposing Fourier decomposition in the periodic homogeneous (streamwise and azimuthal) directions, the resulting modes more precisely adhere to the forms that would be expected to naturally occur given a larger data set. Duggleby & Paul (2010) investigated the differences in POD calculated
with this direct method and the snapshot method with finite data sets of a different, extensively chaotic flow possessing natural symmetries.

6.2 Method

The POD equations relevant to pipe flows are described in Duggleby et al. (2007) and Duggleby (2006). The fundamental equations are summarized below. In general, these equations are similar to those explained in the channel POD study of Moin & Moser (1989), except for the different integration necessary for the cylindrical coordinates and different symmetries that apply to the flow in the pipe geometry.

Defining the POD from a norm that integrates over the volume of the entire periodic pipe domain, the resulting eigenproblem when the the streamwise and azimuthal variations of velocity fluctuations are represented by Fourier expansions is

$$
\int_0^R \Phi_{ij}(k_x, r, r', k_\theta) \hat{\phi}_j(k_x, r', k_\theta) r' dr' = \lambda(k_x, k_\theta) \hat{\phi}_i(k_x, r, k_\theta),
$$

(6.1)
in which the spectral-density tensor is calculated from Fourier-transformed velocity

$$
\hat{u}_i(k_x, r, k_\theta) = \sum_{n=1}^N \hat{a}_n(k_x, k_\theta) \hat{\phi}_i(k_x, r, k_\theta),
$$

(6.3)
in which $n$ is an index assigned to each POD mode. The POD modes are orthonormal with the inner product specifying integration over the domain. POD modes
are typically ordered by decreasing energy content \((n = 1\) being most energetic) for each wavenumber pair \((k_x, k_\theta)\), and the associated eigenvalue \(\lambda_n\) represents the mean energy content of each mode to the flow.

Symmetries improving statistical convergence and reducing storage requirements include

\[
\Phi_{ij}(k_x, r, r', k_\theta) = \Phi_{ij}^*(-k_x, r, r', -k_\theta)
\]

\[(6.4)\]

[c.f. Moin & Moser (1989) (2.5)] and

\[
\Phi_{ij}(k_x, r, r', k_\theta) = \pm \Phi_{ij}(k_x, r, r', -k_\theta),
\]

\[(6.5)\]

with the minus sign used for \(i = 3\) or \(j = 3\) but not both [c.f. Moin & Moser (1989) (2.6b)]. These and other symmetries are described in the context of pipes by Duggleby et al. (2007). When applied to the simulation, the eigenvalue problem \((6.1)\) is numerically integrated using the trapezoidal rule in the radial coordinate. Discrete Fourier transforms on the simulation domain are indexed by integer wavenumber indices defined as \(i_x\) and \(i_\theta\) such that the wavelengths of each Fourier mode are \(\lambda_x = 30R/i_x\) and \(\lambda_\theta = 2\pi/i_\theta\).

6.3 Results

To determine which modes are most important to the present \(Re_\tau = 685\) pipe simulation, the eigenvalue spectra indicate which modes contribute the greatest amounts of energy. For a given \((i_x, i_\theta)\) pair, eigenvalues (energy content) decay rapidly with increasing \(n\), and the \(n = 1\) mode contains a substantial fraction of the energy (at least for low streamwise and azimuthal wavenumbers). Of the 25 most energetic modes in the \(Re_\tau = 150\) pipe POD results of Duggleby et al. (2007), all were for \(n = 1\). For this reason, the \(n = 1\) eigenvalues of the present flow are presented as function of \(i_x\) and \(i_\theta\). (In snapshot POD without imposing Fourier decomposition in the streamwise and azimuthal directions, various scales in these dimensions are
also included in the energy ordering, and consequently the decay would not be as
sharp as in the present case in which \( n \) refers purely to decomposition of the radial
variations.) In figure 6.1, only positive wavenumbers are shown, but the energy
associated with their negative wavenumber pairs is also included. This includes the
energy in the mode with negative \( k_x \) and \( k_\theta \) in (6.4) because the velocity is real-
valued and the mode with negative \( k_\theta \) in (6.5) due to the azimuthal symmetry.

For comparison, the modal energy fractions for the \( n = 1 \) modes listed as the
25 most energetic in Duggleby \textit{et al.} (2007) are plotted in figure 6.2. In com-
paring between the present \( Re_\tau = 685 \) flow and this \( Re_\tau = 150 \) pipe flow, it should
be noted that the domain lengths are different (30R vs. 20R), so the associated
wavelengths are \( \lambda_x = 30R/i_x \) and \( \lambda_x = 20R/i_x \), respectively. The wavenumber
indices for the azimuthal wavelengths are equivalent. The eigenvalue spectra indicate
that the \((i_x, i_\theta, n) = (1, 2, 1)\) and \((2, 3, 1)\) modes are most energetic in the present flow.
Figure 6.2: Eigenvalue (energy content) spectrum of the $n = 1$ (most energetic) modes for each wavenumber index pair of a $Re_\tau = 150$ pipe simulation, obtained from data presented in Duggleby et al. (2007).

whereas $(1, 5, 1)$ is most energetic in the flow of Duggleby et al. (2007). With higher Reynolds number, additional scales are present and, correspondingly, the number of grid points on which velocities are defined are higher (also due to the longer domain). The $Re_\tau = 685$ simulation is defined on $2048 \times 257 \times 1024$ grid points, whereas the $Re_\tau = 150$ simulation is defined on $400 \times 101 \times 64$ grid points $(x, r, \theta)$. With the difference between 539 and 2.59 million grid points, and the corresponding difference in number of modes necessary to span the solution space, it is to be expected that energy would be divided among more modes and the energy in each mode would be lower for higher Reynolds number.

There is agreement between the flows that energy is concentrated at the longest streamwise wavelength, but the higher Reynolds number flow includes larger relative energetic contributions at wider azimuthal wavelength than the lower Reynolds
number flow. The pipe azimuthal scale function indicating the scale at which the azimuthal two-point correlation of streamwise velocity fluctuation decays to a specified level (figure 5.22) appears to grow with increasing Reynolds number, particularly near the wall, but asymptote for high Reynolds numbers. As the Reynolds number of $Re_\tau = 150$ is quite low compared to the Reynolds numbers at which the length scales asymptote, the greater energy at wider azimuthal scales appears consistent with this evidence.

The eigenfunctions are next compared with several displayed in Duggleby et al. (2007). Based on comparing eigenvalue spectra, the modes may no longer play the same roles in each flow, so comparing mode shapes suggests if the modes are capturing modes with the same physical significance. Only the $n = 1$ eigenfunctions are displayed. It is again noted that the domain lengths and therefore the wavelengths for the same streamwise indices are different, but mainly the modes with wavelength equal to the domain length are compared, so these are the most equivalent wavenumber indices. Since the modes are multiplied (in Fourier space) by complex POD coefficients to create a reconstruction, and this produces the proper phase shift, the phase shift of each mode as it appears from the eigensolver may be arbitrary. Thus, the real and imaginary parts, as displayed in Duggleby et al. (2007), may not be directly suitable for comparison (if the phase shifts are different), so the moduli of the complex eigenfunctions $\hat{\phi}_i(k_x,r,k_\theta)$ are compared instead. While the phase information is removed, this presentation is still informative to compare the general behaviour and how the energy is distributed between the different velocity components. The magnitudes are normalized to conform with the normalization of Duggleby et al. (2007). Figures 6.3–6.6 are presented in the same order as the (1,5,1), (2,2,1), (1,1,1), and (1,0,1) modes appear in Duggleby et al. (2007).

In all four modes shown, the streamwise component of fluctuating velocity dominates the contributions for each mode. The (1,5,1), (2,2,1), and (1,1,1)
Figure 6.3: Modulus of the eigenmode for the (1,5,1) mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$.

Figure 6.4: Modulus of the eigenmode for the (2,2,1) mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$.

Figure 6.5: Modulus of the eigenmode for the (1,1,1) mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$. 

245
Figure 6.6: Modulus of the eigenmode for the (1,0,1) mode of the present simulation (solid line) compared with that of Duggleby et al. (2007) (dashed line). Velocity components are: (a) $u$, (b) $u_r$, and (c) $u_\theta$.

modes’ $u$ components rapidly gain magnitude from the wall (where their values must be zero to satisfy the no-slip condition) and, in all of these cases, they decay to zero near the pipe core. All velocity components of these modes behave very similarly between the $Re_\tau = 685$ and $Re_\tau = 150$ flows, despite the considerable difference in Reynolds numbers. A significant difference that is apparent mainly in the $u$ and $u_\theta$ modes is the steeper increase from the wall with higher Reynolds number. In chapter 3, this was observed to occur in one-dimensional ($y$) modes for turbulent channel flows with a range of Reynolds numbers. This behavior is similar to that observed in the modes for the streamwise velocity fluctuation of the channels identified as ‘idiosyncratic.’ Idiosyncratic modes were found to be dependent on the outer length scale except for this thin, near-wall boundary layer behavior. The eigenfunctions for the three-dimensional pipe POD are considerably more complicated that those of the one-dimensional channel POD, as they are complex-valued, dependent on the wavenumber pair for the remaining domain coordinates, and associated with three velocity components for each mode. However, the boundary layer behavior steepens with increasing Reynolds number while the radial dependence for the remaining region of streamwise velocity fluctuation appears to scale in outer units. This similarity with the low $n$ one-dimensional channel modes sug-
gests that these pipe modes may possess other features of the idiosyncratic modes, while adherence to this classification could also depend on $i_x$ and $i_{\theta}$ for the three-dimensional case.

In radial extent, all three of these pipe modes occupy large fractions of the pipe cross-section. This is consistent with the results of Hellström et al. (2011) who noted that VLSMs formed by a reconstruction with the 10 most energetic POD modes had large radial extents inward from the wall. The similarity of the modes between the two Reynolds numbers of the present pipe and Duggleby et al. (2007) suggests that physical mechanisms associated with these large modes may remain similar despite the sizable difference in Reynolds numbers. However, the differences in eigenvalue spectra suggest that the relative magnitudes of their contributions vary with Reynolds number.

The $(1,0,1)$ eigenmode modulus, which is associated with an azimuthally-constant mode, indicates that the $u'$ velocity contribution is strongest in the core of the pipe, in contrast to the three modes previously considered in which these contributions are strongest just above the wall. This mode is therefore related to long streamwise fluctuations in the pipe core. The mode includes streamwise velocity fluctuation contributions up to the wall, but it varies between Reynolds numbers in the higher Reynolds number having the velocity more strongly concentrated at the pipe core.

The other velocity components make weaker contributions for these modes, but other modes making smaller energetic contributions would be expected to exist with the other velocity components more dominant. Despite the relatively weak roles of the other velocity components of the modes compared, significant agreement is apparent in these components between the two Reynolds numbers. In the $(1,5,1)$ mode, for example, the moduli indicate that both the $u$ and $u_r$ components make significant contributions near $y/R = 0.3$, but this is insufficient to determine if this
mode makes significant Reynolds stress ($uu_r$) contributions. The phase relationships between the components would need to be known, since the contributions of each velocity component could be occurring at the same locations or significantly out of phase. Displaying both the real and imaginary parts of each component of the eigenmode instead of modulus would provide this information, but it is difficult to interpret. Three-dimensional isosurface visualizations of the quantities of interest would clarify the phase relations in three dimensions. The present study focuses on the streamwise fluctuation component, since they are commonly used to identify VLSMs, as in the study of Hellström et al. (2011).

The two most energetic modes for the present $Re_T = 685$ pipe simulation are the $(1, 2, 1)$ and $(2, 3, 1)$ modes, whereas the most energetic for the $Re_T = 150$ pipe simulation of Duggleby et al. (2007) was the $(1, 5, 1)$ mode. Three-dimensional visualizations of streamwise velocity for these three modes are displayed in figures 6.7–6.9. The modes are visualized with arbitrary phase and with the isosurfaces generated at half of the peak value (in which red and blue indicate positive and negative). These $n = 1$ modes appear to become more concentrated near the wall as the wavenumber indices become larger (wavelengths decrease). Figure 6.9 displays the $(1, 5, 1)$ mode for which the eigenmode modulus was presented in figure 6.3. Since the streamwise and azimuthal variation is determined by the wavenumber indices, only the radial behavior is determined by the POD analysis, but the phase information also determines the cross-sectional shape of the mode, as is apparent by comparing these visualizations. The additional velocity components and phase relationships between velocity components allows significant freedom in the character of the modes.

As the most energetic modes were observed to be inclined azimuthally (swirling) based on their wavenumber index values and the most energetic modes in Hellström et al. (2011) were observed to recreate the character of the VLSMs
Figure 6.7: Visualization of the streamwise velocity fluctuation for the (1,2,1) eigenmode of the present pipe flow.

Figure 6.8: Visualization of the streamwise velocity fluctuation for the (2,3,1) eigenmode of the present pipe flow.
Figure 6.9: Visualization of the streamwise velocity fluctuation for the (1,5,1) eigenmode of the present pipe flow.

(although they were obtained by snapshot POD), it is of interest how these modes might represent the VLSMs. McKeon & Sharma (2010) showed that combining modes of similar form swirling in opposite directions can create azimuthally- and streamwise-alternating streaks of velocity fluctuation. It is also possible that the modes could make larger contributions corresponding to the dominant azimuthal inclination angles of the streaks. It was shown in chapter 5 (figure 5.16) that azimuthal inclination angles of approximately $5^\circ$ appeared in patterns of the streamwise–azimuthal two-point correlation at $y/R = 0.15$. Over the $30R$ simulation length, $5^\circ$ is a circumferential twist of $2.62R$ or an angular rotation about the pipe axis of $3.09$ radians. The corresponding ratio of wavelengths for the mode to swirl by this amount over the $30R$ domain length is $\lambda_\theta/\lambda_x = 0.1$. In terms of wavenumber indices, $i_x/i_\theta = [30/(2\pi)](\lambda_\theta/\lambda_x)$, so modes with $i_\theta = 2.1i_x$ rotate by $5^\circ$ circumferentially. This suggests that $i_x = 1$ would be associated with $i_\theta = 2$ and $i_x = 2$.
would be associated with \( i_\theta = 4 \) for modes to represent this angle. It was found that 
(1,2,1) and (2,3,1) were the most energetic modes in the eigenspectra, and the angle 
may be somewhat shallower (4° was also measured). While reconstructions of 
modes and additional verifications would be necessary to confirm that these modes 
are closely related to the inclination angles observed in the flow, this suggests one 
possible relation between POD modes and VLSMs.

6.4 Conclusions

The POD decomposition of a \( Re_\tau = 685 \) turbulent pipe indicates significant agree-
ment with modes for the largest scales obtained from an \( Re_\tau = 150 \) pipe. The behav-
ior of these modes near the wall appears consistent with properties of idiosyncratic 
modes observed in a simpler one-dimensional POD analysis of turbulent channel 
data. Much of the flow energy is concentrated in the \( n = 1 \) radial POD mode for 
each streamwise and azimuthal wavenumber pair. The eigenvalue spectra indicate a 
widely distributed of streamwise and azimuthal scales for higher Reynolds numbers, 
and the dominant (most energetic) of these modes change with Reynolds number, 
while the associated POD modes may remain very similar. The evidence also sup-
ports a close relation between the large-scale POD modes and VLSMs, and the re-
lation also observed using snapshot POD in turbulent pipe flow by Hellström et al. 
(2011).
Chapter 7

TURBULENT BOUNDARY LAYER STRUCTURE IDENTIFICATION VIA POD

Proper orthogonal decomposition (POD) is applied to the direct numerical simulation (DNS) of a turbulent boundary layer performed by Wu & Moin (2010), and the resulting POD modes of various scales are examined. The modes include structures resembling those observed in instantaneous flow fields, such as large-scale motions of streamwise velocity with ramp-like wall-normal growth. Other modes correspond closely to near-wall streaks. In addition, POD modes that are constant across the spanwise domain width are observed to grow from the wall with the mean boundary layer thickness. The results support the existence of boundary layer coherent motions described by the hairpin packet model (Adrian, 2007).

Acknowledgements: The authors wish to acknowledge the generous support of the Center for Turbulence Research. This research was also supported by NSF Award CBET-0933848.

7.1 Introduction

Coherent motions in wall-bounded turbulent flows have been the subject of recent study, but questions remain about the organization and form of coherent structures (Adrian, 2007). One method of extracting coherent structures is POD (Lumley, 1981; Holmes et al., 1998). POD extracts modes that are linearly combined to form each flow field snapshot, with the reconstruction by partial sums of POD modes converging faster than by any other set of orthogonal functions in a mean energy sense (Liu et al., 2001).

POD has been successfully applied to wall-bounded turbulent flows including three-dimensional DNS data sets. Moin & Moser (1989) calculated POD modes for a $Re_	au = 180$ turbulent channel DNS simulation and focused on obtaining compact
structures. Since the resulting POD modes span a wide range of spatial scales as large as the domain, they were interpreted using a method proposed by Lumley (1981) in which characteristic eddies were assembled from POD modes with phases chosen to make the eddies compact. The method assumed characteristic eddies to be scattered randomly in the homogeneous streamwise and spanwise coordinates.

Recent attention has focused on large-scale motions (LSMs) and very large-scale motions (VLSMs), also known as superstructures, observed in wall-bounded shear flows (e.g. Kim & Adrian, 1999; Hutchins & Marusic, 2007a). Examining these motions requires interpretations of POD modes that preserve the large scales instead of assembling compact motions. Liu et al. (2001) employed POD to analyze two-dimensional velocity measurements of a turbulent channel including studying the LSMs and VLSMs. They examined individual POD modes and considered projections of fields onto sets of modes (partial reconstructions). Although POD in a homogeneous direction is simply a Fourier mode, this is appropriate for analyzing these motions because LSMs and VLSMs are often defined in terms of Fourier spectra. The results indicated that a set of several large-scale POD modes were associated with large contributions to the turbulent kinetic energy and Reynolds stress. The patterns of motion revealed by projections onto sets of POD modes were consistent with the hairpin packet paradigm (Adrian et al., 2000b), in which hairpin-shaped vortical structures are understood to arrange in streamwise $x$-aligned packets, and thereby contribute to long structures. Adrian et al. (2000b) frequently observed ramps of retarded streamwise velocity with wall-normal growth angles of approximately 10-20° relative to $+x$ as evidence of hairpin packets in turbulent boundary layers.

While POD has also been applied to DNS of turbulent pipes (e.g. Duggleby & Paul, 2010), there have been no recent three-dimensional applications to turbulent boundary layer flows, although structures in transition were studied with POD
by Rempfer & Fasel (1994). Recent incompressible zero-pressure-gradient flat-
plate turbulent boundary layer simulations of Wu & Moin (2009a) periodically in-
troduced blocks of isotropic turbulence into the laminar flow at the inlet and allowed
the boundary layer to progress through transition. Hairpin-shaped vortical struc-
tures were more clearly visible in this simulation compared to previous simulations
with artificially generated turbulent inflows. Wu & Moin (2010) extended the sim-
ulation to a longer streamwise domain so the flow would evolve from \( Re_\theta = 80 \)
 to 1950 and increased the spanwise domain width. With the clear structures observed
in this simulation, POD is an appropriate tool to extract structural information from
this data set.

We apply POD to the entire DNS fields of this simulation including the transi-
tion region. The choice of domain is relevant because the POD equation and
orthogonality between POD modes are defined by an inner product over a speci-
fied domain (\S 7.2). The POD modes decompose the fluctuating velocity field. The
inner product involves all three velocity components, so the resulting POD modes
also include contributions of all velocity components.

7.2 Method

POD is performed on the flow using the conventional norm such that optimality of
convergence exists in the energy sense. The standard POD equation for the three-
dimensional vector field of velocity fluctuation \( \mathbf{u} \) based on an expansion of the form
\[
\mathbf{u}(x) = \sum_{n=1}^{N} a_n \Phi(x)
\]
is

\[
\int_D R(x,x')\Phi(x')dx' = \lambda \Phi(x),
\]
with the two-point spatial correlation \( R(x,x') = \langle u(x) \otimes u^*(x') \rangle \) (Holmes et al.,
1998).

The boundary layer is homogeneous and periodic in \( z \), so POD modes converge
to trigonometric functions in \( z \), and it is appropriate to enforce this behavior by
expressing the velocity components $u_i$ as Fourier series expansions,

$$u_i(x,y,z,t) = \sum_{k_z=-N_z/2+1}^{N_z/2} \hat{u}_i^{(k_z)}(x,y,t) e^{2\pi \sqrt{1-k_z^2} \frac{t}{L_z}}. \quad (7.2)$$

This procedure has been employed in several applications (e.g., Freund & Colonius, 2009; Duggleby & Paul, 2010), and has been shown to improve the statistical convergence of POD because it incorporates information from all possible shifts of the homogeneous coordinate. Then, a POD expansion is used to represent the Fourier coefficients $\hat{u}_i^{(k_z)}(x,y,t)$:

$$\hat{u}_i^{(k_z)}(x,y,t) = \sum_{n=1}^{N} \hat{a}^{(k_z, n)}(t) \hat{\phi}_i^{(k_z, n)}(x,y). \quad (7.3)$$

The inhomogeneity of both $x$ and $y$ results in a direct POD (7.1) problem with a very large correlation matrix of averages calculated over the series of flow field snapshots spaced in time. The number of snapshots is significantly fewer than the number of points involved, so calculating the POD modes using the method of snapshots (Sirovich, 1987) greatly reduces the computation necessary. While mathematically equivalent, the eigenproblem is performed on time correlations instead of spatial correlations. This method is derived (Sirovich, 1987) by expressing the POD modes as linear combinations of the $N_t$ snapshots $\hat{\phi}_i^{(k_z, n)}(x,y) = \sum_{j=1}^{N_t} c^{(k_z, n, t_j)} \hat{u}_i^{(k_z)}(x,y,t_j)$. Then, the eigenproblem (7.1) becomes an eigenproblem to solve for the coefficients $c$

$$\sum_{j=1}^{N_t} M^{(k_z)}_{h_j} c_j^{(k_z, n)} = \lambda^{(k_z)} c^{(k_z)}_{h}, \quad (7.4)$$

where the $M^{(k_z)}_{h_j}$ matrix contains the time correlations. For the present problem, these time correlations are between Fourier coefficients

$$M^{(k_z)}_{h_j} = \int_0^{L_y} \int_0^{L_x} \hat{u}_i^{(k_z)}(x,y,t_j) \hat{u}_i^{*(k_z)}(x,y,t_h) \, dx \, dy = \left( \hat{u}_i^{(k_z)}(x,y,t_j), \hat{u}_i^{(k_z)}(x,y,t_h) \right) \quad (7.5)$$

255
(summation assumed on repeated indices) (Freund & Colonius, 2009). Integration is approximated by the trapezoidal rule, which is satisfactory for POD (Moin & Moser, 1989).

The POD modes obtained from the snapshot method are normalized to satisfy orthonormality as

$$\int_0^{L_y} \int_0^{L_x} \hat{\phi}_i^{(k_z,m)}(x,y) \hat{\phi}_i^{*(k_z,n)}(x,y) \, dx \, dy = \delta_{mn}. $$

The POD coefficients to reconstruct each velocity field in (7.3) are obtained from

$$\hat{a}^{(k_z,n)}(t) = \int_0^{L_y} \int_0^{L_x} \hat{u}_i^{(k_z)}(x,y,t) \hat{\phi}_i^{(k_z,n)}(x,y) \, dx \, dy. \quad (7.6)$$

Due to the conjugate symmetry $\hat{\phi}_i^{(k_z,n)}(x,y) = \hat{\phi}_i^{*(-k_z,n)}(x,y)$, computing only the non-negative $k_z$ modes is sufficient to reconstruct the real-valued velocity. Mode indices $n$ are numbered by decreasing eigenvalue, with $n = 1$ contributing most energy.

As discussed in Wu & Moin (2010), a Blasius profile with momentum thickness $\theta_0$ is specified at the inlet. The computational domain dimensions are $12750\theta_0$, $2250\theta_0$, and $562.5\theta_0$ in the streamwise $x$, wall-normal $y$, and spanwise $z$ coordinates, respectively. The corresponding numbers of grid points are $N_x = 8192$, $N_y = 500$, and $N_z = 256$. The POD modes were calculated from a collection of $N_t = 54$ DNS flow field snapshots separated in time by at least $150\theta_0/U_\infty$. The mean velocities were obtained using frequent sampling during the original simulation run and are better converged than if they were obtained from the 54 fields available. Both time averaging and spatial averaging in the homogeneous $z$ coordinate are used to calculate the mean velocities for obtaining the fluctuation velocities.

### 7.3 Results

The fluctuating velocity of a DNS field is reconstructed by (7.2) and (7.3):

$$u_i(x,y,z) = \sum_{k_z=-N_z/2+1}^{N_z/2} \sum_{n=1}^{N} \hat{a}^{(k_z,n)} \hat{\phi}_i^{(k_z,n)}(x,y) e^{2\pi \sqrt{-1} \frac{k_z z}{L_z}} = \sum_{k_z=-N_z/2+1}^{N_z/2} \sum_{n=1}^{N} u_i^{(k_z,n)}(x,y,z). \quad (7.7)$$

256
Each mode’s contribution is $u_i^{(k_z,n)} = \hat{a}_i^{(k_z,n)} \hat{\phi}_i^{(k_z,n)} e^{2\pi \sqrt{-1} \frac{k_z}{L_z}} = \hat{a}_i^{(k_z,n)} \phi_i^{(k_z,n)}$. Liu et al. (2001) discussed visualization and interpretation of individual POD modes, and similar principles apply to the present results. The POD coefficient $\hat{a}$ specifies the magnitude and phase (which corresponds to a spanwise shift of the entire field in physical space) of each modal contribution, so visualizing only the real part reveals all of the features for each mode. Liu et al. (2001) show that the imaginary part of $\phi_i^{(k_z,n)}$ is equal to the real part of $\phi_i^{(k_z,n)}$ after applying a $\pi/2$ phase shift, which is a spatial shift of $L_z/(4k_z)$. The mode contributions $u_i^{(k_z,n)}$ are linear combinations of the real parts of $\phi_i^{(k_z,n)}$ with and without the shift, so examining only the real part is sufficient. The real part of $\phi_i^{(k_z,n)}$ is equal to the $u_i^{(k_z,n)}$ contribution of a $\pm k_z$ mode pair with $\hat{a}^{(k_z,n)} = 0.5$.

Figure 7.1 contains isosurfaces of streamwise velocity for this real part of a representative selection of individual POD modes. The sign of velocity is arbitrary because the sign is dictated by $\hat{a}^{(k_z,n)}$, and a phase shift of $\pi$ reverses the sign of velocity at a given location. With this sinusoidal dependence, velocity structures of opposite sign and the same shape are present in spanwise locations between the structures shown in the isosurfaces. The oblique perspective from above the flat plate highlights the organization of the $u$ velocity structures that take the form of long streamwise streaks. Figure 7.2 contains similar plots for $k_z = 20$.

Contour plots of $u$ in $x$-$y$ planes for two $k_z = 0$ modes are presented in Figure 7.3. The boundary layer thickness $\delta(x)$, defined as the $y$ where mean velocity is 99% of free-stream, is included as a line. The $n = 1$ mode (a,b), which is responsible for the largest mean energy contribution to the flow, consists of alternating regions of positive and negative $u$ velocity fluctuations with wall-normal growth closely matching the boundary layer thickness. The $n = 2$ mode (not shown) is similar except that the streamwise positions of the structures are shifted by one quarter
Table 7.1: Eigenvalues ($\lambda/E$).

<table>
<thead>
<tr>
<th>$k_z$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
<th>$n = 1-16$</th>
<th>$n = 1-54$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.95%</td>
<td>2.72%</td>
<td>0.38%</td>
<td>0.36%</td>
<td>0.34%</td>
<td>0.30%</td>
<td>9.04%</td>
<td>11.36%</td>
</tr>
<tr>
<td>±1</td>
<td>1.11%</td>
<td>0.77%</td>
<td>0.64%</td>
<td>0.58%</td>
<td>0.53%</td>
<td>0.50%</td>
<td>7.49%</td>
<td>12.64%</td>
</tr>
<tr>
<td>±2</td>
<td>0.67%</td>
<td>0.60%</td>
<td>0.52%</td>
<td>0.49%</td>
<td>0.47%</td>
<td>0.41%</td>
<td>6.27%</td>
<td>11.71%</td>
</tr>
<tr>
<td>±3</td>
<td>0.73%</td>
<td>0.40%</td>
<td>0.37%</td>
<td>0.35%</td>
<td>0.33%</td>
<td>0.32%</td>
<td>4.99%</td>
<td>9.81%</td>
</tr>
<tr>
<td>±4</td>
<td>0.35%</td>
<td>0.32%</td>
<td>0.28%</td>
<td>0.27%</td>
<td>0.25%</td>
<td>0.24%</td>
<td>3.66%</td>
<td>7.85%</td>
</tr>
<tr>
<td>±5</td>
<td>0.23%</td>
<td>0.23%</td>
<td>0.22%</td>
<td>0.21%</td>
<td>0.20%</td>
<td>0.19%</td>
<td>2.85%</td>
<td>6.36%</td>
</tr>
</tbody>
</table>

period, and $n = 1$ and $n = 2$ mode eigenvalues are comparable. This is similar to the sinusoidal modes that would occur if the flow were homogeneous in $x$ (as in the case of a channel) but with wall-normal growth for the boundary layer and decay in strength. These features are also consistent with modes for traveling waves, as discussed by Aubry et al. (1992). Although blocks of isotropic turbulence were periodically introduced at intervals of $3131.45\theta_0/U_\infty$ for this simulation (Wu & Moin, 2010), the streamwise period of the structures in the mode is significantly shorter. The $n = 4$ mode (c,d) includes similar structures of alternating $u$ fluctuation in the transitional region but with shorter wavelength. These structures are less distinct in the second half of the domain (d). This mode also includes contributions associated with the isotropic turbulence blocks that are apparent above the boundary layer thickness.
Figure 7.1: Streamwise velocity isosurfaces of POD modes arranged in sets of constant $k_z$, with $k_z = 1, 5, 10$ from left to right and $n = 1, 5, 10, 20$ within each group. Approximately two spanwise periods are shown, except for the $k_z = 1$ modes where the entire domain is shown. Visualizations of the other fields have the same scale. The modes are viewed obliquely from above the flat plate.
Figure 7.2: Streamwise velocity isosurfaces of POD modes for $k_z = 20$ and $n = 1, 5, 10, 20$ (top to bottom).

Figure 7.3: Planes of the $n = 1$ (a,b) and $n = 4$ (c,d) $k_z = 0$ modes shaded by streamwise velocity for each streamwise half of the domain. The dotted white line indicates the boundary layer thickness $\delta(x)$. 
Figure 7.4 presents vector plots of planes extracted at spanwise locations where the negative $u$ structures are strongest. Boundary layer thickness $\delta(x)$ is also included as a gray line. As spanwise wavenumber index $k_z$ increases, the wall-normal height of $u$ structures decreases for the most energetic mode numbers ($n = 1$ and 5 shown). The shorter ramps are clearly apparent in the $u$ isosurfaces of Figure 7.5. The $k_z = 5$ structures are consistent with the ramp structures associated with hairpin packets, which are discussed by Adrian et al. (2000b). The vector plots (Figure 7.4b,c) indicate the negative velocity structures are associated with positive wall-normal velocity. Steeply inclined regions of these quadrant 2 ejections below a swirling hairpin head pattern are identified as the signature of hairpin vortices which organize to form hairpin packets (Adrian et al., 2000b). These modes contain evidence of such structures with streamwise lengths of several $\delta$ (consistent with large-scale motions), but other modes that include finer scales also contribute to details of the vortices. Although POD is used instead of linear stochastic estimation, this analysis is similar to that of Christensen & Adrian (2001), in that statistically important structures are extracted and shown to be consistent with hairpin packet structures. The present calculation is successful in extracting relevant structures, although statistical convergence of the POD modes would improve with additional snapshots.

For the mode with $k_z = 20$ and $n = 1$, the strongest structures are centered about $y/\theta_0 \approx 5$, which corresponds to $y^+ = 15$ in this region, although they can extend up to $y^+ = 75$. The spanwise wavelength associated with the $k_z = 20$ modes is approximately $100^+$, which is the accepted near-wall streak spacing $\lambda^+$. These features suggest that this mode corresponds to the near-wall velocity streak motions.

The spectrum shown in Figure 7.6 contains the eigenvalues, with values representing the mean energy of pairs with wavenumber indices $\pm k_z$ because modes must contribute in pairs for the velocity to be real valued. The $n$ index is responsi-
Figure 7.4: Mode velocity of (a) $k_z = 1, n = 1$, (b) $k_z = 5, n = 1$, (c) $k_z = 5, n = 5$
Figure 7.4: Mode velocity of (d) $k_z = 10, n = 1$, (e) $k_z = 10, n = 5$, (f) $k_z = 20, n = 1$
Figure 7.5: $u$ isosurfaces for (a) $k_z = 5, n = 1$ and (b) $k_z = 20, n = 1$ POD modes.

Figure 7.6: POD mode eigenvalues.
Figure 7.7: Negative $u$ isosurfaces (shaded by $y$) of one DNS field (a,b) and its reconstruction with $k_z = 0–5$ and $n = 1–16$ POD modes (c,d).

able for the various scales in both the inhomogeneous streamwise and wall-normal directions, whereas $k_z$ represents scale in only the spanwise coordinate. Therefore, the eigenvalue decay is slower in $n$ than in $k_z$. $E$ represents the mean turbulent kinetic energy, and the sum of all $\lambda/E$ displayed is unity. The spectrum indicates much of the energy is contained in low $k_z$ modes, and Table 7.1 summarizes the contributions with $n = 1–54$ including all POD modes.

From the projection of one DNS field onto the POD modes, a partial reconstruction is generated using a set of the most energetic modes with $k_z = 0–5$ and $n = 1–16$. Omitting spanwise Fourier modes effectively applies a low-pass cutoff filter. Figure 7.7 compares negative $u$ fluctuation isosurfaces for the original DNS field and POD partial reconstruction. Since the omitted modes of smaller scales contribute to the velocity peaks, a lower threshold is chosen to plot the isosurfaces of the reconstructed field. The reconstruction indicates how large-scale motions evolve from transition to the fully turbulent regions. It is apparent from the reconstruction that the $k_z = 0$ modes with no spanwise variation (discussed in connection with Figure 7.3) make significant contributions to the flow, which is consistent with their large eigenvalues.
POD modes for a turbulent boundary layer reveal structures that can be identified with features observed instantaneously in the flow. The velocity structures include near-wall streaks and ramps consistent with the hairpin vortex paradigm. The POD modes revealing these structures are useful because they represent patterns of these structures that are statistically significant and likely appear frequently. The negative $u$ ramp segments for the $k_z = 5$ modes shown in Figure 7.1 may indicate characteristic lengths of hairpin packets and the spatial patterns suggest how packets organize. The spanwise drifts and staggering are physically significant and show that POD modes contain useful spanwise information despite the trigonometric behavior. POD also identified modes consistent with traveling waves that decay and grow in scale, and further analysis can address how their form may be affected by the isotropic turbulence introduced at the inlet and how this influences the structure downstream. Tracking the time evolution of the POD mode coefficients and partial reconstructions would reveal further flow physics.
Chapter 8

TURBULENT BOUNDARY LAYER POD MODES: COMPARISON OF INFLOW EFFECTS

Turbulent boundary layers may be simulated with the inflow specified in different methods. While the simulation of Wu & Moin (2010) allowed a laminar inflow with disturbances to undergo transition, the simulation domain can be reduced by introducing a synthetic turbulent flow at the inlet. This significant reduction in domain length leads to a less computationally demanding simulation, but this has the possibility of introducing error with introducing inflow that has undergone a less physically natural evolution. Rescaling the flow that exists at a constant-streamwise-position plane near the outflow and introducing it as the inflow has been a popular method to do this. Methods also include using an auxiliary simulation to allow the flow to be provided as inflow to the main simulation to become more natural (Lund et al., 1998). The simulation of Ferrante et al. (2004) introduced a turbulent inflow by rescaling the outflow of the simulation domain and allowing it to evolve in an auxiliary simulation, yielding statistics for the main simulation that were in good agreement with other results.

As proper orthogonal decomposition (POD) is a method by which to extract structure from turbulent flows, POD is here employed as a means to compare the structures that exist in the Wu & Moin (2010) simulation undergoing a transition from laminar flow with the structures that exist in the Ferrante et al. (2004) simulation. Since orthogonality of POD modes is defined with respect to integration over a particular domain, using identical domain sizes for each case is proper for a comparison. While chapter 7 is based on a full domain POD analysis, only a section of this streamwise domain is needed for an equivalent section to the full simulation domain of Ferrante et al. (2004).
8.1 POD Domain

The simulation parameters and domain sizes are summarized in table 2.1. The determination of equivalent streamwise domains for the two simulations is dependent on how equivalent lengths are defined. Schlatter & Örlü (2010) showed that some statistics of different turbulent boundary layer simulations contained “surprisingly large differences” depending on the simulation methods used, including in parameters relating integral length scales such as shape factor $H_{12}$. For this reason, care was taken in defining equivalent domains.

The method used was to determine the Reynolds number $Re_\theta$ at the streamwise center of the Ferrante et al. (2004) main simulation domain and then calculate the streamwise domain length $L_x$ in terms of $\theta_c$, the boundary layer momentum thickness measured at this location. Then the location in the Wu & Moin (2010) data with the same $Re_\theta$ is found, $\theta_c$ is measured at this location in this channel simulation’s length units, and the streamwise domain is selected with the same $L_x/\theta_c$ length.

The domain is summarized as follows, with $L_z$ representing the spanwise domain width and $\delta_c$ representing the 99% boundary layer thickness measured at the streamwise center of each domain (at which the $Re_\theta$ values were matched). The values in table 8.1 indicate slight differences between the simulations when the domain lengths are normalized by $\delta$ instead of $\theta$, as expected from Schlatter & Örlü (2010).

It also should be noted that the spanwise widths of the two turbulent boundary layers are different regardless of which quantity they are normalized by, indicating the domain widths are slightly different. While only a subset of the wider domain could be used to make the domain widths match, this causes difficulty because the spanwise directions of the simulations are periodic, and the subset would no
Table 8.1: Domain dimensions for turbulent boundary layer comparison.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_x/\theta_c$</td>
<td>129.1</td>
<td>129.1</td>
</tr>
<tr>
<td>$L_x/\delta_c$</td>
<td>15.2</td>
<td>14.4</td>
</tr>
<tr>
<td>$L_z/\theta_c$</td>
<td>32.3</td>
<td>37.4</td>
</tr>
<tr>
<td>$L_z/\delta_c$</td>
<td>3.80</td>
<td>4.18</td>
</tr>
<tr>
<td>$L_z^+$ (center)</td>
<td>1859</td>
<td>2038</td>
</tr>
</tbody>
</table>

Figure 8.1: Comparison of $\lambda_{ci}$ isosurfaces in sample fields of the data sets: (a) Ferrante et al. (2004) and (b) Wu & Moin (2010).

This, in turn, causes difficulty because using a discrete Fourier transform is used to calculate the Fourier spanwise behavior of the POD modes. For this reason, the full spanwise domain widths are used for each of the simulations, with the understanding that spanwise wavenumber indices are not precisely equivalent between the simulations.

Visualizations of three-dimensional swirling strength $\lambda_{ci}$ and streamwise velocity fluctuation $u'$ are displayed for sample fields on the equivalent domains deter-
Figure 8.2: Comparison of negative $u'$ isosurfaces in sample fields of the data sets: (a) Ferrante et al. (2004) and (b) Wu & Moin (2010).

mined for each of the flows. In general, there appears to be finer scales of vortices in figure 8.1 for the Wu & Moin (2010) data compared to that of Ferrante et al. (2004), and this is probably due to the higher resolution of the simulation. In figure 8.2, the negative $u'$ regions appear qualitatively similar.

8.2 POD Methodology

The POD methodology is the same as that applied in §7.2, except with the new domain size.

The velocity fluctuation is represented by a Fourier expansion in the periodic spanwise coordinate, and this implicitly includes all spanwise shifts of fields in ensemble. Applying the snapshot POD method on Fourier coefficients computes POD with no homogeneity imposed on $x$ and $y$. POD is calculated for all 3 components.
of velocity fluctuation

\[
\bar{u}_i^{(k_z)}(x,y,t) = \sum_{n=1}^{N} a^{(k_z,n)}(t) \phi_i^{(k_z,n)}(x,y).
\]  

(8.1)

Modes are indexed by spanwise wavenumber index \( k_z \) and energy ordering \( n \); \( n \) incorporates both streamwise and wall-normal length scales.

8.3 Statistical Convergence Effects to POD Modes

The data set of Ferrante et al. (2004) results in greater statistical convergence with 321 fields to calculate the POD instead of the 54 fields used in the POD calculation for the Wu & Moin (2010) data. Prior to comparing modes from the different simulations, it is necessary to determine if the lesser statistical convergence of one data set would significantly affect the conclusions. To test this, POD is performed on 54 fields selected from the 321 fields of Ferrante et al. (2004). As the fields were obtained at time intervals in the simulation, the 54 fields were chosen to span the time duration for the entire set of 321 fields and be approximately uniformly spaced. This best matches the character of the 54 fields of Wu & Moin (2010) because they are relatively widely spaced in time also.

\( u' \) velocity fluctuation isosurfaces are visualized for POD modes computed from the full 321 field set and 54 field set of Ferrante et al. (2004) data. They are displayed with the flow (and increasing \( x \)) generally from left to right, and the structures generally exist near the wall (\( y = 0 \)) of the boundary layer. The isosurfaces are generated at two values with the same magnitudes but opposite signs, and they are displayed in red and blue. A selection of mode numbers are compared. Here \( k_z \) denotes the spanwise wavenumber index instead of the actual wavenumber. Several of the lowest \( n \) (most energetic) modes are shown, with several angles to display the salient features. The spanwise shift is arbitrary and random for each calculation, so a spanwise shift between the 321 field and 54 cases is to be expected. The modes with 321 fields are smoother than with 54 fields, and the modes contain less small
Figure 8.3: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 1$.

Figure 8.4: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 2$.

scale details These results indicate that modes obtained using 54 fields give good representations of the modes obtained using 321 fields (for the most energetic few modes). From these results, it is concluded that POD on this domain with 54 fields is sufficient to reproduce the behavior of more statistically converged POD.
Figure 8.5: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 3$.

Figure 8.6: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 1$ and $n = 4$.

Figure 8.7: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 1$. 

273
Figure 8.8: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 2$.

Figure 8.9: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 3$. 
Figure 8.10: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 2$ and $n = 4$.

Figure 8.11: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 5$ and $n = 1$. 

275
Figure 8.12: Comparison of $u'$ for POD modes calculated with 321 (left column) and 54 (right column) fields: $k_z = 5$ and $n = 2$.

8.4 Results

POD eigenvalue spectra representing the energy in each mode are displayed for the POD analyses of the Ferrante et al. (2004) (figure 8.13) and Wu & Moin (2010) (figure 8.14) data sets. Important modes that are also visualized are indicated with red dots in the eigenvalue spectra.

Figure 8.15 displays $k_z = 0$ modes. As these modes are constant in the spanwise direction, color contours are visualized in the streamwise–wall-normal plane instead of three-dimensional isosurfaces. These most energetic of the $k_z = 0$ modes contain streamwise-alternating high and low $u'$ regions. The $90^\circ$ phase shift between modes of neighboring $n$ is suggestive of traveling wave behavior. There is similar behavior for the Wu & Moin (2010) POD calculations on the present domain and the full domain of chapter 7. For this reason, the POD modes are shown (for these modes only) on the full domain, with the arrowed red line indicating the domain section for the present study. The dashed line represents the boundary.
Figure 8.13: POD eigenvalue spectrum for Ferrante et al. (2004) data (321 fields).

Figure 8.14: POD eigenvalue spectrum for Wu & Moin (2010) data (54 fields).
layer thickness $\delta$. There is different behavior between the flow simulations: POD performed on Ferrante et al. (2004) (hereafter FE) contains two layers of $u'$ with alternating sign below $\delta$, but POD performed on Wu & Moin (2010) (hereafter WM) contains one layer below and one above $\delta$. This difference is possibly a result of the different inflow/transition specified for each simulation.

Figure 8.16 displays isosurfaces for the real part of $u'$ of POD modes for the lowest positive (nonzero) $k_z$ values. The modes appear to be either streamwise-aligned or significantly inclined in a spanwise sense. Between the simulations, the ordering is different, with the $k_z = 1, n = 2$ FE mode more similar to the $k_z = 1, n = 1$ WM mode (but with a different number of wall-normal layers, as seen for the $k_z = 0$ modes), whereas the $k_z = 1, n = 1$ FE mode has spanwise inclination similar to the $k_z = 2, n = 1$ WM mode. For the inclined modes, it is noteworthy that a streamwise periodicity appears naturally (though the wavelength may vary), as streamwise periodicity (in a Fourier decomposition) would normally be imposed if the flow were homogeneous in the streamwise direction, and would also be expected to occur naturally when the flow is homogeneous.
The $k_z = 5$ mode nominally corresponds to a wavelength of $\lambda_z = 0.8\delta$ and $\Lambda^+ = 400$. The modes appear ramp-like and have distinct breaks in the streamwise direction. A side view of a negative $u'$ ramp isosurface for the $k_z = 5$ and $n = 1$ mode of WM is displayed in figure 8.18(a). A velocity vector plot at a streamwise–wall-normal plane taken through the spanwise center of this structure is shown in figure 8.18(b). Using a less statistically-converged POD calculation using fewer fields on the full domain for WM, the ramps contain more detail of fine structure. Detail of one such ramp is shown in figure 8.19. Ramps from partly converged POD also display the presence of vortices surrounding the ramps; the vortices are likely incorporated into higher order modes with better convergence as small-scale details are averaged out of the larger-scale POD modes. Nonetheless, this is evidence that vortices (matching the hairpin packet paradigm) are associated with these structures. The red lines mark example ramps in the POD mode, and yellow lines indicate the lines oriented approximately 45° up from the wall along which
Figure 8.17: $k_z = 5$ mode comparison.

Figure 8.18: Detail of a ramp in the $k_z n = 1$ mode of WM: (a) isosurface visualization and (b) vector plot through its center.

Q2 vectors are expected if the vortex core corresponds to a hairpin head (Adrian et al., 2000b).
In summary, the dominant forms of energetic low-\(k_z\) modes are, for the lowest \(k_z\), long streamwise structures growing approximately with the boundary layer thickness (spanning the complete \(15\delta\) domain length or nearly so). For the most energetic (low \(n\)) modes with \(k_z\) thereafter, the modes take the form of shorter streamwise structures with distinct ramp shapes of \(u'\) fluctuation.

The spanwise wavelength of \(\lambda^+ = 100\) is a well-documented value for the wavelength of near-wall streak spacing in wall-bounded turbulence. \(\lambda^+ = 100\) corresponds to wavenumber indices of \(k_z = 20.4\) for WM and \(k_z = 18.6\) for FE. Since the spanwise wavenumber indices must be integer values, \(k_z = 20\) is used as a nominal value for both simulations. The \(k_z = 20\) POD modes are compared between the simulations in figure 8.20. One feature that also characterizes other intermediate \(k_z\) modes is that the most energetic modes (low \(n\)) are concentrated near the wall with the intense motions in mode being much shorter than the boundary layer thickness. This is in contrast to the lowest \(k_z\) modes in which the entire boundary layer height \(\delta\) is occupied by the most energetic modes. This reflects the smallest-scale modes being most energetic near the wall, while the large-scale modes encompass large volumes and affect the flow down to the wall.
Among these long streaks visualized in figure 8.20, ramp-like features in streamwise-aligned segments with distinct streamwise lengths are apparent when these fields are visualized at stronger thresholds. A side profile of these motions in FE is visualized in the lower right corner of figure 8.20. It should be noted that these smaller structures are more similar between the two simulations than the large-scale structures are.

8.5 Conclusions

The most energetic of all POD modes for the flow are those with low $k_z$ (and low $n$), and these correspond to large-scale or very large-scale structures. The overall form of modes with low $k_z$ is of long, low momentum and high momentum streaks. Due to the periodicity of the spanwise Fourier modes for a POD decomposition of a flow with spanwise homogeneity, all modes (except for $k_z = 0$) have the form of spanwise alternating motions for each velocity component. For slightly higher $k_z$ values, corresponding to wavelengths on the order of $0.8\delta$ or $400^*$, the most energetic modes of streamwise velocity fluctuation appear as meandering concatenations of ramps. The most energetic modes for higher $k_z$ values correspond to near-wall
motions.

POD extracts different large-scale structures from the two simulations, including $k_z = 0$ modes, possibly as a result of different inflow processes. Smaller scales, however, become more similar between flows, suggesting that these robust features of turbulent boundary layers are successfully identified by the POD decomposition. Many of the features of POD modes have been related to physical phenomena and characteristic structures by comparisons with instantaneous features known to occur in the flows. Since the modes have the same character for positive and negative fluctuations (which are 90° of phase offset in the spanwise directions for each mode), it must be acknowledged that the modes represent an average of both motions and less energetic small scales (not shown in this study) supply the finest details to fill in details. While the inflow process is a large difference between the simulations that is anticipated to significantly affect the large scale modes, it is also possible that other differences between the simulations, including domain size, resolution, and specific numerical method details could affect these modes.
The very wide spanwise and azimuthal modes were found to be energetically important in POD decompositions on typical simulation domains. The two-point correlations that form the basis for these modes are now explored.

9.1 Introduction

For turbulent channels and boundary layers, simulations are generally performed on spanwise-periodic domains. In these simulations, $k_z = 0$ modes (in the simulation algorithm) represent the widest motions (spanwise constant) in the spanwise-periodic simulation domain. In POD analysis, these modes can contain a significant fraction of energy, so it is important to understand their physical meaning. If these motions are physical, their form has implications for large-eddy simulation (LES) involving a spatial filter.

These modes are analogous to $k_\theta = 0$ modes in turbulent pipes, in which they are physically relevant and not a consequence of periodicity imposed in the simulation domain, but naturally occur because velocity is periodic in the azimuthal coordinate due to the geometry. This study explores if the forms of the modes are similar between pipes and channels.

9.2 $Re_\tau = 395$ Channel: Example Planes and Spanwise Mean

In the $Re_\tau = 395$ turbulent channel simulation, the spanwise width is $\pi h$, where $h$ is the channel half-height. $y = 0$ is here defined as the lower wall location. The $k_z = 0$ mode is simply the spanwise average of the channel domain, which is

$$\bar{u}_i(x,y) = \frac{1}{L_z} \int_0^{L_z} u_i(x,y,z) \, dz.$$  \hspace{1cm} (9.1)

Color contours of each velocity component from one example field of the $Re_\tau = 395$ turbulent channel simulation are presented in the left column of figure 9.1. Each
velocity component is shown at the same streamwise–wall-normal plane located at a reference spanwise location $z_{\text{ref}}$. The spanwise average is computed by averaging over all such planes of this instantaneous field, and contours of $\bar{u}_i(x,y)$ are shown for each velocity component in the right column of figure 9.1. Spanwise averages of $u$ and $w$ tend to be inclined structures, whereas spanwise averages of $v$ tend to be vertical column structures. To better visualize the relationship between the spanwise average of the field and the single spanwise plane of the non-averaged velocity, the relevant spanwise averaged velocity is superimposed as line contours in figure 9.2. If the example plane is strongly correlated with the spanwise average, solid lines should fall over red color contours and dashed lines should fall over blue color contours. Of these examples, it appears the $w$ velocity component has the clearest relation or best correlation.

9.3 Energy Contribution

As another method to assess the relevance of the $k_z = 0$ modes, the average energy contribution associated with the $k_z = 0$ or $k_\theta = 0$ mode is calculated for three
Figure 9.2: $Re_{\tau} = 395$ channel: $u_i(x, y, z_{ref})$ color contours and $\bar{u}_i(x, y)$ line contours. Dashed lines indicate negative values.
flows: an $Re_\tau = 395$ channel simulation, an $Re_{\tau,au} = 590$ channel simulation, and an $Re_\tau = 685$ pipe simulation. As the pipe axis is approached, either all or none of the energy is contained in this mode because of the definition of the relevant velocity component in cylindrical coordinates and the decreasing circumference as $r$ approaches zero. The pipe is therefore comparable to the channel only away from the pipe core region. Otherwise, the general trend for both types of flows is increasing fractions of energy in these modes with distance from the wall, which is consistent with greater fractions of energy in narrower scales when near the wall. From Moser et al. (1999), the spanwise width of each channel simulation is $\pi h$. Channel simulations performed on wider domains would be expected to contain less energy in the $k_z = 0$ mode because energy would be in contained in the additional modes corresponding to the wider wavelengths, and yet the average energy per width would be the same. This suggests that the energy in the additional, wider modes would largely be transferred from the $k_z = 0$ mode of the narrower domain. The pipe, however, is spatially constrained and not subject to such widening. Just above $y/R = 0.3$, the $k_\theta = 0$ mode contributes 5% of the $u_\theta$ energy, so these modes make substantial contributions to pipe flows.

9.4 Two-point Correlation Results

As the two-point correlation is the quantity of interest when computing proper orthogonal decompositions, these are computed for the $Re_\tau = 395$ channel and $Re_\tau = 685$ pipe simulations summarized in table 2.1. The two-point correlation

$$R_{ij}(\Delta x, y, y') = \left\langle \frac{1}{L_z} \int_0^{L_z} u_i(x, y, z) u_j(x + \Delta x, y', z) dz \right\rangle_{x, t}$$

is the required two-point correlation when one wishes to calculate POD modes in a two-dimensional streamwise–wall-normal plane of a channel, as in Liu et al. (2001). The contributions (products of velocity at a pair of $(x, y)$ coordinates in a particular $x$–$y$ plane) are averaged over $z$, $x$, and $t$ (time).
Figure 9.3: $k_z = 0$ or $k_\theta = 0$ modal energy contribution fraction for each velocity component.
Conversely, if one wishes to calculate the $k_z = 0$ POD modes in a three-dimensional field, the two-point correlation is equivalent to

$$\tilde{R}_{ij}(\Delta x, y', y) = \left\langle \left[ \frac{1}{L_z} \int_{0}^{L_z} u_i(x, y, z_1) dz_1 \right] \left[ \frac{1}{L_z} \int_{0}^{L_z} u_j(x + \Delta x, y', z_2) dz_2 \right] \right\rangle_{x, t}.$$  (9.3)

The contributions (products of velocity averaged over the spanwise coordinate at a pair of $(x, y)$ coordinates) are averaged over $x$ and $t$.

The analogous equations for turbulent pipe flow are

$$R_{ij}(\Delta x, r, r') = \left\langle \frac{1}{2\pi} \int_{0}^{2\pi} u_i(x, r, \theta) u_j(x + \Delta x, r', \theta) d\theta \right\rangle_{x, t}.$$  (9.4)

and

$$\tilde{R}_{ij}(\Delta x, r, r') = \left\langle \left[ \frac{1}{2\pi} \int_{0}^{2\pi} u_i(x, r, \theta_1) d\theta_1 \right] \left[ \frac{1}{2\pi} \int_{0}^{2\pi} u_j(x + \Delta x, r', \theta_2) d\theta_2 \right] \right\rangle_{x, t}.$$  (9.5)

For comparison, contour lines of each two-point correlation are plotted for each pair of identical velocity components. Figure 9.4 compares the two-point correlations for velocity in the channel at a reference $y$ position of $0.15h$, while figure 9.5 compares the analogous quantities for the pipe at $y_{ref} = 0.15R$. Figures 9.6 and 9.7 compute the same correlations for $y_{ref}$ of $0.50h$ and $0.50R$, respectively. Contour lines are plotted in increments of 0.10 times the peak value, and negative contour lines are dashed.

Generally, the two-point correlations for the reference heights shown of streamwise velocity components are ramp-like and streamwise-elongated. Those of the spanwise or azimuthal components are wall-normally inclined in orientation. These are similar to the characteristics of the instantaneous, planar (unaveraged) motions shown in figure 9.1. For the wall-normal or radial velocity components, the correlation contours have similar dimensions in $x$ and $y$ or, especially when streamwise or azimuthally averaged, are wall-normally or radially elongated. Thus, the character of the two-point correlations are relatively similar whether the fields have been
Figure 9.4: $Re_\tau = 395$ channel two-point correlations for $y_{\text{ref}} = 0.15h$: the left column is the planar correlation of (9.2) and the right column is the correlation of spanwise-averaged velocity in (9.3). The rows correspond to different velocity components for each pair.

Figure 9.5: $Re_\tau = 685$ pipe two-point correlations for $y_{\text{ref}} = 0.15R$: the left column is the planar correlation of (9.4) and the right column is the correlation of azimuthally-averaged velocity in (9.5). The rows correspond to different velocity components for each pair.
Figure 9.6: $Re_T = 395$ channel two-point correlations for $y_{ref} = 0.50h$: the left column is the planar correlation of (9.2) and the right column is the correlation of spanwise-averaged velocity in (9.3). The rows correspond to different velocity components for each pair.

Figure 9.7: $Re_T = 685$ pipe two-point correlations for $y_{ref} = 0.50R$: the left column is the planar correlation of (9.4) and the right column is the correlation of azimuthally-averaged velocity in (9.5). The rows correspond to different velocity components for each pair.
spanwise/azimuthally averaged or they are calculated directly from planes in instantaneous fields. One major difference is the relatively strong region of negative correlation for streamwise velocity that appears above the positively correlated region, especially in the pipe flow, at $y_{ref} = 0.15R$. For $y_{ref} = 0.50R$, negatively-correlated regions occur both above and below the positive correlation peak. The characteristics appear relatively similar between pipe and channel flows. For the wall-normal/radial and spanwise/azimuthal velocity component correlations, the similarities in correlations of spanwise/azimuthally-averaged velocities for both flow persist deeply toward the centerlines, indicating they are relatively unaffected by the difference in curvature.

9.5 Conclusions

$k_z = 0$ modes in turbulent channel and boundary layer simulations contain strong similarities to $k_\theta = 0$ modes in turbulent pipes. This observation suggests physical behavior is similar between artificially periodic domains and physically periodic (but curved) pipes, particularly when near the walls. The motions’ characters, revealed through two-point correlation, retain characteristics of motions in unaveraged $x$–$y$ planes. The difference is significant between azimuthal averaging and spanwise averaging in the region near the centerline, particularly for correlations of the streamwise velocity component, and this would probably cause significant differences in POD modes between the two geometries for the $k_z = 0$ and $k_\theta = 0$ modes. Two-point correlations for planar velocity measurements are relatively similar between channel and pipe geometries for the reference radii shown.
Chapter 10

CONCLUSIONS

Coherent structures in turbulent structures are complex entities that require multiple perspectives and analysis techniques to advance toward their compact description in ways that lead to better understanding and predicting the behavior of turbulent flows. To this end, several techniques have been applied in the present study to further characterize and understand the structures that occur in wall-bounded turbulence, including proper orthogonal decomposition. As it is important to deeply understand the meaning and physical implications of POD modes in their application to more complex problems, this study has also sought to characterize these properties. In the simple case of one-dimensional POD in turbulent channels, comparisons of modes between Reynolds numbers and with another turbulent flow elucidated the effects of flow physics on the POD modes and led to the classification between idiosyncratic highly-energetic larger-scale modes and self-similar smaller modes. This observation also led to a formulation that generates synthetic modes that are similar in convergence properties to POD modes and superior to other orthogonal polynomials.

An $Re_\tau = 685$ pipe flow simulation with very long streamwise domain was studied. First, its statistical properties were characterized and compared with experiments. Energy spectra were also analyzed and compared with experiments extensively, and the results support the experimental evidence that very large-scale motions are associated with large contributions of streamwise turbulent kinetic energy and Reynolds shear stress. Additional aspects including convection velocities of long motions were also analyzed, and the results were discussed in the context of the issues relating to errors introduced into experimental measurements of energy spectra due to the variations in convection velocities for different scales. The
net force spectra were also spectrally decomposed to describe how net turbulent stresses influence the mean flow profile. This study also revealed new insight into their behavior at the wall that was not available from prior experiments. This work also extended the net force decompositions to two dimensions, and this reveals a simpler picture of the spectral contributions of net Reynolds force that accelerate and decelerate the mean flow.

Having established the importance of the very long scales of motion in this flow, the associated instantaneous structures were described by several means. The vortical structures were also characterized and compared to results from other flows. The results suggest that the long meandering motions of streamwise velocity characteristic of this flow are composed of smaller motions. The forms of the smaller motions were compared with the motions theorized to be present according to the vortex packet model, and the results generally support the organization associated with vortex packets. However, the vortical structures in the pipe flow frequently vary significantly from the idealized shapes of hairpins. Organizational patterns were also studied from the perspective of the streamwise velocity two-point correlation. The results support the concatenation of LSMs to form VLSMs. The azimuthal merging of structures with wall-normal distance is in good agreement with experiments and is similar to that of channels when within 0.1R of the wall.

As an additional perspective of the structures, POD analysis was also extended to three dimensions in this pipe flow at higher Reynolds number than has been previously studied with POD. The results indicate that the modes with large scales contain significant similarities with Reynolds numbers varying by a factor of four. The results also suggest that the mode classification developed in one-dimensional channel flow POD is applicable to more complicated three-dimensional POD.

POD was also applied using the snapshot method to several turbulent boundary layer simulations. The POD modes were found to be related to instantaneous
flow structures in the fields, and this relationship is also useful in interpreting the 
POD modes. Comparison of POD modes for two simulations with different inflow 
methods indicates that smaller-scale structures agree closely while more significant 
differences appear in larger-scale structures. Therefore, POD draws attention to the 
differences in the flow that occur as a result of different flow conditions.

Overall, the results demonstrate the applicability of POD to a variety of turbu-
 lent flows in geometries conducive to clearly interpreting the modes. The insights 
gained form a framework for understanding POD analysis in more complex turbu-
 lent flows. Additional results obtained from other analysis methods for pipe flows 
reveal new insights of the structures and organization in pipe flows.
REFERENCES


304


THEODORSEN, T. 1952 Mechanism of turbulence. In *Proceedings of the Midwestern Conference on Fluid Mechanics*. Ohio State University, Columbus, OH.


APPENDIX A

ADDITIONAL ASPECTS OF ONE-DIMENSIONAL POD MODES OF
TURBULENT CHANNELS
Additional detail on several aspects of the methods used in chapter 3 is provided below.

A.1 Further Details on Phase/Envelope Decomposition Methods

As noted in §3.4, the phase and envelope decomposition is not unique and it must further be specified how the signal is divided into the amplitude and phase functions. In other applications, it is common to apply the Hilbert transform as described below for generating the analytic function to determine the envelope for a signal that includes oscillation at a constant ‘carrier frequency.’ In the case of analyzing a POD mode, however, the oscillation spatial ‘frequency’ must be determined and varies with y.

A.1.1 Hilbert Transform Method

As previously noted, Fernandes (2001) employed the Hilbert transform to generate the imaginary part of an analytic signal (also known as a pre-envelope, Dugundji, 1958), from which the phase and envelope are extracted. The Hilbert transform of a real-valued function \( \phi(y) \) is denoted by \( \mathcal{H}\{\phi(y)\} \) and is defined by

\[
\mathcal{H}\{\phi(y)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\eta)}{y-\eta} d\eta,
\]

where the Cauchy principal value of the integral is used. (Typically, Hilbert transforms are applied to signals varying as a function of time and the variable of integration is customarily denoted by \( \tau \).)

Recognizing this form as a convolution leads to the result that the transform may be computed by multiplication in Fourier space with

\[
\hat{\mathcal{H}}(\phi)(k_y) = \begin{cases} 
-i\hat{\phi}(k_y) & k_y > 0 \\
0 & k_y = 0 \\
i\hat{\phi}(k_y) & k_y < 0
\end{cases},
\]

where \( \hat{\cdot} \) denotes the Fourier transform in \( y \) with wavenumber \( k_y \) and \( i = \sqrt{-1} \). Mathis et al. (2009a) and Dugundji (1958) provide further details.
Thus, the Hilbert transform shifts the phase of each Fourier component of the signal by 90°. In a sense, the Hilbert transform ‘fills in’ the oscillations of the original signal so the analytic signal

\[ Z(y) = \phi(y) + i\mathcal{H}\{\phi(y)\} \quad \text{(A.3)} \]

contains an imaginary peak where the original signal has a zero crossing for each Fourier mode contributing to the signal. Since the envelope function is defined as the modulus of the analytic signal

\[ A(y) = \sqrt{\phi(y)^2 + \mathcal{H}\{\phi(y)\}^2} \quad \text{(A.4)} \]

this formulation interpolates between the peaks of the original function to create the envelope function. The result for an infinite-length or periodic signal is a smooth envelope function such that the sinusoidal term in the decomposition is responsible for the small-scale oscillation. The phase is obtained from the arctangent of the complex analytic signal components, and 2\pi phase increases are added so the phase smoothly increases when the arctangent reaches the boundaries of its \([-\pi/2, \pi/2]\) range, resulting in a continuous phase function. Extracting the phase and envelope for form (3.6) using the Hilbert transform has been applied to turbulence to analyze long time signals of velocity by Mathis et al. (2009a) and also references therein.

Employing this method to decompose POD modes faces several issues. The POD modes are equal to zero at the walls due to the no-slip boundary conditions, but their derivatives are typically steepest at the walls. Since POD modes are defined on a finite domain, the modes must either be periodized in some manner or assumed to be surrounded by zero in order to compute the transform. Periodizing this signal and interpreting the Hilbert transform as shifting each Fourier component as discussed above results in the slope discontinuity due to the wall behavior inducing high-frequency oscillations in the Fourier representation. Evaluating various
zero-padding and periodic extension schemes reveals these treatments significantly affect the phase results, particularly when the domain length is on the same order as the wavelength of the oscillations. These properties make the precise properties of this decomposition method uncertain and suggests that a more robust decomposition method be used for this application in a finite domain. (Many oscillations occur in the long time signals analyzed in Mathis et al. (2009a), so these issues are not significant for their circumstances.)

A.1.2 Hilbert Transform Testing

The Hilbert transform algorithm implemented in MATLAB that was previously described is tested on synthetic functions under precisely the same circumstances as when it is applied to the POD modes. In summary:

1. The function of choice is evaluated on the same grid as for the POD modes (Chebyshev Gauss-Lobatto).

2. This is passed to the Hilbert transform routine using the Fourier transform and zero padding exactly as a POD mode is, and with a 257-point uniform grid that the routine interpolates onto with spline interpolation (as for the POD modes).

3. The output phase and envelope are compared with the known phase and envelope of the test function.

The test functions are

1. Sine Wave:

   \[ \phi^{(n)}(y) = \sin(n\pi/2)(1 + y) \]  

   The phase function is \( \beta^{(n)}(y) = n\pi/2(1 + y) \), and the envelope function is \( A(y) = 1 \).
2. Synthetic POD Mode: The following function has the approximate appearance of a POD mode; it is obtained by deriving synthetic modes assuming \( A(y) \) is independent of mode number and requiring that multiplication by \( y \) of a mode and subtracting the mode before that in the recursion relation produce the subsequent mode. An issue with this function is that the rapid oscillations near the walls are much more rapid than for real POD modes; it should not be expected that the Hilbert transform would correctly interpret these (or that they are even resolved correctly). Nonetheless, this case is interesting to examine the behavior near \( y = 0 \). In

\[
\phi^{(n)}(y) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-y^2}} \sin(n(\pi - \arccos(y))),
\]

(A.6)

the phase function is \( \beta^{(n)}(y) = n(\pi - \arccos(y)) \) and the envelope function is \( A(y) = \frac{\sqrt{2/\pi}}{\sqrt{1-y^2}} \).

![Phase Function – Pure Sine Wave – n=20](image)

Figure A.1: Hilbert transform test on \( n = 20 \) sinusoidal function: phase
Figure A.2: Hilbert transform test on $n = 20$ sinusoidal function: envelope

Figure A.3: Hilbert transform test on $n = 20$ sinusoidal function: phase error
Figure A.4: Hilbert transform test on $n = 2$ sinusoidal function: phase

Figure A.5: Hilbert transform test on $n = 2$ sinusoidal function: envelope
Figure A.6: Hilbert transform test on $n = 2$ sinusoidal function: phase error

Figure A.7: Hilbert transform test on $n = 50$ sinusoidal function: phase
Figure A.8: Hilbert transform test on $n = 50$ sinusoidal function: envelope

Figure A.9: Hilbert transform test on $n = 50$ sinusoidal function: phase error
Figure A.10: Hilbert transform test on $n = 20$ synthetic POD function: phase

Figure A.11: Hilbert transform test on $n = 20$ synthetic POD function: envelope
Figure A.12: Hilbert transform test on $n = 20$ synthetic POD function: phase error

Figure A.13: Hilbert transform test on $n = 2$ synthetic POD function: phase
Figure A.14: Hilbert transform test on $n = 2$ synthetic POD function: envelope

Figure A.15: Hilbert transform test on $n = 2$ synthetic POD function: phase error
Figure A.16: Hilbert transform test on \( n = 50 \) synthetic POD function: phase

Figure A.17: Hilbert transform test on \( n = 50 \) synthetic POD function: envelope
A.1.3 Phase Zero Method

Desiring that an envelope curve be smooth and positive-valued in the domain, (3.6) indicates that the zeros of the POD mode must be matched by the zeros of the sinusoidal function in the mode decomposition. For the sinusoidal function in (3.6), each zero occurs when the phase function is an integral multiple of $\pi$. Assuming that $\beta$ increases monotonically, each zero encountered in a mode as $y$ increases therefore is associated with the phase function increasing by $\pi$ from the phase at the last zero. Therefore, starting with an initial phase of $\beta^{(n)}(y = -h) = 0$ and finding the $y$ at each $\phi^{(n)}(y) = 0$ location, a $(y, \beta^{(n)}(y))$ pair can be obtained for each zero. Since there is a zero in each mode at $y = h$ due to the wall boundary conditions, the maximum phase of each mode is $\beta^{(n)}(h) = \pi n$. Accurate zeros are obtained computationally by using a cubic spline to interpolate the mode in the region of each zero on a finer grid, followed by linear interpolation to estimate $y$ for the zero crossing.

If the mode has numerous zero crossings, the zeros form a good representation
of \(\beta_i^{(n)}(y)\). However, modes with low \(n\) have few zero crossings, so this provides little information about \(\beta_i^{(n)}(y)\). The method outlined in §3.4.2 using the analytic signal from the Hilbert transform results in a phase function that exactly satisfies the zero crossing locations because the phase is determined from an arctangent of the analytic function with the original signal as the real component.

### A.1.4 Envelope Spline Method

Since the goal of the decomposition is to concentrate the oscillations into the sinusoidal term, thereby forming a smooth amplitude function, making this decomposition unique can be approached from the opposite perspective by specifying a smooth amplitude function. Rearranging (3.6) to

\[
\sin \left( \beta_i^{(n)}(y) \right) = \frac{\phi_i^{(n)}(y)}{A_i^{(n)}(y)}
\]

(A.7)

indicates that when the envelope is specified, the phase function is determined and therefore the decomposition is unique. Again, a positive-valued envelope forces the sinusoidal function to represent the oscillations, and therefore \(\beta_i^{(n)}(y)\) is an integer multiple of \(\pi\) at each mode zero, which is consistent with the method of obtaining phase based on the zero locations (A.1.3). Choosing \(A_i^{(n)} = \max|\phi_i^{(n)}(y)|\) and solving (A.7) for \(\beta_i^{(n)}(y)\) would guarantee that form (3.6) would reproduce the mode everywhere in the domain, but would result in phase function discontinuities at each peak of the mode. If the \(\sin(\beta(y))\) term oscillates through \(\pm 1\) with \(\beta\) increasing smoothly, then the resulting phase function is continuous in this region. This requirement is satisfied if the envelope function is tangent to each of the mode’s peaks because differentiating (3.6) yields \(d|\phi_i^{(n)}(dy) = dA_i^{(n)}/dy\) at the sine function peaks \((\beta_i^{(n)}(y) = \pi/2 + m\pi, m = 0, 1, 2, \ldots)\).

As noted in chapter 3, “selecting a smooth envelope function with minimal oscillation that is tangent to each peak of the mode results in a decomposition with a (generally) continuous phase function.” When there are few peaks, there is more...
latitude in specifying the envelope while satisfying this requirement, but when several peaks are present there is little flexibility. The present approach will be to “concentrate the variations in the mode into the sinusoidal function by generating an envelope function with a low-order polynomial spline fit to the mode peaks to smoothly interpolate” between each of the peaks. An iterative procedure is used to select the point in the vicinity of each peak that makes the spline tangent to each peak. As an initial guess, the true peak (where $|\phi^{(n)}_i(y)|$ attains a local maximum) locations are determined via interpolation and a spline fit is constructed for these points. A point is found near each $\phi^{(n)}_i$ peak where the slope of $|\phi^{(n)}_i(y)|$ matches the slope of the envelope spline at the same $y$. These points become the new points through which the spline must pass, and this is repeated until the set of points defining the spline converges with one point lying on $|\phi^{(n)}_i(y)|$ near each peak where the slope is equal to that of the envelope function at that point.

Since the number of peaks is equal to the mode number, the envelope curve is interpolated based on $n$ points. However, (A.7) indicates the correct phases to attain a zero at each of the $n + 1$ mode zero crossings are also obtained by this method. Therefore, this method calculates the phase function using approximately twice as much information (data points) as pure phase zero interpolation (A.1.3) and also guarantees envelope smoothness. The peak locations are slightly less certain than the zero locations because they depend on the slope of the envelope function, but the dependence is weak.

Nonetheless, the sparsity of data points (zeros and peaks) for the first few modes with low $n$ make the phase function “less constrained and more dependent on the interpolation function” (spline). Intuitively, the envelope should contain no strong variations except as required to match the peaks, and splines with natural endpoint conditions fulfill this requirement. To examine the dependence on the spline, results
employing splines of different polynomial order must be compared to establish the convergence of this decomposition scheme.

As noted in chapter 3, “after establishing the envelope, solving (A.7) yields the phase function.” However, simply calculating an arcsine produces a $\beta^{(n)}_i(y)$ in the range $[-\pi/2, \pi/2]$. The phase should increase between each zero crossing as $y$ increases, in the same way as with the phase zero and Hilbert transform methods. The phase function should also be continuous and smooth, but this requirement must be relaxed for a special case that arises.

If a mode were a purely sinusoidal function, the raw phase function obtained from the arcsine would be a triangle wave. To obtain a continuously increasing phase, the first segment increasing linearly from $\beta_{raw} = 0$ to $\pi/2$ would be left intact. The segments following alternate between linearly decreasing and increasing by $\pi$. This indicates that an equivalent but monotonically increasing phase function may be obtained by applying shifts of integer multiple of $2\pi$ and replacing segments of decreasing phase with increasing segments using the identity $\sin(\beta) = \sin(\pi - \beta)$. The raw phase functions of actual POD modes behave similarly, so they can be converted to increasing functions in the same manner by splitting the raw phases into increasing and decreasing segments. It is emphasized that these adjustments to $\beta$ leave $\sin(\beta)$ unchanged, yet are critical to properly scaling the phase functions.

This procedure is implemented by searching for points across which the sign of $d\beta_{raw}/dy$ changes, indicating a change between increasing and decreasing raw phase regimes. As illustrated in the figure A.19 insets, the derivative sign change occurs at a grid point indicated by an arrow in the figure and the ‘reflected’ raw phase begins at this point or the next, based on where $\pi/2$ is reached relative to the grid points on which the function is sampled. In the purely sinusoidal mode example shown, the phase increases linearly, but for an actual POD mode the phase derivative changes as a function of $y$. However, the phase function is assumed to
Figure A.19: Example raw phase obtained from the arcsine using the envelope spline method ($Re_T = 395$ channel for $n = 5$ mode) with dots indicating the values at the grid points. The function is split into segments according to if it is increasing or decreasing, and the center segment is split by the dashed line located at the local minimum. The insets depict two example scenarios for a purely sinusoidal mode where the raw phase switches between increasing and decreasing regimes. The points connected by solid black line show $\beta_{\text{raw}}$ sampled on grid points and the dotted lines show the raw phase depiction of continuous phase increasing at a constant rate. Each arrow indicates the point across which the derivative based on the sampled phase changes signs.

change smoothly so an estimate of phase based on linear extrapolation from the previous two points is expected to closely match the actual phase. If phase is calculated based on the raw phase with switches between increasing/decreasing regimes assumed to occur at the grid point of derivative sign change and the following point, these values can be compared to the linear extrapolated phase to determine which point yields the smoothest phase and is therefore the correct point of the switchover.

These switchovers are shown for an example raw phase function in figure A.19 from a POD mode for the $u$ velocity component of the turbulent channel at $Re_T = 395$, where the raw phase approaches $\pm \pi/2$ and changes slope abruptly at $y = \{\pm 0.96, \pm 0.76, \pm 0.31\}$. The phase is modified to continuously increase for the segments to yield the desired phase function. This mode ($n = 5$) was chosen to
illustrate a situation that rarely occurs: the raw phase function contains a local extremum at $y = 0$, in contrast to the expected continuously increasing or decreasing between $-\pi/2$ and $\pi/2$. This corresponds to a local extremum in the mode with the concavity opposite the $\phi = 0$ axis, whereas sinusoidal behavior is that of peaks concave toward the $\phi = 0$ axis. Such extrema are observed in only a few modes with low $n$ for the channel. The issue is that the desired phase function must increase by $\pi$ between the zeros on each side of this occurrence, so the phases of segments 3 and 6 in the figure must be increasing to smoothly reach the correct phases. Segment 4 must also be increasing, which the algorithm above predicts because segment 3 approaches $\pi/2$ before changing direction with similar slope. Segment 5 should be increasing also to match segment 6 (which has been converted to increasing), so a discontinuity at the local minimum is necessary. Placing the discontinuity at this point satisfies the symmetry $\beta(y) = \beta(1) - \beta(-y)$ that is expected for a POD mode computed from a two-point correlation that is symmetric in $y$.

Thus, the final rule governing transforming raw phase to the phase function is that raw phase is adjusted such that phase be smooth and nondecreasing with the exception of a discontinuous decrease where a local extremum occurs in the mode with the concavity directed away from $\phi = 0$. In summary, the algorithm finds points where the derivative sign changes and determines the proper action based on if the phase is changing at a sufficient rate to exit the $[-\pi/2, \pi/2]$ range. If so, it determines if the switch is to an increasing or decreasing raw slope and determines the point at which this occurs based on extrapolation of the previous phase. If the phase would not exit the range, it concludes a local extremum exists and creates a discontinuity. Results show that this logic provides a phase function consistent in important properties with the phase zero and Hilbert transform methods.
A.2 Recursive Relation for POD Modes

The following summarizes the application of a recursion relation between 1D POD modes of streamwise velocity fluctuations for an $Re_\tau = 180$ turbulent channel.

A.2.1 Theory

A.2.1.1 Motivation

According to Hesthaven et al. (2007), Jacobi polynomials are “the polynomial eigenfunctions of the singular Sturm-Liouville problem.” From Hesthaven et al. (2007), Theorem 4.2: “All Jacobi polynomials, $P_n^{(\alpha, \beta)}$, satisfy a three-term recurrence relation of the form

$$x P_n^{(\alpha, \beta)} = a_{n-1,n}^{(\alpha, \beta)} P_{n-1}(x) + a_{n,n}^{(\alpha, \beta)} P_n(x) + a_{n+1,n}^{(\alpha, \beta)} P_{n+1}(x).$$

(A.8)

Here, $\alpha$ and $\beta$ are constants in the singular Sturm-Liouville problem.

In particular, Legendre polynomials (a case of Jacobi polynomials) follow the recurrence relation

$$x P_n(x) = \frac{n}{2n+1} P_{n-1}(x) + \frac{n+1}{2n+1} P_{n+1}(x).$$

(A.9)

Chebyshev polynomials, which are related to Jacobi polynomials, obey the recurrence relation

$$x T_n(x) = \frac{1}{2} T_{n-1}(x) + \frac{1}{2} T_{n+1}(x).$$

(A.10)

It is not clear that POD eigenmodes would follow any similar recurrence relation because the Sturm-Liouville problems that Jacobi and Chebyshev polynomials satisfy are different than the integral Fredholm equation for the POD eigenmodes. However, both equations are eigenvalue problems that produce sets of orthogonal modes so it is not unreasonable that some properties may be similar. The precise relationship between a mode with one sequency and the mode with an adjacent sequency is not clear from the Fredholm equation; the orthonormality property is useful in providing some intuition relating mode shapes.
Proceeding with the assumption that POD eigenmodes obey a recursion relation similar to that of other orthogonal polynomials, the following relation is proposed:

\[ \phi^{(n+1)}(y) = \alpha_n y \phi^{(n)}(y) - \beta_n \phi^{(n-1)}(y). \]  
(A.11)

For the purpose of testing this relation and potentially determining the constants it is more convenient to rearrange it to

\[ \phi^{(n+1)}(y) = \beta_n \left[ \frac{\alpha_n}{\beta_n} y \phi^{(n)}(y) - \phi^{(n-1)}(y) \right]. \]  
(A.12)

Here it is clear that \( \beta \) is directly related to the amplitude (and therefore the energy contained in the mode), and so \( \beta \) is easily determined by the orthonormality condition (the energy in each mode must equal 1). Therefore, the problem is essentially reduced to finding the proper \( \alpha_n/\beta_n \), assuming the modes conform to the proposed formula.

Rearranging the recursion formula for Legendre polynomials,

\[ P_{n+1}(x) = \frac{n+1}{n+2} \left[ \frac{2n+1}{n} x P_n(x) - P_{n-1}(x) \right] \Rightarrow \beta = \frac{n+1}{n+2}, \quad \frac{\alpha}{\beta} = \frac{2n+1}{n}. \]  
(A.13)

Rearranging the recursion formula for Chebyshev polynomials,

\[ T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \Rightarrow \beta = 1, \quad \frac{\alpha}{\beta} = 2. \]  
(A.14)

It is observed for Legendre polynomials in the limit of large \( n \) that \( \alpha/\beta \) approaches 2 and \( \beta \) approaches 1, and these are the values of these constants for the Chebyshev polynomials.

A.2.1.2 Recursion Relation for Synthetic POD

The form of synthetic POD modes is

\[ \phi^{(n)}(y) = \sqrt{\frac{2}{\pi}} \frac{d\beta(y)}{dy} \sin(n \beta(y)). \]  
(A.15)

It should be noted that \( \beta(y) \) refers to the scaled phase function that is used for every synthetic POD mode while \( \beta_n \) is the constant for the recursion formula and is unrelated to the scaled phase.
Assuming that $\beta_n = 1$ and rearranging the resulting recursion relation (A.11), the equation becomes

$$\alpha_n y \phi^{(n)}(y) = \phi^{(n+1)}(y) + \phi^{(n-1)}(y). \quad (A.16)$$

Next, substituting the synthetic POD modes in the right-hand side of the above equation,

$$\phi^{(n+1)}(y) + \phi^{(n-1)}(y) = \sqrt{\frac{2}{\pi}} \frac{d\beta(y)}{dy} \sin((n+1)\beta(y)) + \sqrt{\frac{2}{\pi}} \frac{d\beta(y)}{dy} \sin((n-1)\beta(y)). \quad (A.17)$$

Using a trigonometric identity, this simplifies to

$$2 \cos(\beta(y)) \sqrt{\frac{2}{\pi}} \frac{d\beta}{dy} \sin(n\beta(y)). \quad (A.18)$$

This implies that $\alpha_n = 2$ and the multiplication by $y$ that would be expected for other orthogonal polynomials should be replaced by $\cos(\beta(y))$. Recall that $\beta(y)$ ranges from 0 to $\pi$, so $\cos(\beta(y))$ ranges from $-1$ to 1, just as $y$ does. It is interesting to note that if $\beta(y) = \pi - \arccos(y)$ then the recursion formula prescribes multiplication by $\cos(\pi - \arccos(y)) = y$ just as in the case of orthogonal polynomials such as Chebyshev or Legendre.

### A.2.1.3 Generalization

The previous derivation suggests that the $\phi^{(n)}(y)$ mode need not be multiplied by $y$, but in general could be multiplied by some function of $y$, $f^{(n)}(y)$. Solving the recursion relation (A.11) for $\alpha_n f^{(n)}(y)$,

$$\alpha_n f^{(n)}(y) = \frac{\phi^{(n+1)}(y) + \beta_n \phi^{(n-1)}(y)}{\phi^{(n)}(y)}. \quad (A.19)$$

With this more general form, there is much more freedom than the previous case with multiplication by $y$ specified, where only $\alpha_n$ and $\beta_n$ could be adjusted. In
fact, it might appear that by solving for \( f^{(n)}(y) \) for each \( n \), one would always get the desired \( \phi^{(n+1)}(y) \) so this general recursion relation is saying very little about the modes and one could choose any \( \beta_n \). However, the division by \( \phi^{(n)}(y) \) would cause problems if the zeros of \( \phi^{(n+1)}(y) + \beta_n \phi^{(n-1)}(y) \) do not match up with the zeros of \( \phi^{(n)}(y) \) and the resulting \( f^{(n)}(y) \) would have sharp discontinuities. Therefore, \( \beta_n \) is important to positioning the zeros of \( \phi^{(n+1)}(y) + \beta_n \phi^{(n-1)}(y) \) to their correct locations and if the resulting \( f^{(n)}(y) \) is smooth, this has important implications about how the modes are related.

### A.2.1.4 Orthogonality in Recurrence Relation

Squaring both sides of the proposed general recurrence relation (A.19) and integrating over the channel,

\[
\int_{-1}^{1} \phi^{(n+1)}(y) \phi^{(n+1)}(y) dy = \\
\alpha_n^2 \int_{-1}^{1} \left[ f^{(n)}(y) \phi^{(n)}(y) \right]^2 dy + \beta_n^2 \int_{-1}^{1} \left[ \phi^{(n-1)}(y) \right]^2 dy - 2 \alpha_n \beta_n \int_{-1}^{1} f^{(n)}(y) \phi^{(n)}(y) \phi^{(n-1)}(y) dy
\]

Using the orthonormality property of the modes, this simplifies to

\[
1 = \alpha_n^2 \int_{-1}^{1} \left[ f^{(n)}(y) \phi^{(n)}(y) \right]^2 dy + \beta_n^2 - 2 \alpha_n \beta_n \int_{-1}^{1} f^{(n)}(y) \phi^{(n)}(y) \phi^{(n-1)}(y) dy. \quad (A.21)
\]

By multiplying both sides by either \( \phi^{(n+1)} \), \( \phi^{(n)} \), or \( \phi^{(n-1)} \) and then integrating over the channel height, it can be seen that:

\[
\alpha_n \int_{-1}^{1} f^{(n)}(y) \phi^{(n+1)}(y) \phi^{(n)}(y) dy = 1 \quad (A.22)
\]

\[
\alpha_n \int_{-1}^{1} f^{(n)}(y) \phi^{(n)}(y) \phi^{(n)}(y) dy = 0 \quad (A.23)
\]

\[
\alpha_n \int_{-1}^{1} f^{(n)}(y) \phi^{(n)}(y) \phi^{(n-1)}(y) dy = \beta_n. \quad (A.24)
\]

Substituting the third equation into (A.21), it can be seen that

\[
\int_{-1}^{1} \left[ f^{(n)}(y) \phi^{(n)}(y) \right]^2 dy = \frac{1 + \beta_n^2}{\alpha_n^2}. \quad (A.25)
\]
Several conclusions can be drawn from the orthonormality property of the modes. First, in the recursion formula, each term would be orthogonal to every other term, were it not for the \( f^{(n)}(y) \) that is multiplied by the \( \phi^{(n)}(y) \) term. This multiplication of \( \phi^{(n)}(y) \) results in a linear combination of \( \phi^{(n+1)}(y) \) and \( \phi^{(n-1)}(y) \). Second, \( f(y) \) behaves like a weight to the inner product of \( \phi^{(n)}(y) \) with itself so that when unweighted, the inner product is 1 (by orthonormality), when weighted by \( f^{(n)}(y) \) it is 0, and when weighted by \( \left[f^{(n)}(y)\right]^2 \) it is \( (1 + \beta_n^2)/\alpha_n^2 \). The Jacobi and Chebyshev polynomials have the property that this occurs for a \( f^{(n)}(y) \) that is independent of \( n \), namely \( y \). Note that in the case of \( f^{(n)}(y) = y \), \( [\phi^{(n)}(y)]^2 \) is (evenly) symmetric about \( y = 0 \) if \( \phi^{(n)}(y) \) is precisely even or odd so it follows that the inner product weighted by odd \( f^{(n)}(y) = y \) must be zero when integrated over the channel.

\[ \text{A.2.2 Implementation} \]
\[ \text{A.2.2.1 y-Multiplied Results} \]

The goal is to test how well the actual computed modes fit the recursion formula (A.11), which requires determining the coefficients \( \alpha_n \) and \( \beta_n \). To determine these coefficients, it is more convenient to use the form (A.12). As previously discussed, \( \beta \) sets the amplitude so \( \alpha/\beta \) is the parameter to be determined that affects the shape. Therefore, a sequence of the \( \alpha/\beta \) parameter values are used and the resulting predicted modes are compared with the true \( \phi^{(n+1)}(y) \) mode. It is found that the agreement is best at the center of the channel (near \( y = 0 \)) so the amplitude (i.e. \( \beta_n \)) parameter is set based on matching the energy contained in the several points around \( y = 0 \) for the candidate predicted mode by the recursion relation (with this \( \alpha/\beta \)) and the actual next mode. (It is possible to calculate \( \beta_n \) based on the orthonormality of the modes but a potential disadvantage is the fact that the behavior of the function elsewhere, such as incorrect amplitude near the walls, could affect the amplitude of the center region of interest for calculating the best match \( \alpha/\beta \).) After determining the optimal \( \beta_n \), the error norm over the middle 11 points (or the entire channel
height) is calculated based on the difference between for the candidate predicted mode by the recursion relation and the actual next mode. This error norm is also stored so the error norms over all values of $\alpha/\beta$ can be compared and the value that minimizes the error norm is chosen as optimal.

In summary, after ordering by sequency (instead of by energy contribution) and ensuring the derivative of each mode at $y = +1$ is positive by multiplying by $-1$ if it is not (this behavior is is true for Chebyshev polynomials), the algorithm proceeds as follows for each mode number:

1. Pick next $\alpha/\beta$ in sequence

2. Calculate recursion relation $\phi^{(n+1)}(y)$ with this $\alpha/\beta$ and assuming $\beta = 1$

3. Calculate the energy in the central 5 points for both the recursion relation $\phi^{(n+1)}(y)$ and the actual $\phi^{(n+1)}(y)$, and then calculate the $\beta_n$ value that makes them equal

4. Compute and save the error norm (from difference between recursion relation and actual $\phi^{(n+1)}(y)$) for the center-most 11 points

5. Proceed with next $\alpha/\beta$ (Step 1)

The above steps calculate the best parameters based on the center-most points. In practice, the optimal $\alpha/\beta$ is the same when based on the error norm calculated over the middle 11 points or the entire channel (calculating $\beta$ for the center few points in both cases).

The plots below show the results of this procedure on the $Re_\tau = 180$ turbulent channel streamwise velocity fluctuation. The first two plots show the results when $n = 40$ to predict the $n = 41$ mode using the recurrence relationship. Figure A.20 compares the predicted mode and the actual $n = 41$ mode. Note that the match is
best near the center and the $\alpha/\beta$ and $\beta$ parameters were selected for the best match at the center. The locations of the zeros appear correct all across the channel but the amplitude of the oscillations is not correct as the walls are approached. Figure A.21 shows the nature of error (difference between the predicted and actual mode).

Figures A.22 and A.23 show the optimal $\alpha_n$ and $\beta_n$ values. The values seem reasonable from $n = 18$ to 85; likely the matches are not good outside of this range. Note that $\alpha_n$ increases from approximately 2 to 2.5 but $\beta_n$ seems constant at 1. These values compare closely to the Chebyshev and Legendre polynomials considered earlier with values of $\alpha = 2$ and $\beta = 1$ (at least for high $n$).

Since it was determined that the recursion formula for synthetic POD involves multiplication by a function other than $y$, it seems reasonable to next investigate multiplying by a different function (the generalization).

![Figure A.20: Mode 41](image-url)
Figure A.21: Mode 41 Error by Recurrence Relation with $y$-multiplication and Optimal Parameters

Figure A.22: $\alpha_n$ that provides optimal $\phi^{(n+1)}(y)$ vs. $n$
A.2.2.2 Generalized Recursion Formula Results

Next it is assumed that the true function $f(y)$ in the recursion formula approaches $f(y) = y$ in the center region of the channel. This means that the $\alpha$ and $\beta$ calculated by using the procedure described above with $f(y) = y$ would be accurate since this calculation is based on only the center few points. Therefore, the true $f(y)$ can be calculated using (A.19).

This is calculated for the $n = 40$ case shown in the previous example and the points values near 0 are removed because the division in (A.19) can result in large errors when dividing two near-zero values. The resulting $f(y)$ is plotted in Figure A.24 and the center section does appear nearly linear. The function near the walls does, however, level out. This could be related to the lower amount of stretching near the walls observed by Carbone & Aubry (1996). It is not clear how multiplying by $f(y) = y$ compresses the function. It is interesting that the slope of $f(y)$ is greater than 1 in the regions of $y = \pm0.5$ before becoming near 0 at the walls. The relatively
smooth \( f(y) \) is consistent with the zero locations being calculated approximately correctly for the \( \alpha \) value specified.

![Graph of \( f(y) \) for \( n=40 \) assuming optimal \( \alpha \) and \( \beta \) calculated from \( f(y) = y \).](image)

Figure A.24: \( f(y) \) for \( n=40 \) assuming optimal \( \alpha \) and \( \beta \) calculated from \( f(y) = y \)

Next, consider \( n = 5 \) to predict the 6th mode. Now the match from the recursion relation with \( f(y) = y \) is very poor, particularly when considering the entire channel height. It was observed in Figures A.22 and A.23 how the multiplication constants vary wildly for low mode numbers \( n \). To examine how the values of these parameters affect the multiplication function, \( \alpha_n f^{(n)}(y) \) is computed directly from (A.19) and \( \beta \) is systematically varied (in effect, changing the \( \alpha/\beta \)). As previously discussed, the important property of \( f(y) \) is that it is smooth, meaning the zeros are determined correctly. (In plotting this the points determined when \( \phi^{(n)}(y) \) has very small magnitude, in this case less than 0.05, are again removed.) The plots in Figure A.2.2.2 were manually picked when the plot of \( \alpha_n f^{(n)}(y) \) appeared smooth. The modes \( n = 5, 15, 40, 70 \) are considered.
Figure A.25: Recursion relation $\alpha_n f^{(n)}(y)$: (a) $n = 5$ First $\beta$, (b) $n = 5$ Second $\beta$, (c) $n = 15$, (d) $n = 40$ First $\beta$, (e) $n = 40$ Second $\beta$, and (f) $n = 70$

It is clear that the zeros obtained from this procedure do not match exactly with the actual zeros. However, in regions the match is good. For example, plots (a) and (b) with $n = 5$ show that a fairly smooth function is obtained with a small disconti-
nuity that shows up in different location based on the value of $\beta$. Plots (d) and (e) for $n = 40$ show that one value of $\beta$ makes the near-wall regions smooth (d) but a different value makes the center region smooth (e). In either case the overall values of $\alpha_n f^{(n)}(y)$ are similar if one imagines the curves smoothed and $\beta_n$ has changed very little (from 0.930 to 0.995). It is possible that there are different regimes for the near wall region and the center region requiring two different values of $\beta$ but in some cases, such as plot (c), the worst discontinuities in $\alpha_n f^{(n)}(y)$ occur midway between the wall and the centerline. Finally, recall the fitting procedure based on assuming $f(y) = y$ performs the fit considering only the match in the middle. Therefore, the optimal values reported should be similar to fitting manually to the centerline region with this procedure if the $f(y)$ is linear in the centerline region. The plots in A.2.2.2 give the impression of the linear region occupying more and more of the channel with increasing mode number $n$. 
B.1 Energy Normalization

The spectrum normalization follows Guala et al. (2006), in which it is stated that

\[ \overline{u_i' u_j'} = \int_{-\infty}^{\infty} S_{ij}(k_x) \, dk_x = \int_{0}^{\infty} \Phi_{ij}(k_x) \, dk_x. \]  

(B.1)

The computed spectra (via discrete Fourier transforms) are normalized so the contributions at discrete wavenumbers satisfy (B.1) such that

\[ \overline{u_i' u_j'} = \sum_{n_x=-N_x/2+1}^{N_x/2} S_{ij}(n_x) \Delta k_x = \sum_{n_x=0}^{N_x/2} \Phi_{ij}(n_x) \Delta k_x, \]  

(B.2)

where \( \Delta k_x \) is the spacing between wavenumbers, which is equal to \( L_x/(2\pi) \), \( L_x \) is the \( x \) domain length, and \( N_x \) is the number of grid points in \( x \). The normalization in (B.1) is also consistent with that of other authors, including Perry & Abell (1975) and Nickels & Marusic (2001).

If the integral in (B.1) were instead approximated using the trapezoidal rule, the result is similar to (B.2), except the weighting of the first and last points are different (half). The difference is negligible if the energy has decayed to a small level at the longest wavelength (i.e., the sample is very long to capture all of the important long scales) and at the shortest wavelength (i.e., the signal is well resolved so the smallest viscous scales are resolved). The trapezoidal rule also neglects any energy in the \( k_x = 0 \) mode because it does not appear in the energy spectrum (see §B.2). Integration numerically approximated with the trapezoidal rule can be performed by integrating the premultiplied spectrum with respect to the logarithm of the \( x \)-axis (and then the result multiplied by \( \ln(10) \)), or the premultiplied energy can be converted to energy and then directly numerically integrated with widely changing wavenumber spacings.

The \( \ln(10) \) factor arises when integrating premultiplied energy with respect to the natural logarithm of \( k_x \). Defining the new integration variable as \( m \equiv \log(k_x) \),
it follows that $k_x = 10^m$ and $dk_x = 10^m \ln(10) \, dm$, so the integral in (B.1) to recover $u'_i u'_j$ becomes

$$u'_i u'_j = \int_0^\infty \Phi_{ij}(k_x) \, dk_x = \ln(10) \int_{-\infty}^\infty k_x \Phi_{ij}(k_x) \, d[\log(k_x)].$$

(B.3)

For the DNS, the domain is short enough that there is significant energy in the $k_x = 0$ mode and the first mode ($k_x = 2\pi/L$, where $L$ is the domain length). The choice of using the summation or numerical integration significantly changes the computed total energy value for the DNS. The difference is not as severe for experimental spectra if the domain/time record is very long and the motions are well resolved.

To test the method of checking the normalization that will be applied to the experimental spectra by numerically integrating the digitized values of a spectrum plot, this same method is applied to a spectrum plot for the DNS spectrum. The plot is also digitized, and after converting to energy and applying the sum in (B.2), when the 0-mode energy is introduced, the correct streamwise turbulence intensity is recovered to within 1% of the actual value.

For the DNS, at $y/R = 0.1$, $u_{rms}^+ = 1.91$ or $\langle u^2 \rangle / u_r^2 = 3.65$, and 4.27% of the energy is in the 0-mode. Integrating using the trapezoidal rule and neglecting the 0-mode energy, the energy is underestimated with an error of 14.2% when integrating the premultiplied spectrum with respect to the logarithm of $k_x R$ and 13.7% when directly integrating energy with respect to $k_x R$. In this example, applying the trapezoidal rule with the issues of the spectrum not decaying to a low value at long wavelength and the wavenumbers being very widely spaced at long wavelength leads to significant error. Using the trapezoidal rule for this spectrum is inconsistent with the normalization in (B.2), so the half weight applied to the highly-energetic longest wavelength mode by the trapezoidal rule and neglecting the 0-mode energy
are responsible for this error. It is likely that some experimental spectra are normalized using the trapezoidal rule, and as discussed above, the error is much smaller for greater spectral resolution and spectra that decay to small magnitudes at long wavelength, so this is not a concern.

B.2 Energy in the $k_x = 0$-modes

The non-zero mean energy for the pipe DNS is a result of calculating the mean statistics for all of the fields and then subtracting this long-time mean from each field, instead of subtracting the instantaneous mean from each field separately, which would result in no energy in the $k_x = 0$ mode. This was chosen because motions longer than the periodic pipe domain presumably exhibit themselves as time fluctuations in the mean, and placing these motions at an infinite wavelength would likely lead to a more accurate cumulative spectrum energy distribution. This is also
expected to be more comparable to experiments, such as hot-wire measurements, in which long time series are acquired that correspond (by Taylor’s hypothesis) to very long wavelength motions. In this case, the $k_x = 0$ energy for experiments are probably negligible, as supported by the decay to low magnitude in the longest wavelength contributions, which does not occur for the DNS with limited domain length. It is therefore less significant how the mean is subtracted for experiments. Thus, the $k_x = 0$ motions for the DNS are intended to represent the energy in the wavelengths present in the experiment that are longer than the pipe DNS domain length.

B.3 Definition of Reynolds Numbers

The most common definition of Reynolds number $Re_D = U_{\text{bulk}} D / \nu$ is based on pipe diameter $D$, bulk velocity $U_{\text{bulk}}$, and kinematic viscosity $\nu$. Some papers define $Re_D$ based on centerline mean velocity $U_{\text{cl}}$, and in this manuscript, this Reynolds number is explicitly named as $Re_{D,cl} = U_{cl} D / \nu$. The friction Reynolds number is most commonly defined as $Re_{\tau} = u_{\tau} R / \nu$, which is equal to the Kármán number $R^+$, but some papers instead define $Re_{\tau}$ using the pipe diameter so the value is twice that of the more common definition. All such values are converted to the more common definition based on radius herein.

B.4 Kim & Adrian (1999) Reynolds Numbers

Prior to examining the spectra of Kim & Adrian (1999), hereafter referred to as KA99, the Reynolds numbers reported are considered. While it is noted that the Reynolds number $Re_D$ is based on centerline velocity with KA99 instead of bulk velocity, comparison with other similar pipe flows with consistently-defined $Re_D$ values for indicates the corresponding $Re_{\tau}$ or $R^+$ values vary significantly from those of KA99.
From calculations performed by the authors of KA99 (separate from the KA99 article), the kinematic viscosity is assumed to be $\nu = 0.155 \times 10^{-4}$ m$^2$/s. Using Sutherland’s formula for viscosity and the ideal gas law for density, this kinematic viscosity corresponds to air at 296 K (73° F) and 1 atm pressure.

For the “$Re_D = 33$ 800” pipe flow listed, the KA99 article reports $U_{cl} = 4.12$ m/s. Calculating the Reynolds number based on the pipe diameter and centerline velocity, $Re_{D,cl} = U_{cl}D/\nu = (4.12 \text{ m/s})(0.127 \text{ m})/(0.155 \times 10^{-4} \text{ m}^2/\text{s}) = 33757$. When rounded, this matches the reported value.

The viscous length scale is $y^* = \nu/u_\tau$. Using the $u_\tau$ value given in both the KA99 article and this sheet, $u_\tau = 0.201$ m/s for the “$Re_D = 33$ 800” case. Based on this and $\nu = 0.155 \times 10^{-4}$ m$^2$/s, $y^* = 7.71 \times 10^{-5}$ m. This is also the value on the sheet. In contrast, the published article states $y^* = 6 \times 10^{-5}$ m. This corresponds to using $\nu = 0.12 \times 10^{-4}$ m$^2$/s (although the article apparently uses $\nu = 0.155 \times 10^{-4}$ m$^2$/s for calculating $Re_{D,cl}$, based on the calculation above), and this is consistent for all of the Reynolds number cases published in the KA99 article. This (0.12 × 10$^{-4}$) value of kinematic viscosity corresponds to air at a temperature of 257 K (2° F).

If the article used $\nu = 0.155 \times 10^{-4}$ m$^2$/s for this calculation as well, then the resulting $R^+ = R/y^*$ would be 823 (the same as written on the sheet), instead of the $R^+ = 1058$ value reported in the article. Still, the $R^+$ value is somewhat higher than would be expected based on other pipe flows with similar $Re_D$ values.

Using the $u_\tau$ value reported, the $U_{cl}/u_\tau$ (or $U_{cl}^+$) is used in conjunction with a relationship obtained from other flows for $Re_D$ and $Re_\tau$. In order to get the $U_{cl}/u_\tau = (4.12 \text{ m/s})/(0.201 \text{ m/s}) = 20.5$ value based on the reported velocities (which were used in both the article and calculation sheet), the relationship suggests that a $Re_\tau(=R^+) = 287$ would have this $U_{cl}/u_\tau$ value. The $Re_{D,cl} = 33757$ found above is estimated to occur for $Re_\tau = 746$, with a $U_{cl}/u_\tau$ value of 22.62.
If the reported value of $U_{cl} = 4.12$ m/s is correct, then $U_{cl}/u_\tau = 22.62$ would be obtained if $u_\tau = 0.182$ m/s instead of the 0.201 m/s reported. Thus, a 9.4% overestimate of $u_\tau$ would increase the calculated value of $R^+$ by 10%.

Conversely, the $u_\tau$ value could be correct with some error in $U_{cl}$. Since the experiment uses the local mean $U$ measurement for Taylor’s hypothesis, the agreement of the wavenumbers at which peaks in the spectra occur when compared to other experiments suggests that these measurements are correct.

The scenario presented in which the $u_\tau$ value is overestimated by 9.4% and the $U_{cl}$ is correct would mean that the spectrum is correct (except for the magnitude normalization), but the reported $Re_\tau$ and $y^+$ values are not correct (also because of the two values of the viscosity used). Then it would be approximately correct to compare at $y/R = 0.10$ the spectra between this experiment (after renormalization) and the DNS. In this comparison, the true Reynolds number difference between the experiment and DNS would be $\sim 10\%$ in $Re_D$ and $\sim 9\%$ in $Re_\tau$.

### B.5 Rescaling of KA99 Spectra

KA99 includes energy spectra for pipes at three Reynolds numbers. Superimposing spectra from the pipe DNS reveals that the KA99 spectra are significantly larger in magnitude. Therefore, the KA99 spectra should be compared with the normalization to verify if they match this normalization. The wavenumber axis appears to align similarly for these experiments and the DNS. For simplicity, chapter 4 and Wu et al. (2012) include a comparison of the DNS spectra with additional spectra obtained by the authors of KA99 but not published in KA99. For this experimental spectrum renormalization was not required. However, the normalization of the spectra published in the KA99 article is here considered.
B.5.1 Outer Scaled Spectra

In the KA99 figure, the curves often fall on top of each other, particularly at high wavenumber, so it is impossible to identify each individual curve. The lowest and the highest curves at long wavelength were digitized for each Reynolds number. At the high wavenumbers, it is not possible to pick out curves, so points following the average roll-off trend are selected.

Comparing the two methods of applying the trapezoidal rule, the results are $\sim 1\%$ higher when directly numerically integrated instead of integrating premultiplied energy with respect to the logarithm of $k_x R$, and this direct integration method is used for the results below.

The line with lowest magnitude at low wavenumber was chosen from the $Re_D = 33,800$ spectrum curves because it is intended to be matched with the DNS at $y/R = 0.1$, and this is the maximum $y$ of the curves in KA99. The curve with maximum $y$ (furthest above the pipe wall) is expected to have the least energy of those in the KA99 figure. The $u_{rms}^+ = 1.91$ to which it is compared matches the pipe DNS. It is also within the scatter for this $y/R$ of the $Re_D = 49,645$ pipe measurements of Lekakis (1988) (there is relatively good agreement at this $y/R$ over a range of Reynolds numbers). The result is an integrated energy 18.5% greater than the energy indicated by the recorded RMS value. (Using the other integration method, it would be 17.1% greater).

There is a significant amount of uncertainty in digitizing the spectrum because the KA99 curves fall on top of each other, and using the lowest curve in all points of lower wavenumber than the high wavenumber peak may underestimate the true amount of energy shown in KA99. Using the center of the band of lines in the high wavenumber roll off region is another source of error. To establish the bounds of the rescaling factor, envelope lines at the highest and lowest values were extracted.
Figure B.2: KA99 premultiplied spectra plot with extracted lines (dashed red) representing lower and upper envelopes of the $Re_{D,cl} = 33\,800$ family of spectrum lines.

and superimposed in Figure B.2. These lines vary from from the lines for the previous calculations in that only the lowest or highest magnitude values for the high wavenumber peak and high wavenumber roll off regions are selected, whereas the previous calculations used an average between the envelopes for both of the digitized curves.

Using these new lower and upper bound curves, the calculated energies from integration are compared with the DNS energies. More refined estimates of the radii included are now used. It has been estimated that, assuming the reported $Re_{D,cl} = 33\,800$ is correct, the estimated value for $Re_\tau$ is 748, which is significantly lower than the value of 1058 that is reported in KA99. The figure in KA99 specifies that these spectra are for $y^+ > 100$ and $y/R < 0.1$. Using the value of $Re_\tau = 748$, at $y^+ = 100$, $y/R = 0.134$ and no radial locations satisfy $y/R > 0.134$ and $y/R < 0.1$.  

347
Using the value that KA99 specifies of $Re_\tau = 1058$, at $y^+ = 100$, $y/R = 0.0945$, and then all of the measurements would be between $y/R = 0.0945$ and $y/R = 0.10$.

DNS spectra indicate that for a band of spectra at these radii ($y/R$ approaching 0.1), as $y$ increases the spectra become smaller in magnitude at essentially all wavelengths (particularly the smaller-wavelength peak and decay region), and the roll-off is completed at a lower wavenumber. (Conversely, decay persists to smaller scales when closer to the wall.)

Although the KA99 $Re_D$ is 10% greater than that of the DNS, based on the reported $Re_{D,cl}$ value in KA99, the DNS $u_{rms}^+$ values are used to provide the energy values to compare the integrated spectra against for the same $y/R$ positions. At $y/R = 0.0945$, the low end of the band of radii determined above, $u_{rms}^+ = 1.911$, and at the high end, $y/R = 0.10$, $u_{rms}^+ = 1.939$. To assess the sensitivity of Reynolds number, the Wu & Moin (2008) $Re_D = 44,000$ pipe DNS RMS values at these same $y/R$ locations are within 1% of the $Re_D = 24,580$ DNS values. Therefore, the difference in Reynolds number is not expected to be a significant source of error. This is consistent with the observations above when comparing to the values in Lekakis (1988).

The result when comparing the integrated value from the lowest energy spectrum envelope to the DNS value at $y/R = 0.10$ is the energy spectrum has 9.6% greater energy (consistent between integration methods). The integrated energy from the highest envelope is 25.5% greater than the DNS energy at $y/R = 0.0945$. (The integrated energies are $\overline{u^2}/u_\tau^2 = 4.00$ and 4.72 for the lower and upper envelopes, respectively, and the DNS energies are 3.65 and 3.76 for $y/R = 0.10$ and $y/R = 0.0945$, respectively.) While the curves oscillate so the envelopes don’t match any particular curve, these percent values establish bounds on how strongly the original KA99 spectra overestimate the true magnitudes. The 17% calculated using the
peak and roll off region estimate in between the two envelope extremes is likely a more accurate estimate.

Another set of spectra (§B.7) exists at similar Reynolds number to the middle (dashed) set of spectra in KA99, and the spectrum matches the KA99 spectrum closely, indicating they are normalized the same way. The more accurate digitized points for this spectrum are calculated to overestimate the energy by a higher percentage for this Reynolds number. An intermediate value of 23% is used to rescale the y axis for the KA99 spectrum, and this is the value used for one version of the plots (Figure B.12). It appears that different factors for each Reynolds number are necessary to more accurately normalize the energy spectra curves in KA99.

### B.5.2 Inner Scaled Spectra

KA99 also includes the same spectra compared with inner scaling of the wavenumber as $k_x \nu/u_\tau$. Based on the outer scaled plots, in which the behavior as a function of wavenumber compares well with other spectra, the viscous decay (roll-off) for the lowest Reynolds number has the spectrum closely approach zero for a nominal wavenumber of $k_x R = 100$. To convert this from outer to inner scaling,

$$\frac{k_x \nu/u_\tau}{k_x R} = \frac{\nu}{u_\tau R} = \frac{1}{Re_\tau}. \quad (B.4)$$

For $k_x R = 100$ and $Re_\tau = 1000$ (roughly), an estimate of where the spectrum approaches zero with inner scaling is $k_x \nu/u_\tau = 0.1$. However, the original figure in KA99 (Figure 2) has the roll-off approaching zero at $k_x \nu/u_\tau \approx 2$. This suggests that the original inner scaled figure in KA99 displays a wavenumber roughly 20 times the true wavenumber.

To obtain a more exact estimate, prominent features of the spectra in both the outer and inner scaled figures in KA99. Since the ratio between them should be $Re_\tau$ according to (B.4), the measured ratio between the wavenumbers where these
features exist can be compared determine a scaling to correct the spectra. Averaging over a total of 8 different features involving all of the Reynolds numbers, the ratio of the true wavenumber to the displayed wavenumber in Figure 2 of KA99 is 0.052 based on the Re$_\tau$ values reported by KA99. However, if the reported Re$_{D,cl}$ values in KA99 are correct, then the Re$_\tau$ values are different. Using the estimated Re$_\tau$ values based on the reported Re$_{D,cl}$ values, the factor between true and displayed wavenumber is 0.07365. There is significant scatter in ratio for each feature measured (from 0.069 to 0.078), so 0.07365 is the average value.

Figure B.3 compares these spectra with the wavenumbers corrected with this 0.07365 ratio. The Guala et al. (2006) spectrum converted to inner scaling with (B.4) assuming the flow was at Re$_\tau = 1425$ is also included. The energy spectra magnitudes were not corrected. While this complicates the comparison with the present pipe DNS spectrum for $y/R = 0.1$ that is also included, it appears that the drop-off region of the pipe DNS spectrum is shifted to the high-wavenumber extreme of the curves that are included. This is further considered in §B.13.

B.6 Spectra Comparison of KA99 with Perry & Abell (1975)

Perry & Abell (1975), hereafter PA75, report following the energy normalization in (B.1). Their pipe flow is at Re$_{D,cl} = 120000$ and Re$_\tau = 2325$. While much higher than the present DNS, this Reynolds number is suitable for comparison with the highest Reynolds number in KA99, Re$_{D,cl} = 115400$.

The PA75 spectra at $y^+ = 100$ and 200 are compared with the channel DNS of Hoyas & Jiménez (2006) at Re$_\tau = 2003$ in del Álamo & Jiménez (2009). Examining the shorter wavelengths where error due to Taylor’s hypothesis is expected to be negligible, the experimental spectrum for $y^+ = 100$ is very close but generally slightly larger than the DNS. del Álamo & Jiménez (2009) note the $y^+ = 200$ spectrum is generally larger for the experiment than the simulation (to a greater extent
Figure B.3: KA99 premultiplied spectra plot with inner scaled wavenumber. The inner scaled wavenumber is scaled from the original plot in KA99 by a factor of 0.07365. The energy spectrum magnitude is not rescaled (although the discussion of the outer scaled KA99 spectrum indicates it should be), and this should be kept in mind when comparing the spectrum at $y/R = 0.1$ ($y^+ = 69$) from the present DNS (blue line). The red line is the spectrum extracted from Guala et al. (2006) that should correspond to a $Re_{D,cl} = 66400$ spectrum in KA99.
Figure B.4: Premultiplied energy spectra from KA99 pipe experiments with the vertical axis at the original scale. Additional data are included from PA75 at $Re_{D,cl} = 120\ 000$, comparable to the KA99 solid lines for $Re_{D,cl} = 115\ 400$, and from Guala et al. (2006) at $Re_{D,cl} = 68\ 781$, comparable to the dashed lines at $Re_{D,cl} = 66\ 400$.

than the $y^+ = 100$, although the intensities could be affected by the differences in flow geometries or Reynolds numbers.

Figure B.4 clearly shows the solid black lines of KA99 have larger magnitudes than the data of PA75. It is difficult to discern values for each line of the KA99 data, but comparing estimates in the higher-wavenumber peak of the bimodal spectrum with values for the PA75 spectra suggests the KA99 magnitudes are 30–45% larger than the PA75 magnitudes. Adjusting the axes so the majority of the spectrum
matches suggests the KA99 magnitudes are 43% greater than the PA75 spectrum. The spectrum agreement shown in del Álamo & Jiménez (2009) supports the conclusion that the PA75 data is correctly scaled (although possibly slightly large). The significant disagreement with KA99 then indicates that the KA99 data is too large to follow the scaling in (B.1). Again, the wavenumber axis appears to align similarly between the data sets.

The points displayed are sparsely distributed among wavenumbers, so considerable error in numerical integration of the PA75 data is to be expected. Integrating the PA75 spectra using the trapezoidal rule directly on wavenumber and energy (non-premultiplied), the resulting energies are 7.8 and 12 percent larger than the energies based on the RMS values for $y^+ = 100$ and 200, respectively. Given the good agreement with del Álamo & Jiménez (2009), this may be, at least in part, attributable to numerical integration inaccuracy with the wide spacing of points. The Guala et al. (2006) spectrum considered next is digitized to provide spectral data with much finer spectral resolution.

### B.7 Spectra Comparison of KA99 with Guala et al. (2006)

Guala et al. (2006) states that their spectra are normalized according to (B.1). The example premultiplied spectrum of a pipe flow at $Re_{D,cl} = 68781$ in their Figure 1(b) was digitized and included here superimposed on the KA99 plot in Figure B.4. This measurement at $y/R = 0.084$ is located at $y^+ = 120$ based on the estimated $Re_\tau$ value of 1425. There is excellent agreement between this spectrum and one of the dashed lines of KA99 for $Re_{D,cl} = 66400$, indicating these spectra are normalized in the same way for both KA99 and Guala et al. (2006). While Guala et al. (2006) states their spectra obey (B.1), the higher spectral resolution and ability to clearly discern the curve relative to the KA99 curves allows accurate numerical integration to verify that this normalization is satisfied.
The spectrum wavenumber density is very good for this spectrum, and consequently both integration methods are consistent with each other, yielding a value of \( \overline{u''u'} = 5.42 \). However, using intensity measurements from Hultmark et al. (2010) and linearly interpolating for this \( Re_D \) (estimated to be 56 114 based on bulk velocity), the intensity is \( \overline{u''u'} = 3.93 \) (the value at this radius for \( Re_D = 75 500 \) is 4.19). The numerically integrated Guala et al. (2006) spectrum yields an intensity 37.7% larger than the interpolated value using the measured intensity profiles of Hultmark et al. (2010).

### B.8 Spectra Calculations for Balakumar & Adrian (2007)

Although this is not a pipe flow, the channel energy spectra of Balakumar & Adrian (2007) are experimental hot film spectra obtained shortly after Guala et al. (2006) and produced with the same methods. Carefully obtaining the streamwise RMS values for these constant temperature anemometry (CTA) experiments from Balasubramaniam (2005) (his Figure 3.3), the energies implied by these values closely match the values obtained by numerical integration of the digitized spectra (calculated using the same method as for the previous flows). The \( Re_\tau = 531 \) with \( y/h = 0.12, 0.32 \) and \( Re_\tau = 960 \) with \( y/h = 0.11, 0.31 \) spectrum energies each match the energies based on the RMS values to within 2%, with three of them within 1%. Using energy values less carefully obtained from other sources, values had errors approaching 10%. Since the energy is the RMS value squared, it is clear that a small error in the RMS value compared against can results in a sizable error in the energy. This comparison demonstrates the issue that the errors observed could result from relatively small errors in the calculated RMS values used to normalize the spectra, and the accuracy of the RMS values used when doing the present comparisons is of great importance. Significant scatter exists among RMS values presented for various measurements.
B.9 KA99 Scaling Summary

In summary, the estimate of the integral of the KA99 spectrum for the lowest Reynolds number is 17% greater than the estimated energy at this $y/R$ taken from the DNS. This estimated energy from the DNS is obtained at $y/R = 0.1$. Uncertainty in the radius where the measurement was taken and the uncertainty in picking out the line on the plot to digitize contribute to significant uncertainty in this number. Assuming it is true that $Re_D,cl = 33800$ and the spectra presented were obtained between $y/R = 0.0945$ and 0.10, there is good confidence that the energy spectra for this Reynolds number contain between 9.6% and 25.5% more energy than comparable DNS statistics (calculated from RMS).

The estimate for the middle Reynolds number is 37.7% greater when the matching Guala et al. (2006) spectrum is integrated and compared to values from other experiments, and this comparison has much better certainty. The highest Reynolds number in KA99 is estimated to have energy 43% greater than the PA75 spectrum, as estimated by rescaling the axis for the best match (while the PA75 spectrum may be slightly large).

B.10 Guala et al. (2006) Spectra and Cumulative Wavelengths

While the pipe flow spectra measurements of Guala et al. (2006) were obtained at significantly higher Reynolds numbers than those of the present DNS, the study of Guala et al. (2006) is relevant because cumulative spectra for pipes were established from these experiments. The associated wavelengths are much longer for Guala et al. (2006) than for the present DNS, and the reasons for this are here considered.

Figure B.5 includes the $Re_\tau = 3005$ experimental pipe spectrum map of Monty et al. (2009) with digitized PA75 experimental spectra at $Re_\tau = 3451$ as superimposed magenta lines and digitized spectra from Guala et al. (2006) at $Re_\tau = 3815$ ($Re_D = 192000$ based on centerline velocity) as superimposed white lines. The data
Figure B.5: $Re_\tau = 3005$ experimental pipe spectrum map of Monty et al. (2009) with digitized PA75 experimental spectra at $Re_\tau = 3451$ as superimposed magenta lines and digitized spectra from Guala et al. (2006) at $Re_\tau = 3815$ ($Re_D = 192,000$ based on centerline velocity) as superimposed white lines. Line contour levels are 0.1 to 1.5 in increments of 0.1.

are digitized from the plotted one-dimensional premultiplied spectra for various $y$ values in each of these publications. The contours therefore involve a significant amount of interpolation due to the relatively widely-spaced $y$ positions for which spectra are available. The Monty et al. (2009) and PA75 contours match well, but the Guala et al. (2006) spectra indicate a distinct lack of energy at short wavelengths.

To further validate the PA75 pipe spectra used in the comparison, they are compared in figure B.6 for $y/R = 0.2$ to $Re_\tau = 2003$ channel DNS spectra calculated by Hoyas & Jiménez (2006) (once available at http://torroja.dmt.upm.es/channels/data/). The comparison indicates these PA75 spectra are generally consistent (particularly in the high-wavenumber region) with a channel at similar Reynolds number, and a similar comparison is also included in del Álamo & Jiménez (2009). This further establishes the PA75 data for compar-
Figure B.6: Comparison of streamwise velocity spectra for PA75 pipe experiments at two Reynolds numbers and Hoyas & Jiménez (2006) channel DNS at $Re_\tau = 2003$ for $y/R$ or $y/h$ of 0.2.

ison with Guala et al. (2006) and suggests the difference between the two spectra sets, particularly at high wavenumbers, is due to Guala et al. (2006).

To more specifically compare the one-dimensional premultiplied streamwise spectra of PA75 and Guala et al. (2006), line plots for $y/R = 0.05$ and 0.1 are included in figure B.7. The line plots are obtained from digitized spectra in the relevant publications. Irrespective of the magnitude normalization, it is clear that the Guala et al. (2006) spectra have a lack of energy at the shortest wavelengths and appear to roll off at lower wavenumbers than the PA75 spectra.

To compare the cumulative spectra wavelengths corresponding to chosen energy fractions, these wavelengths for 0.5 and 0.8 fractions are computed for a variety of other pipe experiments and compared with those reported by Guala et al. (2006). The wavelengths of Figure B.8 are those for which all shorter wavelengths
Figure B.7: Comparison of streamwise velocity spectra for PA75 pipe experiments at two Reynolds numbers and Hoyas & Jiménez (2006) channel DNS at $Re = 2003$ for $y/R$ or $y/h$ of 0.2.

contribute the specified fraction of streamwise velocity energy. Since they are a fraction of the full energy for all wavelengths, these quantities are not affected by the magnitude normalization. Thus, while a lack of energy at high wavenumbers in Guala et al. (2006) would result in these calculated wavelengths being longer, the long-wavelength sections of the spectra could be reliable but affected in the normalization by missed energy at shorter wavelength. In this case, the wavelengths at which the long-wavelength peaks occur (which were compared in Guala et al., 2006) would be unaffected.


den Toonder & Nieuwstadt (1997) includes spectra obtained from laser-Doppler velocimetry measurements of turbulent pipe flows at various Reynolds numbers,
including one at that of the present DNS. They plot their spectra as a function of the natural logarithm of $f^+ = f\nu/u_\tau^2$, a dimensionless temporal frequency. Their premultiplied spectrum $\Psi_{u_\alpha u_\alpha}(f^+)$ is equivalent to the conventional premultiplied spectrum when plotted as a function of wavenumber. The normalization involves integration with respect to $\ln(f^+)$ and is closely related to (B.3), but the integration with respect to the natural logarithm eliminates the $\ln(10)$ factor.

To compare the spectra as a function of wavenumber, Taylor’s hypothesis is used to relate these quantities assuming the convection velocity $U_c$ is the mean velocity at the measurement radius. The relationship is $k_x = 2\pi f/U_c$. Since $f^+ = f\nu/u_\tau^2$, it follows that $f = f^+(u_\tau/R)Re_\tau$. Then $k_x = 2\pi(f^+/R)(u_\tau/U_c)Re_\tau$. Since
\[ U_c^+ = U_c/u_\tau, \] the final result is

\[ k_x R = 2\pi \frac{Re_\tau}{U_c^+} f^+. \] (B.5)

The wall-normal locations included in den Toonder & Nieuwstadt (1997) are \( y^+ = 12 \) and 30. At \( y^+ = 12 \), their LDV measurements indicate that the mean velocity is \( U^+ = 9.7058 \), and the present DNS has \( U^+ = 9.4768 \). At \( y^+ = 30 \), their LDV measures \( U^+ = 13.5308 \), and the DNS has \( U^+ = 13.4646 \). Using the LDV measurements for \( Re_\tau = 690 \) (they report 1380 based on diameter), (B.5) relates the nondimensionalized wavenumber to inner scaled frequency as \( k_x R = 446.68 f^+ \) at \( y^+ = 12 \) and \( k_x R = 320.41 f^+ \) at \( y^+ = 30 \).

### B.12 Comparison of DNS with Experiments

Having established the convention that the data sets are to be normalized according to, the present pipe DNS is compared with several spectra from experiments. Hultmark et al. (2010) used the convention in (B.1) with the trapezoidal rule to normalize their hot wire spectra.

The comparisons in figures B.9–B.11 focus on low \( y \) values because more experimental measurements are available for comparison in this region. At the nearest wall location (\( y^+ \approx 12 \) in Figure B.10), the match is good between both experiments and the DNS to within the noise of the measurements. At slightly higher \( y^+ \approx 14 \), a slight shift in the high wavenumber roll-off between the DNS and Hultmark et al. (2010) measurements is visible. Further from the wall, at \( y^+ \approx 30 \) (Figure B.11), a greater shift in high wavenumber roll-off is visible, with the hot wire spectrum rolling off at lowest wavenumber, followed by the den Toonder & Nieuwstadt (1997) LDV measurements, and the the DNS spectrum decays at the highest wavenumber.

This also occurs somewhat for the comparison between DNS and hot wire for channel flow in Monty & Chong (2009), but the intersection occurs at a somewhat
Figure B.9: Adapted from Figure 5(b) of Hultmark et al. (2010) showing the energy spectra at the inner peak radius, including unfilled circles representing the $Re_D = 25000$ measurements. The spectrum for the pipe DNS at the inner peak location ($y^+ = 14.4$) is superimposed as a red line.

Figure B.10: Comparison of energy spectra for the present DNS, den Toonder & Nieuwstadt (1997) LDV measurements also at $Re_D = 24580$, and Hultmark et al. (2010) hot wire measurements for $Re_D = 25000$ pipe flow, all for $y^+ \approx 12$. 
higher $y^+$ value and the shift may be less pronounced when viewed inner units. This comparison is complicated by the higher Reynolds number of the experiment ($Re_\tau = 1040$) compared to the DNS ($Re_\tau = 934$), but the inner scaling is expected to give the fairest comparison for the viscous cutoff. This paper presents the one-dimensional spectrum lines, and the inner scaled comparisons show a noticeable shift particularly at $y^+ = 50$, but the outer scaled presentation obscures the shift so it appears they are rolling off at the same wavelengths with the Reynolds number difference. The outer scaled spectrum map makes the high wavenumber roll-offs appear similar for $y/h > 0.02$. Returning to their inner scaled map, the lowest energy contour lines at high wavenumber converge together above $y^+ = 100$ and are very close for $y^+ > 200$. 

Figure B.11: Comparison of energy spectra for the present DNS, den Toonder & Nieuwstadt (1997) LDV measurements also at $Re_D = 24\,580$, and Hultmark et al. (2010) hot wire measurements for $Re_D = 25\,000$ pipe flow, all for $y^+ \approx 30$. 

362
Figure B.12: Premultiplied streamwise velocity spectra are compared for the present DNS, KA99 measurements, and Hultmark et al. (2010) measurements. The families of black lines extending to the lowest wavenumbers are the premultiplied streamwise velocity spectra adapted from the hot-film measurements of KA99 with \( y^+ > 100 \) and \( y/R < 0.1 \). Dotted line: \( Re_D = 33\,800 \); dashed line: \( Re_D = 66\,400 \); solid line: \( Re_D = 115\,400 \). The thick solid lines are the present DNS, with the blue line at \( y^+ = 69 \) (\( y/R = 0.1 \)) and red line at \( y^+ = 96 \) (\( y/R = 0.14 \)). The thin solid lines with symbols are hot-wire spectra obtained by Hultmark et al. (2010), with the blue line and squares at \( y^+ = 69 \) (\( y/R = 0.1 \)) and the red line and diamonds at \( y^+ = 96 \) (\( y/R = 0.14 \)).

The shift in roll-off wavenumber for the present pipe spectra relative to the Hultmark et al. (2010) hot wire spectra near the logarithmic layer region is clearly evident in Figure B.12. The Hultmark et al. (2010) spectra roll off on the higher-wavenumber boundary of the KA99 spectra. The Hultmark et al. (2010) used a short hot-wire length so high-wavenumber attenuation due to spatial averaging should be minimal.
B.13 Comparison of Pipe DNS with Channels

To further compare the wavelengths where roll-off occurs, the pipe DNS is compared with spectra obtained from channels. At much higher Reynolds number \( (Re_\tau = 3005–3015) \), Monty et al. (2009) compared hot wire spectra for turbulent channel and pipe flow, and they find “no obvious or significant differences.” The high wavenumber roll-off occurs at the same wavenumbers for their comparisons. It is therefore expected that comparing the pipe DNS to channel DNS should reveal similar high-wavenumber decay.

Figure B.13 compares the spectra at \( y/h \) and \( y/R \) values of 0.1, and Figure B.14 compares for values of 0.3. They are presented in both inner and outer scaling of wavenumber. The inner scaled plots reveal how the cutoffs compare. Hultmark et al. (2010) shows collapse at high wavenumber over a range of Reynolds numbers when this scaling is applied to each spectrum obtained at the location of the inner energy peak for its Reynolds number, although the roll off begins at different inner scaled wavenumber, but the roll off ends at the same scaled wavenumbers. In the present inner scaled plots, the \( y/R \) positions are the same for each Reynolds number, so the inner scaling is not complete and less agreement is expected. In Figures B.13 and B.14, the overall behaviors agree well, but at 0.1, the pipe DNS rolls off at noticeably higher viscous scaled wavenumber. At 0.3, the pipe DNS also rolls off at noticeably higher viscous scaled wavenumber. In addition, the \( Re_\tau = 531 \) channel measurements roll off at somewhat higher viscous scaled wavenumber than the \( Re_\tau \approx 950 \) spectra both from measurement and simulation. Some of this difference could be due to using outer scaling in selecting the same \( y/R \) values for different Reynolds numbers. With viscous scaling, the Hultmark et al. (2010) pipe spectra roll off at generally higher wavenumber than the \( Re_\tau \approx 950 \) channel spectra,
Figure B.13: Comparison of energy spectra at $y/R = 0.1$ and $y/h = 0.1$ for the present DNS, channel measurements of Balakumar & Adrian (2007), a channel hot-wire measurement of Monty & Chong (2009), and a spectrum calculated from a single channel field of del Álamo et al. (2004). The frames have outer and inner scaling of wavenumber.

but at a wavenumber similar to the $Re_\tau \approx 531$ hot film channel spectra (and lower wavenumber than the present pipe DNS spectra).

Better agreement with inner scaling is expected if the different Reynolds numbers are compared for the same viscous scaled $y^+$ location. Figure B.15 compares...
Figure B.15: Comparison of channel DNS (with $Re_\tau$ values indicated in the legend) energy spectra at $y^+ \approx 100$ with inner scaling of wavenumber.

...a set of channel spectra at $y^+ \approx 100$ with viscous scaling of the wavenumber. The scaled wavenumbers at which the spectra have rolled off to small magnitude are very similar with a slight tendency to higher scaled wavenumber as Reynolds number increases. As shown in §B.5.2, KA99 also find that pipe measurements in bands of radii with $y^+ > 100$ and $y/R < 0.1$ for different Reynolds numbers with this scaling agree similarly for the high wavenumber roll-off. Figure B.15 shows how the range of wavenumbers over which the roll-off occurs widens as Reynolds number increases.

Figures B.16 and B.17 compare the inner scaled energy spectra for the present pipe flow with those of various other pipe and channel flows at $y^+ = 100$ and 300. In both cases, the pipe DNS energy throughout most of the roll-off region is larger in magnitude than all of the other spectra included. At the highest wavenumbers, however, the present DNS pipe spectra decays to a smaller level than some of the
Figure B.16: Comparison of energy spectra at $y^+ \approx 100$ with inner scaling of wavenumber for the present DNS, a similar pipe flow hot wire measurement of Hultmark et al. (2010), and several channel simulations and measurements.

Other included spectra. This is also consistent with the comparison with inner scaled KA99 data in §B.5.2. At $y^+ = 300$, the Hultmark et al. (2010) hot wire pipe spectra fits between the Balakumar & Adrian (2007) channel hot film spectra for $Re_\tau$ values surrounding the pipe’s Reynolds number. The Hultmark et al. (2010) hot wire pipe spectra magnitudes are near or slightly above the $Re_\tau = 960$ Balakumar & Adrian (2007) spectra near $y^+ = 100$. The effect of hot wire length on the channel spectra has not been compared, but the wires are short for accurate high wavenumber measurements with the pipe experiments. The validity of energy spectrum comparisons between turbulent pipe and channel flows is supported by the experimental comparisons of Monty et al. (2009) and del Álamo & Jiménez (2009) using PA75 pipe measurements to compare with channel DNS.

The premultiplied spectrum of present pipe DNS is noticeably larger in the decay region than that of Hultmark et al. (2010) at the same Reynolds number. Points
Figure B.17: Comparison of energy spectra at $y^+ \approx 300$ with inner scaling of wavenumber for the present DNS, a similar pipe flow hot wire measurement of Hultmark et al. (2010), and several channel simulations and measurements.

obtained from the spectrum map for the Chin et al. (2010) $Re_{\tau} \approx 500$ pipe DNS with both $12\pi R$ and $20\pi R$ simulation domain lengths match well in the roll off region with the Hultmark et al. (2010) pipe measurements. The Chin et al. (2010) points also match well in this region with channel DNS, further supporting comparison between these two geometries.

In the $y^+ = 100$ comparison, the present pipe DNS roll-off appears shifted to lower wavenumber relative to the Chin et al. (2010) simulation at the beginning of the roll off just beyond the peak. When the energy has decayed to a lower level, the pipe DNS shift to higher wavenumber is apparent. In summary, the pipe DNS appears to decay at slightly higher wavenumber relative to other experiments and simulations that are expected to be comparable.
Uncertainty of Energy Spectra

The pipe spectrum was computed using all fields available, and in the process the sample standard deviation of the individual fields that are averaged to make the mean (where each individual field has the azimuthal average taken to generate the spectrum) was also computed (for each y and wavenumber). The final spectrum value (for each y and wavenumber) may be viewed as a mean of the spectrum values of each field. From this it is possible to compute a 90% confidence interval of the mean.

According to Figliola & Beasley (2000) (pp. 121–125), statistical estimates of finite data sets $x_i, i = 1, N$ can be calculated for the sample mean

$$\bar{x} = \frac{1}{N} \sum_{n=1}^{N} x_n \quad \text{(B.6)}$$

and sample variance

$$S_x^2 = \frac{1}{N-1} \sum_{n=1}^{N} (x_n - \bar{x})^2. \quad \text{(B.7)}$$

The standard deviation of the means is then $S_\bar{x} = S_x/N^{1/2}$. The estimate of the mean with the $P\%$ precision interval based on a finite data set is given by

$$x' = \bar{x} \pm t_{\nu, P} S_\bar{x}. \quad \text{(B.8)}$$

Applying this to the energy spectrum, each energy spectrum point for a given field $n$, here denoted by $E_n$ with the velocity components and wavenumber omitted, is averaged over the available fields to yield the final spectrum $\bar{E}$. The confidence interval for this mean energy is $\pm t_{\nu, P} S_\bar{E}$. The $t$ estimator using the Student-t distribution is $t_{91, 90} = 1.662$ for a $P = 90\%$ confidence interval. The resulting confidence intervals calculated using the computed sample variances for each point are included in the plot of streamwise spectra are shown in Figure B.18.

The confidence intervals statistically support the validity of at least the 15 $R$ peaks at $y/R = 0.3$ and 0.4 by examining the worst cases of the interval bars. It is
Figure B.18: Pipe DNS energy spectra at several radii with 90% confidence interval bars.

also unlikely that a better-converged spectrum would be at the bottom of the bar for the peak location and the top of the bars at the neighboring points (the worst case for reducing the peak). The slight peak that could be associated with an outer site possibly beginning to appear in the contour map exists at these radii.
The net force balance that appears in chapter 4 (§4.8) for turbulent pipes is explained in greater detail, and the terms are compared to those of the net force balance for turbulent channels.

C.1 Channel Mean Axial Momentum Equation

The net force balance in turbulent channel flow is described by the Reynolds-averaged mean streamwise momentum equation in Cartesian coordinates:

\[
\frac{DU}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} + \frac{\partial}{\partial y} \left[ \frac{-\bar{u}'\bar{v}'}{\partial y} \right],
\]

in which $\frac{DU}{Dt}$ represents the total derivative of Reynolds averaged axial velocity $U$, $y$ is the wall-normal coordinate, $u'$ is streamwise velocity fluctuation, $v'$ is wall-normal velocity fluctuation (and $v$ has zero mean), and $\partial P/\partial x$ is the mean pressure gradient (Pope, 2000, (7.8)). The total derivative $\frac{DU}{Dt}$ represents the effect of net forces that accelerate or decelerate the flow, but must balance each other so the total derivative is equal to zero for a statistically stationary turbulent channel flow. The Reynolds force term is related to Reynolds stresses that appear from the Reynolds averaging.

C.2 Pipe Mean Axial Momentum Equation

For a turbulent pipe, the mean flow profile is described by the Reynolds-averaged axial momentum equation in cylindrical equations assuming only the axial velocity component has a nonzero mean:

\[
\frac{DU}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial (\bar{U})}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} \left[ r \bar{u}'u'_r \right],
\]

where $\frac{DU}{Dt}$ represents the total derivative of Reynolds averaged axial velocity $U$. $u'$ and $u'_r$ are velocity fluctuations in the $x$ axial and $r$ radial directions, respectively, and $\partial P/\partial x$ is the mean pressure gradient. For the flow in a pipe, $\frac{DU}{Dt} = 0$. This
form applies only to streamwise-homogeneous, non-swirling flows. This equation is simplified from more complex forms in Pope (2000) (5.45–48) and Shiri (2010). For analogy to channel flows, it is convenient to define \( v \) as the wall-normal velocity component, which is oriented opposite of the radial direction, so \( v = -u_r \).

Then the equation becomes:

\[
\frac{\bar{D}U}{\bar{D}t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\nu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \bar{U}}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{u}' \bar{v}' \right).
\]  

(C.3)

For analogy to channel flows, it is also convenient to cast the equation as a function of wall-normal coordinate \( y \). With an outer pipe radius \( R \), \( y \) is defined as \( y \equiv R - r \). The mean axial momentum equation becomes:

\[
\frac{\bar{D}U}{\bar{D}t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \left( \nu \frac{\partial^2 U}{\partial y^2} - \frac{\nu}{R-y} \frac{\partial \bar{U}}{\partial y} \right) + \left( \frac{\partial -\bar{u}' \bar{v}'}{\partial y} + \frac{\bar{u}' \bar{v}'}{R-y} \right).
\]  

(C.4)

Note the equation is identical to (C.1) for channels, except for an additional viscous term and an additional Reynolds force term. Each of the additional terms is due to the cylindrical coordinates.

C.3 Total Shear Stress

It is useful to introduce total shear stress \( \tau \) as

\[
\tau \equiv \rho \nu \frac{dU}{dy} - \rho \bar{u}' \bar{v}',
\]  

(C.5)

and \( \tau \) is a function only of \( y \) (Pope, 2000, (7.10)). This is the typical definition for a channel. For the pipe, shear stress is typically presented in a different form with radial derivatives for cylindrical coordinates,

\[
\tau_{\text{pipe}} \equiv \rho \nu \frac{dU}{dr} - \rho \bar{u}' \bar{u}_r',
\]  

(C.6)

according to Pope (2000) (7.99). For the purpose of analogy with channels, the definition in (C.5) will also be used for the pipe, but the cylindrical coordinate form
(C.6) will be used to establish the form of the total shear stress in the pipe. The ‘pipe’ subscript therefore is used to indicate this alternate definition.

Introducing total shear stress (C.5) into (C.1) for the channel, the result is

$$\frac{d\tau}{dy} = \frac{dP}{dx}.$$ \hfill (C.7)

Since $\tau$ is only a function of $y$ and $P$ is only a function of $x$, it can be argued that $d\tau/dy$ and $dP/dx$ are each constant because they are equal to each other (Pope, 2000). Integrating, the result is that shear stress varies linearly with $y$ as

$$\tau = \frac{dP}{dx} y.$$ \hfill (C.8)

At $y = -h$, the lower wall where $\tau = \tau_w$, it follows that

$$\frac{dP}{dx} = \frac{-\tau_w}{h} = -\frac{\rho u_r^2}{h},$$ \hfill (C.9)

since $u_r \equiv \sqrt{\tau_w/\rho}$. This result is consistent with writing a net force balance for a volume of channel with shear stress on the upper and lower walls balancing a force acting on the flow cross-section due to the axial pressure gradient.

For the pipe flow, (C.2) may be rearranged to

$$\frac{\bar{D} U}{\bar{D}t} = -\frac{dP}{dx} + \frac{1}{r} \frac{d}{dr} \left[ r \left( \rho \frac{dU}{dr} - \rho u' u' r \right) \right].$$ \hfill (C.10)

For the statistically stationary turbulent pipe, this is

$$\frac{dP}{dx} = \frac{1}{r} \frac{d}{dr} \left( r \tau_{pipe} \right).$$ \hfill (C.11)

Using similar reasoning as for the channel, with $\tau_{pipe}$ only a function of $r$ and $P$ only a function of $x$, the left hand side and right hand side are each constant. Integrating, the result is that shear stress varies linearly with $r$ as

$$\tau_{pipe} = \frac{r}{2} \frac{dP}{dx}.$$ \hfill (C.12)
At the pipe wall where $r = R$ and $\tau_{\text{pipe}} = -\tau_w$, it follows that

$$\frac{dP}{dx} = -2\frac{\tau_w}{R} = -2\frac{\rho u^2}{R}. \quad (C.13)$$

This result is consistent with writing a net force balance for the volume of an axial pipe segment with shear stress on the wall balancing a force acting on the flow cross-section due to the axial pressure gradient.

Another relevant result is that, according to (C.12), $\tau_{\text{pipe}}$ varies linearly with $r$. Since $y = R - r$, i.e. $y$ also varies linearly with $r$, it follows that both $\tau$ (as defined for the channel) and $\tau_{\text{pipe}}$ vary linearly with $y$. They are simply related to each other by, using $v = -u_r$ and $y = R - r$,

$$\tau_{\text{pipe}} = \rho \nu \frac{\partial U}{\partial r} - \rho u_r u'_r$$

$$= -\rho \nu \frac{\partial U}{\partial y} - \rho (-u'_r v')$$

$$= -\left(\rho \nu \frac{\partial U}{\partial y} - \rho u'_r v'\right)$$

$$= -\tau. \quad (C.14)$$

Thus, $\tau_{\text{pipe}}(y) = -\tau(y)$.

### C.4 Far-wall Mean Force Balance for Channel

In the mean axial momentum equation (C.1), when far from the wall, the viscous term makes a minor contribution because the velocity profile is relatively constant. Therefore, the balance is between the pressure and Reynolds force terms.

The value of the Reynolds force term far from the wall is obtained as follows: From (C.8) and (C.9), it follows that $\tau = -\rho u^2_y$. From the total shear stress definition (C.5), with the viscous term neglected far from the wall,

$$\rho \nu \frac{dU}{dy} - \rho u'_r v' = -\rho u^2_y \rightarrow u'_r v' = u^2_y. \quad (C.15)$$
The relevant Reynolds force term in the mean axial momentum equation is \(d(-\overline{u'v'})/dy\), so this is
\[
\frac{d(-\overline{u'v'})}{dy} = \frac{-u_T^2}{h}. \tag{C.16}
\]

C.5 Far-wall Mean Force Balance for Pipe

Considering the individual terms in (C.4) for the pipe, both viscous terms involve derivatives of mean streamwise velocity \(U\), so these can be assumed to be small when far from the wall. The first Reynolds force term is analogous to the channel Reynolds force term previously considered. Using the equations for cylindrical coordinates, combining (C.12) and (C.13) yields
\[
\tau_{\text{pipe}} = \frac{-r}{R} \rho u_T^2. \tag{C.17}
\]

From the total shear stress for pipes defined in (C.6), it follows that
\[
\rho \frac{dU'}{dr} = \rho \overline{u'u'_r} = -\frac{r}{R} \rho u_T^2 \rightarrow \overline{u'u'_r} = \frac{u_T^2 r}{R} \rightarrow \overline{u'v'} = -\frac{u_T^2 r}{R}. \tag{C.18}
\]

The first net Reynolds force term is
\[
\frac{d(-\overline{u'v'})}{dy} = \frac{d}{dy} \left(\frac{u_T^2 R - y}{R}\right) = -\frac{u_T^2}{R}. \tag{C.19}
\]

The second net Reynolds force term is
\[
\frac{\overline{u'v'}}{R - y} = \frac{\overline{u'v'}}{r} = -\frac{u_T^2}{R}. \tag{C.20}
\]

The pressure term is, from (C.13),
\[
-\frac{1}{\rho} \frac{\partial P}{\partial x} = 2 \frac{u_T^2}{R}. \tag{C.21}
\]

Thus, in the net force balance, the accelerating pressure contribution of \(2u_T^2/R\) opposes the decelerating Reynolds force contributions of \(-u_T^2/R\) for the first term and \(-u_T^2/R\) for the second term when far from the wall.

C.6 Pipe DNS Results

For the \(R^+ = 685\) pipe DNS, the net force contributions are as follows.
Figure C.1: Net force balance in which the solid grey line is \(-1/\rho \partial P/\partial x\), solid light blue is \(\partial (u'v')/\partial y\), dash-double-dot green is \(u'v'/(R-y)\), dashed magenta is \(\nu \partial^2 U/\partial y^2\), dash-dot orange is \(-[\nu/(R-y)] \partial U/\partial y\). They sum to zero. The dotted light blue line also depicts the component of \(\partial (u'v')/\partial y\) with \(\lambda_x \geq 6R\).

Figure C.2: Net force balance in which the solid grey line is \(-1/\rho \partial P/\partial x\), solid light blue is \(\partial (u'v')/\partial y\), dash-double-dot green is \(u'v'/(R-y)\), dashed magenta is \(\nu \partial^2 U/\partial y^2\), dash-dot orange is \(-[\nu/(R-y)] \partial U/\partial y\). They sum to zero. The dotted light blue line also depicts the component of \(\partial (u'v')/\partial y\) with \(\lambda_x \geq 6R\).
Figure C.3: Net force balance in which the solid grey line is \(-(1/\rho)\partial P/\partial x\), solid light blue is \(\partial(-u'v')/\partial y\), dash-double-dot green is \(\bar{u}'\bar{v}'/(R-y)\), dashed magenta is \(\nu\partial^2 U/\partial y^2\), dash-dot orange is \(-[\nu/(R-y)]\partial U/\partial y\). They sum to zero. The dotted light blue line also depicts the component of \(\partial(-u'v')/\partial y\) with \(\lambda_x \geq 6R\).
APPENDIX D

EXAMPLE $x-r$ PLANES OF PIPE SIMULATION
Additional example $x-r$ planes of the turbulent pipe flow are included in the same format as those included in chapter 5. Only several examples were included therein for brevity, but additional examples used to form the conclusions of the study are here presented. Negative $u$ fluctuation streaks are selected from $x-s$ ($x-\theta$) planes at $y^+ = 101$. They are selected to be long, straight, and strong in magnitude. The velocity fluctuation contours at this $y^+$ and also at a location nearer the wall ($y^+ = 60$) are shown for each selected streak. The planes (slices) for these streaks focus on the low speed regions relatively far from the wall, with positions approximately corresponding to (or slightly above) the log law region. The planes at the two $y^+$ values provide an indication of how well the streak patterns match between the two radii. The azimuthal width scales (in an average sense) are expected to grow noticeably between these two radii. The extracted streak lines are not comprehensive, but are the result of picking several of the straightest, longest, strongest streaks from the field. Examples are also chosen to avoid crossing the periodic streamwise domain edges. Therefore, low speed streaks are even more common than the number of chosen examples indicates. Due to wavering, the straight extracted lines often miss some of the negative $u$ fluctuation peaks in azimuthal position. The extracted lines do generally contain negative $u$ fluctuation regions, however.

The presentation of $x-r$ planes follows the method of Adrian et al. (2000b) and Adrian et al. (2000a), in which vectors are presented showing velocity (not fluctuation) with a selected streamwise convection velocity subtracted. The velocity is interpolated onto a uniform grid with the same spacing in each direction. This results in a presentation that is more easily interpreted and more analogous to PIV fields. The vector spacings used are $4.5^+$ and $6.75^+$ in viscous units (comparable with pipe PIV figures of Urushihara et al., 1993). Vortex cores are identified using $3D \lambda_{ci}$ signed by azimuthal vorticity $\omega_\theta$ to indicate the direction of swirling, and color contours of this quantity are included in each plot. The color legend is pre-
Figure D.1: Color contour levels for $u$ fluctuation $x-s$ plane plots and $x-r$ plane vector plots; nondimensionalizations are by radius $R$ for length scale and $U_{bulk}$ for velocity scale.

Presented in figure D.1. 3D $\lambda_{ci}$ represents the true swirl, whereas 2D $\lambda_{ci}$ can falsely identify regions that are not truly swirling. However, 3D $\lambda_{ci}$ is sensitive to swirl that is not about an axis normal to the plane, and the $\omega_\theta$ signing can cause this quantity to change signs across a single contour region. The local velocity at the center of each core identified by a peak in signed 3D $\lambda_{ci}$ is used to set the subtracted convection velocity for each frame. The vortex used for this purpose is marked by a red arrow in each $x-r$ plane plot. The vortices shown and indicated with arrows focus on (but are not all instances of) those that give a circular vortex pattern when their $u$ at the center is subtracted. The circular vortex pattern is an indication that the swirl is normal to the page. Complex, large regions of signed $\lambda_{ci}$ (particularly those that change signs rapidly) are often associated with predominantly out-of-plane oriented swirl and vorticity.

The $x-r$ plots are extracted along low speed streaks because this is where evidence of packets is expected to be found. However, as shown by $x-\theta$ plots, there is considerable wavering so other behavior will be observed along the lines of extracted $x-r$ planes. For that reason, the $x-r$ plots show full lengths along the lines to give a more complete picture of what occurs in the flow and the frequency of features expected to commonly occur. The examples shown are obtained from a single field at $t = 252R/U_{bulk}$, identified also as field 00035 based on the numbering scheme of the fields saved by the simulation.
Figure D.2: $u$ fluctuation color contours for $t = 252R/U_{bulk}$ field
D.1 Streak A of $t = 252R/U_{\text{bulk}}$ field

Figure D.3: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak A

Figure D.4: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_q$. 
Figure D.5: A section of streak A ($t = 252R/U_{bulk}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_{th}$.

Figure D.6: A section of streak A ($t = 252R/U_{bulk}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_{th}$.
Figure D.7: A section of streak A ($t = 252R/U_{bulk}$) with vectors at coarse ($6.75^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.

Figure D.8: A section of streak A ($t = 252R/U_{bulk}$) with vectors at coarse ($6.75^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 

385
Figure D.9: A section of streak A \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_\theta\).

Figure D.10: A section of streak A \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.9.
Figure D.11: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$ and red line contours of 2D $\lambda_{ci}$ signed by $\omega_\eta$.

Figure D.12: A section of streak A ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 
Figure D.7 includes what appears to be a tall packet-like arrangement (growing), but with growth at a steeper angle than would be expected for a hairpin packet ($27^\circ$ vs. $11^\circ$). Figure D.8, using a lower convection velocity, displays another relatively low (in $y$) vortex in the packet and also shows a retrograde vortex (indicated by the right arrow).

In figure D.9, the region from $x/R = 4.8$ to 5.1 is consistent with swirl about an axis that is largely streamwise-aligned. The same frame with azimuthal vorticity presented in figure D.10 shows an upward-inclined layer of azimuthal vorticity from $x/R = 4$ to 4.5, which extends up to $69^+$, where it appears a vortex core (consistent with a head) detaches or breaks off from the other inclined vorticity. The small circular regions (heads) in figure D.9 (whereas vorticity is shown in figure D.10) appear in a pattern like that of growth of a small packet. It is difficult to obtain circular vortices in the velocity vectors by subtracting the local streamwise velocity at the centers, but the presence of the shear layer (seen in the vorticity plot in figure D.10) could obscure this.

2D $\lambda_{ci}$ is included in figure D.11. It can be seen how 2D and 3D versions compare. This frame does not include a clear example of a packet. The right vortex with an arrow shows 3D $\lambda_{ci}$ that is evidently in a vortex tube not aligned normal to the page. This is indicated by the pattern of changing $\omega_\theta$ sign that leads to the signed $\lambda_{ci}$ having different sign so the contours appear as yellow below and cyan-blue above. When 2D $\lambda_{ci}$ is calculated, the peak values in this region occur above the 3D $\lambda_{ci}$ identified.

Figure D.12 shows a complex arrangement of vortices; two examples of retrograde vortices exist. In figure D.3 at $y^+ = 60$ below the retrograde vortex near $x/R = 6.65$, a decreased magnitude in the negative $u$ fluctuation is visible. $y^+ = 101$ is above this vortex, so the negative $u$ fluctuation at this location should be strengthened.
D.2 Streak B of $t = 252R/U_{\text{bulk}}$ field

Figure D.13: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak B

Figure D.14: A section of streak B ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 

389
Figure D.15: A section of streak B \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \(\lambda_c\) signed by \(\omega_\theta\).

Figure D.16: A section of streak B \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.15.
The $x-\theta$ contour plot of $B$ (figure D.13) at $y^+ = 60$ shows the extracted plane marked by the line crossing peaks of strong negative $u$ fluctuations at $x/R = 3.75$, 4.03, and 4.30 (as well as further locations downstream). If these motions are associated with hairpins, it is expected to see hairpin signatures near these locations, with heads above (in a wall normal sense) and downstream the velocity peak locations. Examining figure D.14, evidence of vortex heads at approximately $x/R = 3.77$, 4.05, and 4.3 exists. The signatures of Q2 vectors along an inclined angle to each head are also visible. The vortex evidence near $x/R = 4.05$ is quite complex. The subtracted advection velocity is slightly mismatched (too low) for the vortex near 4.3, so the vortex core velocity signature appears slightly too low. There are complex motions below and slightly upstream the $x/R = 4.05$ vortex.

Examining figure D.15, many complex vortices are apparent in the location of the strong negative $u$ fluctuation visible in the $x-\theta$ contour plot from $x/R = 4.9$ to 5.3. To better understand the flow, figure D.16 shows the same plane with azimuthal

![Figure D.17: A section of streak B ($t = 252R/U_{bulk}$) with vectors at fine (4.5$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.](image-url)
vorticity. The vorticity makes clearly visible the shear layers that are consistent with the observed strong negative $u$ fluctuation from $x/R = 4.9$ to 5.3. The vorticity near the top of the shear layer near $x/R = 5.0$ to 5.1 is complex and does not clearly match the idealized hairpin head shape. When downstream 5.3, even though the peak negative $u$ fluctuation is slightly shifted azimuthally from the extracted plane, the presence of vortical structures continue in the $x-r$ plane of figure D.15.

Figure D.17 also shows complex arrangements of vortices. The general arrangement includes the presence of significant azimuthal vorticity (into the page) above the negative $u$ fluctuations that are observed in the $x-\theta$ contour plot of $B$ at $y^+ = 60$.

D.3 Streak C of $t = 252R/U_{\text{bulk}}$ field

Figure D.18: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak C

Figure D.19 displays complex arrangements of vortices and a large region of retrograde azimuthal vorticity. Figure D.20 shows vortices above the strong low-speed regions and, while complicated by other vortices and with vortices in complex shapes, the two identified vortices are arranged in a train so it is plausible that
Figure D.19: A section of streak C \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_\theta\).

Figure D.20: A section of streak C \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_\theta\).
Figure D.21: A section of streak C ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.

Figure D.22: A section of streak C ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 

394
this arrangement is responsible for the streamwise extent of the low-speed region. Subtracting the local velocity at the core indicated by \( \lambda_{ci} \) at \( x/R = 11.15 \) and \( r/R = -0.76 \) does not result in a clear core signature at that location because there is radial flow inward to the centerline.

Figure D.21 shows several vortices far from the wall. Figure D.22 focuses on a vortex nearer to, but still relatively far from, the wall. Plotting contours of azimuthal vorticity for the same plane in figure D.23 indicates the presence of a shear layer not identified by \( \lambda_{ci} \) that extends far from the pipe wall, up to 0.34\( R \) (233\( ^{+} \)) above the wall at \( x/R = 12.6 \). This shear layer seems strongly related to the negative \( u \) region, but no clear hairpin heads are associated with the end (downstream extent) of this example, as the shear layer that is part of the hairpin vortex signature would suggest. However, the head-like vortex near \( x/R = 12.2 \) (figure D.22) also appears associated with the negative \( u \) region. This head-like vortex is located above and upstream much of the shear layer.

![Figure D.23: A section of streak C (\( t = 252R/U_{bulk} \)) with vectors at fine (4.5\( ^{+} \)) spacing with color contours of \( \omega_{\theta} \). The field view is the same as figure D.22.](image-url)
D.4 Streak D of $t = 252R/U_{bulk}$ field

Figure D.24: $u$ fluctuation color contours of $t = 252R/U_{bulk}$ field low momentum streak D

Figure D.25: A section of streak D ($t = 252R/U_{bulk}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 

396
Figure D.26: A section of streak D ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$.

Figure D.27: A section of streak D ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5$^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 

397
Figure D.25 includes several small, apparent heads with $r/R$ between −0.9 and −0.8 for $x/R = 16.8$ to 16.9. These heads appear to be associated with the negative $u$ region below. At $x/R = 17.25$ there is a hairpin vortex signature with an elongated region of weaker swirl below and upstream the head (red arrow). The negative $u$ region near $x/R = 17.4$ is below a shear layer that lifts up to about $0.05R$ above the wall ($34^+$). There is a small head-like region of swirl at its upper tip, near $x/R = 17.47$. Figure D.26 includes a large number of distinct vortices, and those with the orientation of heads are relatively near the wall.

Figure D.26 includes several inclined regions of azimuthal vorticity (shear layers) that lift up from the wall up to approximately $0.1R$ ($69^+$) above the wall. The $\lambda_{ci}$ region identified at $x/R = 19.8$ is immediately below a shear layer. Figure D.27 shows many vortices in a complex arrangement with many retrograde vortices, including one unusually near the wall. True (not fluctuating) vorticity sign is used to determine the sign of the signed $\lambda_{ci}$, and this highlights the strength of the retrograde vortex.

D.5 Streak E of $t = 252R/U_{bulk}$ field

Figure D.29 includes a classic signature of the heads of a growing hairpin packet with the blue $\lambda_{ci}$ contours from $x/R = 19.6$ (the smaller region below the stronger head) to $x/R = 19.9$. Figure D.30 shows the shear layer that connects the three heads (presumably) identified in the previous figure. From figure D.29, the vortices elsewhere (including downstream) are more complicated, and this could occur if there were some azimuthal shift or meandering. Figure D.30 also shows other shear layers that grows in $y$ as $x$ increases. A vortex slightly above the shear layer is indicated in figure D.31 (by the red arrow, also indicating the convection velocity is chosen for this vortex).
Figure D.28: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak E

Figure D.29: A section of streak E ($t = 252R/U_{\text{bulk}}$) with vectors at fine (4.5+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_{\theta}$. 
Figure D.30: A section of streak E ($t = 252R/U_{\text{bulk}}$) with vectors at fine ($4.5^+$) spacing with color contours of $\omega_\theta$. The field view is the same as figure D.29.

Figure D.31: A section of streak E ($t = 252R/U_{\text{bulk}}$) with vectors at fine ($4.5^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 

400
Figure D.32: A section of streak E \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \(\lambda_{c\theta}\) signed by \(\omega_\theta\).

Figure D.33: A section of streak E \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \(\lambda_{c\theta}\) signed by \(\omega_\theta\).
Figure D.34: A section of streak E ($t = 252R/U_{bulk}$) with vectors at fine (4.5\textsuperscript{+}) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_{\theta}$.

Figure D.35: A section of streak E ($t = 252R/U_{bulk}$) with vectors at fine (4.5\textsuperscript{+}) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_{\theta}$.
Figure D.36: A section of streak E \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.35.

Figure D.37: A section of streak E \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_\theta\).
The $x-\theta u$-fluctuation color contour plot for $E$ at $y^+=60$ shows a series of strong negative $u$ fluctuation peaks that the line for the extracted plane passes through at $x/R=20.65, 20.81,$ and $20.95$. This relatively straight arrangement of fluctuations seems like an ideal candidate for the flow pattern associated with a hairpin packet, so this is a good region to examine what vortical structures are present. However, this $x-\theta u$-fluctuation plot also indicates the line for $E$ is slightly offset azimuthally from the center of each individual fluctuation, particularly for the first streamwise half of the plane near $x/R=20.65$. Before examining the details nearer the wall, figure D.32 shows several vortices fairly high above the wall, and the vector pattern surrounding them suggests that the environment near the wall may be one of low momentum fluid, making the near-wall negative $u$ fluctuations particularly strongly low-momentum. Another layer of vortices appears between $r/R=-0.9$ and $-0.8$, but these either are in tubes not normal to the plane shown or do not result in a clear vortex core signature when their center velocity is subtracted. The vortical structure near the negative $u$ observed at 20.65 appears to be related to a tube not normal to this plane, and that is to be expected with the negative fluctuation peak at a azimuthal offset to this plane. The negative $u$ regions at 20.81 and 20.95 also appear related to vortices with orientations different than heads in this plane. Figure D.29 shows this plane with a subtracted velocity more suitable for the near-wall structures. Based on the cut in this plane, there is not clear evidence of a hairpin packet with vortices relatively near the wall, and a 3D examination would shed more light on this flow pattern.

For the next streamwise segment up to $x/R=22.3$, the strong negative $u$ fluctuation peak is azimuthally offset from the extracted plane. Figure D.34 shows the presence of vortical structures in this plane that are predominantly not oriented normal to the plane, yielding a large number of segments with retrograde components.
of azimuthal vorticity (but the majority of the vorticity is probably oriented not out
of the plane).

After this region, the plane with $x/R$ ranging from 22.3 to 23.2 slices through a
strongly negative $u$ fluctuation region, and through the center of the negative $u$ fluc-
tuation peak (at $y^+ = 60$) for $x/R = 22.6$ to 22.9 and 23.0 to 23.1. This is another
good candidate for a packet. However, vortical structures consistent with packets
are not clearly visible in figure D.35. Subtracting the streamwise velocity at the cen-
ters of several cores (as indicated by $\lambda_{ci}$) does not yield clear velocity signatures of
cores because there is radial velocity towards the pipe centerline. Figure D.36 does
indicate the presence of inclined shear layers (without clear vortex heads) above the
strong negative $u$ regions. A 3D examination of the vortices in this vicinity would
reveal if vortices in different azimuthal positions are associated with the apparent
radial inward flow in this region, as evidenced by the radial inward vectors and
lifting from the wall likely associated with the shear layer. Figure D.37 is the last
section in the extracted plane.

D.6 Streak F of $t = 252R/U_{\text{bulk}}$ field

Figure D.39 includes the flow passing through strong negative $u$ fluctuations at
$x/R = 20.0$ and 20.55 at $y^+ = 60$. There is evidence of nearby complex vortices at
each of these locations. The relatively low subtracted frame convection velocity to
down the cores at this height ($> 69^+$ from the wall) suggests that large-scale motions
of some type without obvious, strong, head-like vortex cores in this plane are con-
tributing significantly to the negative $u$ fluctuation in the frame shown (particularly
at distances such as $y^+ = 101$ above the wall).

The plane shown in figure D.40 passes through strong negative $u$ fluctuations in
the x-theta inset at $x/R = 21.28$ and 21.59 for $y^+ = 60$. There is evidence of vortices
near these regions, but the presence of azimuthal vorticity both in and out of the

405
Figure D.38: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak F

Figure D.39: A section of streak F ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 
Figure D.40: A section of streak F \( (t = 252R/U_{\text{bulk}}) \) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \( \lambda_{ci} \) signed by \( \omega_\theta \).

Figure D.41: A section of streak F \( (t = 252R/U_{\text{bulk}}) \) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \( \lambda_{ci} \) signed by \( \omega_\theta \).
plane for individual regions of strong $\lambda_{ci}$ indicates their orientations are different from those of heads. Figure D.41 corresponds to a region of weakly negative $u$ fluctuation. The vortex identified is associated with a single region of strong negative $u$ fluctuation nearer the wall than the $x-\theta$ plane that was extracted. There is an inclined shear layer upstream that lifts up to this vortex head signature. Overall, this pattern matches the hairpin vortex signature well. There is another region of swirl nearby that is nearer the wall, and another shear layer connects this to the larger shear layer that was discussed. Other vortical structures also exist in this plane. Figure D.42 contains the last section along the extracted plane, and this region contains weak to moderate positive $u$ fluctuation.

D.7 Streak G of $t = 252R/U_{\text{bulk}}$ field

Figure D.43 at $y^+ = 60$ contains a series of negative $u$ peaks between $x/R = 24.5$ and 25.5, followed by a region of positive $u$ fluctuation. Figure D.44 shows that this positive $u$ fluctuation region occurs slightly above a head near $x/R = 25.7$ (indicated

Figure D.42: A section of streak F ($t = 252R/U_{\text{bulk}}$) with vectors at fine ($4.5^+$) spacing with color contours of $3D \lambda_{ci}$ signed by $\omega_\theta$. 

408
Figure D.43: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak G

Figure D.44: A section of streak G ($t = 252R/U_{\text{bulk}}$) with vectors at coarse (6.75+) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 
Figure D.45: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.44.

Figure D.46: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_\theta\).
Figure D.47: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.46.

Figure D.48: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_\theta\). The field view is the same as figure D.46 but the subtracted velocity is different.
Figure D.49: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_p\). The field view is the same as figure D.46 but the subtracted velocity is different.

Figure D.50: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_p\). The field view is the same as figure D.46 but the subtracted velocity is different.
Figure D.51: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_{\theta}\).

Figure D.52: A section of streak G \((t = 252R/U_{\text{bulk}})\) with vectors at fine \((4.5^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_{\theta}\).
by the right arrow), and there is evidence of additional vortices above this with predominantly retrograde azimuthal swirl. At the higher $y^+ = 101$, the flow is a negative $u$ fluctuation, so the contour in the $y^+ = 101$ $x-\theta$ plane shows a continuous negative $u$ fluctuation region. There are many complex vortices (apparently having orientations other than normal to the plane) in the negative $u$ fluctuation region for both $y^+$ locations at lower $x/R$. There is evidence of higher vortices consistent with orientation normal to the plane based on $\lambda_{ci}$ patterns, and also evidence of a shear layer that extends around $0.4R$ wall-normal from the pipe wall. This large shear layer could be important to the overall low speed character of the fluid in this region.

In the next section of the $x-r$ plane, the appearance matches the classic hairpin packet shape more closely. The first example of this frame (figure D.46) has the same convection velocity subtracted as the previous figure. Figure D.47 shows several inclined shear layers rising up to approximately $0.15R$ ($100^+$) above the wall. In addition, there appears to be detached azimuthal vorticity above and downstream the shear layers, consistent with the idea of hairpins rolling up and concentrated vorticity detaching from near-wall layers. However, subtracting the velocities at the centers of several different vortex cores does not reveal a clear core vector pattern (figures D.48, D.49, and D.50). The elongated $\lambda_{ci}$ contours suggest that several of the vortex tubes may not be normal to the plane. Figure D.50, with the lowest convection velocity subtracted, shows substantial radial upward velocity in the vicinity of $x/R = 26.5$ within $0.2R$ from the wall.

The next frame along the extracted plane, figure D.51, displays few clear vortices and some weak shear layers. There are vortices of uncertain orientation near the regions of positive fluctuation (most of the frame contains weak negative $u$ fluctuation). For the last frame, figure D.52, there is also a lack of lifted up shear layers, and the velocity fluctuations tend to be weak in this region.
D.8 Streak H of $t = 252R/U_{\text{bulk}}$ field

Figure D.53: $u$ fluctuation color contours of $t = 252R/U_{\text{bulk}}$ field low momentum streak H

Figure D.54: A section of streak H ($t = 252R/U_{\text{bulk}}$) with vectors at coarse ($6.75^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 
Figure D.55: A section of streak H \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{\text{ci}}\) signed by \(\omega_\theta\).

Figure D.56: A section of streak H \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.55.
Figure D.57: A section of streak H \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_{\theta}\).

Figure D.58: A section of streak H \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of \(\omega_{\theta}\). The field view is the same as figure D.57.
Figure D.59: A section of streak H \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of 3D \(\lambda_{ci}\) signed by \(\omega_\theta\).

Figure D.60: A section of streak H \((t = 252R/U_{\text{bulk}})\) with vectors at coarse \((6.75^+)\) spacing with color contours of \(\omega_\theta\). The field view is the same as figure D.59.
Figure D.54 contains many small-scale vortex cores with both prograde and retrograde orientations. In the plane section considered in figure D.54, there is a strong negative $u$ fluctuation near $x/R = 24.4$ at $y^+ = 60$. There is evidence of several vortices nearby that could have the correct orientation to be a head, but other nearby vortices also exist.

The next section of this plane is shown in figure D.55. The relatively strong positive $u$ fluctuation near $x/R = 25.85$ appears to be above a vortex head and below a vortex tube with orientation not predominantly normal to the plane. The strong negative $u$ fluctuation at 26.27 and upstream has several nearby vortices (regions of $\lambda_{ci}$) that do not seem to be clear heads. Shear layers appear to be related to the negative $u$ fluctuations nearer the wall (figure D.56). The detected negative $u$ region at $y^+ = 60$ ($r/R = -0.912$) may be related to the upper tip of a shear layer, the vortex (of some orientation) at $x/R = 26.26$, and the higher vortex at $x/R = 26.1$.

Figure D.61: A section of streak H ($t = 252R/U_{bulk}$) with vectors at fine ($4.5^+$) spacing with color contours of 3D $\lambda_{ci}$ signed by $\omega_\theta$. 

419
In the cross-section corresponding to figure D.57, there is mainly a region of weak negative $u$ fluctuation and a negative $u$ peak near $x/R = 27.07$. Figure D.57 shows few (if any) strong vortices within $0.3R$ of the wall. Figure D.58 shows the azimuthal vorticity, and shear layers are clearly evident, with evidence of a vortex (based on the 3D $\lambda_{ci}$) next to the upper tip of a shear layer, and the vortex appears to be oriented not normal to the plane. The negative $u$ fluctuations that are visualized in $x-\theta$ occur under shear layers.

Figure D.59 uses the same convection velocity as the previous figures. Note the wall-normal position where vectors have zero-streamwise velocity components compared to the previous figures. This frame contains significant positive fluctuation, but the previous section contains negative $u$ fluctuations. This may be evidence of a large scale “background” flow contributing to the overall flow for the region shown in each frame or possibly other large-scale vortices that are not clearly discerned. For this figure D.59, in the positive $u$ fluctuation regions there are very few vortices near the wall, and the shear layers are smooth (figure D.60), not lifting up from the wall as often occurs in other fields with significant negative $u$ fluctuations. Figure D.61 is the last in the series for this extracted plane H.
E.1 Author Permissions for Dissertation Material

The majority of the text in chapter 3 was published in Baltzer & Adrian (2011) with co-author R. J. Adrian. He has given his permission for the manuscript to be used in this dissertation.

The majority of the text in chapter 4 is a manuscript currently in press as Wu et al. (2012), prepared with co-authors X. Wu and R. J. Adrian. They have given their permission for the manuscript to be used in this dissertation.

The majority of the text in chapter 5 is a manuscript currently being prepared with co-authors R. J. Adrian and X. Wu. They have given their permission for the manuscript to be used in this dissertation.

The majority of the text in chapter 7 was published in Baltzer et al. (2010) with co-authors R. J. Adrian and X. Wu. They have given their permission for the manuscript to be used in this dissertation.
E.2 Publisher Permissions for Dissertation Material

Copyright permission for chapter 3:

<table>
<thead>
<tr>
<th>Field</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>License Number</td>
<td>2878630217490</td>
</tr>
<tr>
<td>Order Date</td>
<td>Mar 30, 2012</td>
</tr>
<tr>
<td>Publisher</td>
<td>American Institute of Physics</td>
</tr>
<tr>
<td>Publication</td>
<td>Physics of Fluids</td>
</tr>
<tr>
<td>Article Title</td>
<td>Structure, scaling, and synthesis of proper orthogonal decomposition modes of inhomogeneous turbulence</td>
</tr>
<tr>
<td>Author</td>
<td>Jon R. Baltzer, Ronald J. Adrian</td>
</tr>
<tr>
<td>Online Publication Date</td>
<td>Jan 26, 2011</td>
</tr>
<tr>
<td>Volume number</td>
<td>23</td>
</tr>
<tr>
<td>Issue number</td>
<td>1</td>
</tr>
<tr>
<td>Type of Use</td>
<td>Thesis/Dissertation</td>
</tr>
<tr>
<td>Requestor type</td>
<td>Author (original article)</td>
</tr>
<tr>
<td>Format</td>
<td>Print and electronic</td>
</tr>
<tr>
<td>Portion</td>
<td>Excerpt (&gt; 800 words)</td>
</tr>
<tr>
<td>Will you be translating?</td>
<td>No</td>
</tr>
<tr>
<td>Title of your thesis / dissertation</td>
<td>Structure and Proper Orthogonal Decomposition in Simulations of Wall-bounded Turbulent Shear Flows with Canonical Geometries</td>
</tr>
<tr>
<td>Expected completion date</td>
<td>May 2012</td>
</tr>
<tr>
<td>Estimated size (number of pages)</td>
<td>300</td>
</tr>
<tr>
<td>Total</td>
<td>0.00 USD</td>
</tr>
</tbody>
</table>

Table E.1: Copyright permission for chapter 3.

Terms and Conditions American Institute of Physics – Terms and Conditions:
Permissions Uses American Institute of Physics (“AIP”) hereby grants to you the non-exclusive right and license to use and/or distribute the Material according to the use specified in your order, on a one-time basis, for the specified term, with a maximum distribution equal to the number that you have ordered. Any links or other content accompanying the Material are not the subject of this license.
Copyright permission for chapter 4:

[Note: at the time of obtaining copyright permission, the article was available online (FirstView), without content date and volume/issue information available.]

This is a License Agreement between Jon Baltzer (“You”) and Cambridge University Press (“Cambridge University Press”) provided by Copyright Clearance Center (“CCC”). The license consists of your order details, the terms and conditions provided by Cambridge University Press, and the payment terms and conditions.

All payments must be made in full to CCC. For payment instructions, please see information listed at the bottom of this form.

TERMS & CONDITIONS

Cambridge University Press grants the Licensee permission on a non-exclusive non-transferable basis to reproduce, make available or otherwise use the Licensed content ‘Content’ in the named territory ’Territory’ for the purpose listed ’the Use’ on Page 1 of this Agreement subject to the following terms and conditions.

1. The License is limited to the permission granted and the Content detailed herein and does not extend to any other permission or content.

2. Cambridge gives no warranty or indemnity in respect of any third-party copyright material included in the Content, for which the Licensee should seek separate permission clearance.

3. The integrity of the Content must be ensured.

4. The License does extend to any edition published specifically for the use of handicapped or reading-impaired individuals.

5. The Licensee shall provide a prominent acknowledgement in the following format: author/s, title of article, name of journal, volume number, issue number, page references, reproduced with permission.

424
Table E.2: Copyright permission for chapter 4.
E.3 Permissions for Figures Reproduced from Other Publications

Copyright permission for figure 5.9:

<table>
<thead>
<tr>
<th>License Number</th>
<th>2878861096653</th>
</tr>
</thead>
<tbody>
<tr>
<td>License date</td>
<td>Mar 30, 2012</td>
</tr>
<tr>
<td>Licensed content publisher</td>
<td>Springer</td>
</tr>
<tr>
<td>Licensed content publication</td>
<td>Experiments in Fluids</td>
</tr>
<tr>
<td>Licensed content title</td>
<td>Analysis and interpretation of instantaneous turbulent velocity fields</td>
</tr>
<tr>
<td>Licensed content author</td>
<td>R. J. Adrian</td>
</tr>
<tr>
<td>Licensed content date</td>
<td>Sep 1, 2000</td>
</tr>
<tr>
<td>Volume number</td>
<td>29</td>
</tr>
<tr>
<td>Issue number</td>
<td>3</td>
</tr>
<tr>
<td>Type of Use</td>
<td>Thesis/Dissertation</td>
</tr>
<tr>
<td>Portion</td>
<td>Figures</td>
</tr>
<tr>
<td>Author of this Springer article</td>
<td>No</td>
</tr>
<tr>
<td>Order reference number</td>
<td></td>
</tr>
<tr>
<td>Title of your thesis / dissertation</td>
<td>Structure and Proper Orthogonal Decomposition in Simulations of Wall-bounded Turbulent Shear Flows with Canonical Geometries</td>
</tr>
<tr>
<td>Expected completion date</td>
<td>May 2012</td>
</tr>
<tr>
<td>Estimated size(pages)</td>
<td>300</td>
</tr>
<tr>
<td>Total</td>
<td>0.00 USD</td>
</tr>
</tbody>
</table>

Table E.3: Copyright permission for figure 5.9.
Copyright permission for figure 1.1:

Figure E.1: Copyright permission for figure 1.1.